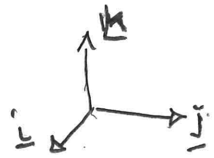
14/12/2015

Vektor operatorerer og deres tolkning

(MEK 1100 — "Vector calculus" P.C. Matthews)

Nabla operatoren $\nabla(\ ) = [\partial/\partial x, \partial/\partial y, \partial/\partial z] \rightarrow$ Vektor



$\underline{(\ )}$ = vektor rotasjon, $\underline{\underline{(\ )}}$ = tensor

Divergens:

enhetsvektoren $\underline{e} = [\underline{e}_x, \underline{e}_y, \underline{e}_z] = [\underline{i}, \underline{j}, \underline{k}]$

→ Essensiell for alle bevarelses lover

$$\nabla \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Eks.:

a) Strømning gitt av $\underline{u} = u\,\underline{i}$

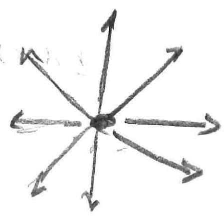
$\Rightarrow \nabla \cdot \underline{u} = \frac{\partial u}{\partial x}$ dersom $u = u(y,z,t) \rightarrow$ divergens fri strømning $\frac{\partial u}{\partial x} = 0$, konstant volum

$\frac{\partial u}{\partial x} \gtrless 0$ → volum økning  → volum minkning

b) $\underline{u} = k\underline{r} = kx\,\underline{i} + ky\,\underline{j} + kz\,\underline{k}$

$\nabla \cdot \underline{u} = 3k$

Ikke divergens fri!



Strømning ut av origo/kilde, $|\underline{u}| \uparrow\ r \uparrow$ $k > 0$

$u \downarrow\ r \uparrow$ $k < 0$

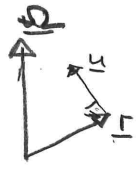
c) $\underline{u} = \underline{\Omega} \times \underline{r}$, $\underline{\Omega} = \Omega\,\underline{k}$

(Noter $\underline{\Omega} \times \underline{r}$ = kryssprodukt. I læreboka Acheson $\wedge$)

$\underline{u} = \Omega\underline{k} \times (x\underline{i} + y\underline{j} + z\underline{k}) = \Omega x\,\underline{j} - \Omega y\,\underline{i} \rightarrow u = -\Omega y,\ v = \Omega x,\ w = 0$

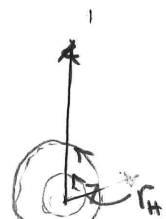
$$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Geometrisk tolkning



$|\underline{u}| = \sqrt{u^2 + v^2} = \Omega\sqrt{x^2 + y^2} = \Omega r_H$

Rotasjon som fast legeme



Hvirvling : $\nabla \times \underline{A} \rightarrow \boxed{\text{vektor}}$ $\quad \nabla \times (\,) = \text{curlen}$

$$\nabla \times \underline{A} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\underline{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\underline{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\underline{j}$$

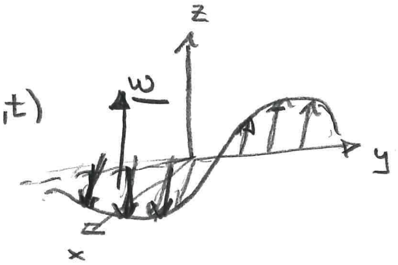
Hvirvling av en strømning dvs **vortisiteten** $= \boxed{\underline{\omega} \equiv \nabla \times \underline{u}}$

(Noter; Hvirvling → komp. Gjevik; $\underline{\zeta}$, Acheson → $\underline{\omega}$)

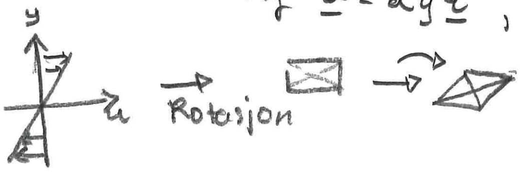
Eks.

a) $\underline{u} = u\underline{i}$, $u = u(x,y,z,t)$, $v = 0$, $w = 0$

$\underline{\omega} = \nabla \times \underline{u} = \frac{\partial u}{\partial z}\underline{j} - \frac{\partial u}{\partial y}\underline{k}$ $\quad$ antar $u = u(x,y,t)$



Selv om hastighetsfeltet er divergensfritt kan det ha virvling $\underline{u} = ay\underline{i}$, $\underline{\omega} = -a\underline{k}$

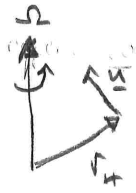


c) $\underline{u} = \underline{\Omega} \times \underline{r}$, $\underline{\Omega} = \Omega\underline{k}$ $\quad (\underline{r} = x\underline{i} + y\underline{j} + 0\cdot\underline{k},\ |r| = r_H)$

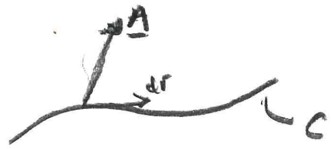
$\underline{u} = \Omega x\underline{j} - \Omega y\underline{i}$ $\quad \wedge\ \Omega = \text{konstant}$

$$\underline{\omega} = \nabla \times \underline{u} = \left(\cancel{\frac{\partial w}{\partial y}} - \cancel{\frac{\partial v}{\partial z}}\right)\underline{i} + \left(\cancel{\frac{\partial u}{\partial z}} - \cancel{\frac{\partial w}{\partial x}}\right)\underline{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{k}$$

$$\boxed{\underline{\omega} = 2\underline{\Omega} \Rightarrow \underline{\Omega} = \frac{1}{2}\underline{\omega}}$$

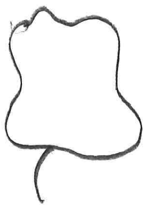


Rotasjon som stivt legeme, hvirvlingen $= \frac{1}{2}$ vinkelhastigheten



Sirkulasjonen $\Gamma(\underline{r},t)$

[Notasjon; Komp. Gjevik – $C(\underline{r},t)$, Acheson: $\Gamma(\underline{r},t)$]



C = lukket kurve

Kurve integral; $\int_C \underline{A}\cdot d\underline{r}$

$$\boxed{\text{sirkulasjonen} = \Gamma \equiv \oint_C \underline{A}\, d\underline{r}}$$

(skalar)

Integralet avhenger av kurven C

$$\Gamma = \oint (A_x dx + A_y dy + A_z dz)$$

Dersom $\underline{A}$ er en konstant vektor (i rommet) $\Rightarrow \oint \underline{A}\, d\underline{r} = A_x \oint dx + A_y \oint dy + A_z \oint dz = 0$

$\rightarrow \oint d\underline{r} = 0$

Hvis $\underline{A} = \nabla\beta$ og β er entydig får vi

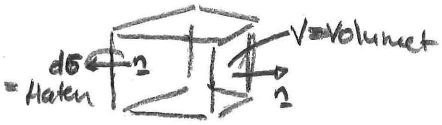
$$\boxed{\Gamma = \oint_C \nabla\beta\, d\underline{r} = 0}$$

(β få start verdien β_0 ved slutten av kurven)

Integral satser;

Gauss; Divergens-integral

$$\boxed{\begin{array}{c} \int_V \nabla\cdot\underline{A}\, dV = \int_\sigma \underline{A}\cdot\underline{n}\, d\sigma \\ \text{volum int.} = \text{flate int.} \end{array}}$$



σ = flatene

$\underline{A}\cdot\underline{n}\, d\sigma$ = fluxen gjennom flaten $d\sigma$

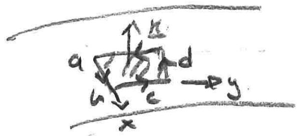
Green; setter $\underline{A} = \beta\nabla k \quad \wedge \quad \beta = \beta(\underline{r}),\ k(\underline{r})$

(ser at $\nabla k\cdot n = \frac{\partial k}{\partial n}$)

$$\int_V \nabla\cdot(\beta\nabla k)\, dV = \int_V \nabla\beta\cdot\nabla k\, dV + \int_V \beta\nabla^2 k\, dV = \int_\sigma \beta\nabla k\cdot\underline{n}\, d\sigma$$

$$\Rightarrow \boxed{\int_V \nabla\beta\cdot\nabla k\, dV = -\int_V \beta\nabla^2 k\, dV + \int_\sigma \beta\nabla k\cdot\underline{n}\, d\sigma}$$

Stokes



Hvirvlingen til et vilkårlig flate element $d\underline{\sigma} = dx\,dy\,\underline{k}$ (x-y planet)

Integrerer over hele overflaten gir

$$\int \nabla \times \underline{A} \cdot d\underline{\sigma} = \iint \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) dx\,dy = \underbrace{\int_b^c A_y\,dy - \int_a^d A_y\,dy - \int_d^c A_x\,dx + \int_a^b A_x\,dx}_{\text{sirkulasjonen av vektor}}$$

$$\boxed{\int \nabla \times \underline{A} \cdot d\underline{\sigma} = \oint \underline{A} \cdot d\underline{r}}$$