Jan. 4, 2015

Potensial strømning → redusert Euler's ligning – ikke linjær – u, v, w, P, 4 variabler

Linear ligningsett med **1** ϕ variabel!

$$\boxed{\nabla^2\phi = \nabla^2\psi = 0 \quad , \quad \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \quad , \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}}$$

→ hvordan bestemme vilkårlig ϕ, ψ ?

Grensebetingelser?

1) solid

$\underline{n} = \frac{\nabla f}{|\nabla f|}$

Ingen flux gjennom solidet;

$\underline{u} \cdot \underline{n} = 0 \Rightarrow \frac{\partial\phi}{\partial n} = 0$ eller $\frac{\partial\psi}{\partial s} = 0$ (⇒ veggen er en strømlinje!)

Viktig: Potensialstrømning ved solid $\underline{u} \cdot \underline{t} \neq 0$ ⇒ fluidet "slipper" solidet.
Kun sant når $\mu = 0$, dersom $\mu \to 0$ (singulær pert.) $\underline{u} \cdot \underline{t} = 0$! no-slip!

2) "Far-field", $\underline{x} \to \infty$,

Ofte konstant hastighet dvs. $\frac{\partial\phi}{\partial x} = U \wedge \frac{\partial\psi}{\partial y} = U$

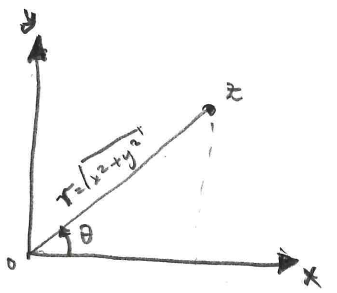
$\nabla^2\phi$ ($, \nabla^2\psi$) + 1, 2) → utfordrende analytisk (numerisk løsning)

Løsningen på en Laplace ligning kan beskrives som en sum av harmoniske funksjoner → lineær, prinsipp superposisjon (sum av løsninger).
→ finner funksjonen som oppfyller grensebetingelsene!

Komplekse potensialet

Cauchy-Riemann relasjonene gitt av løsning via kompleks-variabel teori

Komplekse tallet; $z \equiv \underbrace{x}_{\text{reell}} + i\underbrace{y}_{\text{imaginær}} = re^{i\theta} = r(\cos\theta + i\sin\theta)$, husk at $i^2 = -1$



$$\ln(z) = \ln(r) + i\theta$$

Løsning: Komplekse hastighetspotensialet ; $\boxed{w = \phi + i\psi}$ ^ $\nabla\phi, \nabla\psi$ er kontinuerlig
er en løsning på C-R. relasjonen

Hastighetskomponentene gitt av, $\boxed{\frac{dw}{dz} = \frac{\partial}{\partial x}(\phi + i\psi) = u - iv}$

(komplekse hastigheten)

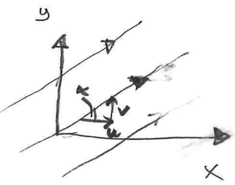
δz = paralell med x-akse
→ samme resultat dersom ∥ y-akse

Hastigheten, $\underline{\left|\frac{dw}{dz}\right| = q = (u^2+v^2)^{\frac{1}{2}}}$

Typiske fluid strømninger → "komponenter" av hele strømningen

Eks;

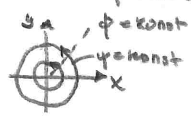
i) strømning med vinkel α - xaksen , $u = U\cos\alpha$, $v = U\sin\alpha$



Hastighetspotensialet? $\frac{dw}{dz} = u + iv = U(\cos\alpha + i\sin\alpha) = Ue^{-i\alpha}$

$\Rightarrow \underline{w = Uze^{-i\alpha}}$

ii) Linje sirkulasjon (aksesymmetrisk) , $\underline{u} = \frac{\Gamma}{2\pi r}\underline{e_\theta}$ ^ Γ = sirkulasjonen



$\underline{u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = 0}$, $\underline{u_\theta = -\frac{\partial\psi}{\partial r} = \frac{\Gamma}{2\pi r}}$

$\Rightarrow \underline{\psi = \frac{-\Gamma}{2\pi}\ln(r)}$,

$u_r = \frac{\partial\phi}{\partial r} = 0$, $u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\Gamma}{2\pi r}$ $\Rightarrow \underline{\phi = \frac{\Gamma\theta}{2\pi}}$

Komplekse pot. $w = \phi + i\psi = \frac{\Gamma}{2\pi}(\theta - i\ln(r)) = \frac{-i\Gamma}{2\pi}\underbrace{(\ln(r) + i\theta)}_{\ln(z)} = \underline{\frac{-i\Gamma}{2\pi}\ln(z)}$

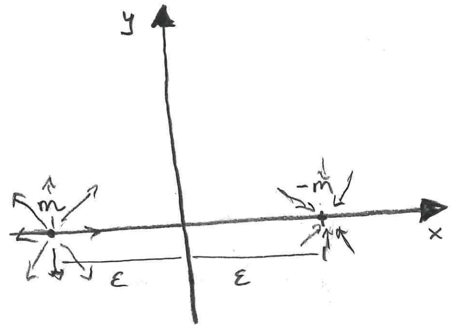
iii) Kilde / sluk

$u_r = \frac{m}{2\pi r}$, $u_\theta = 0$ $\Rightarrow$ $\phi = \frac{m}{2\pi}\ln(r)$, $\psi = \frac{m}{2\pi}\theta$

$w = \frac{m}{2\pi}(\ln(r) + i\theta)$

$\Rightarrow \underline{w = \frac{m}{2\pi}\ln z}$

iv) **Dipol**: spesialtilfelle der sluk/kilde sammenfaller på samme punkt



$x = -\varepsilon$, kilde med styrke m

$x = \varepsilon$, sluk med styrke $-m$

Ved hjelp av superposisjon;

$$w(z) = \frac{m}{2\pi} \ln(z-(-\varepsilon)) + \frac{-m}{2\pi} \ln(z-\varepsilon)$$

$$= \frac{m}{2\pi} \ln\left(\frac{(z+\varepsilon)\cdot 1/z}{(z-\varepsilon)\cdot 1/z}\right) = \frac{m}{2\pi} \ln\left(\frac{1+\varepsilon/z}{1-\varepsilon/z}\right) = \frac{m}{2\pi} \ln\left(\left(1+\frac{\varepsilon}{z}\right)\underbrace{\left(\frac{1}{1-\varepsilon/z}\right)}_{f(\varepsilon)}\right)$$

Taylor rekkeutvikling

Taylor rekke; $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \dots$

utvikle $f(\varepsilon)$

$$f(\varepsilon) = \frac{1}{1-\varepsilon_0/z} + (\varepsilon-\varepsilon_0)\cdot\frac{1/z}{(1-\varepsilon_0/z)^2} + \mathcal{O}(\varepsilon^2)$$

$\varepsilon_0 = 0 \Rightarrow \underline{f(\varepsilon) = 1 + \frac{\varepsilon}{z}} + \mathcal{O}(\varepsilon^2)$

Insatt i $w(z) = \frac{m}{2\pi} \ln\left(\left(1+\frac{\varepsilon}{z}\right)\left(1+\frac{\varepsilon}{z}\right)\right) = \frac{m}{2\pi} \ln\left(1+\frac{\varepsilon}{z} + \mathcal{O}(\varepsilon^2)\right)$ $\qquad \varepsilon^2 \ll 1$

Husker at; $\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} \dots -1 \leq x \leq 1$

$$\underline{w(z) = \frac{m}{2\pi} \ln\left(1+\frac{\varepsilon}{z}\right) = \frac{m\varepsilon}{2\pi z} + m\cdot\mathcal{O}(\varepsilon^2)}$$

For at sluk/kilde skal overlappe lar vi $\varepsilon \to 0$. For at $\underline{u} \equiv 0$ lar vi $m \to \infty$

Grensen $\lim_{\substack{\varepsilon\to 0\\ m\to\infty}} (m\varepsilon) = \Gamma$ (konstant) $\Rightarrow$ $\boxed{w(z) = \lim_{\substack{\varepsilon\to 0\\ m\to\infty}} \left\{\frac{m}{2\pi}\left(\frac{\varepsilon}{z} + \mathcal{O}(\varepsilon^2)\right)\right\} = \frac{\Gamma}{2\pi z}}$

Finn ϕ og ψ fra $w(z)$,

$$w(z) = \frac{\Gamma}{2\pi(x+iy)}\left(\frac{x-iy}{x-iy}\right) = \frac{\Gamma x - \Gamma i y}{2\pi(x^2+y^2)} = \phi + i\psi$$

$$\underline{\phi = \frac{\Gamma x}{2\pi(x^2+y^2)}}$$

$$\underline{\psi = \frac{-\Gamma y}{2\pi(x^2+y^2)}}$$

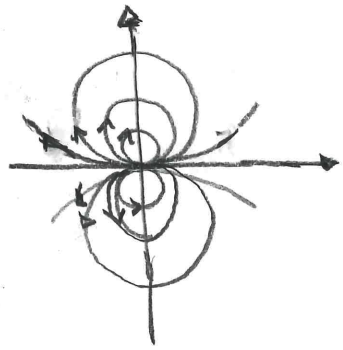
Omskriver ψ; $\psi = \dfrac{-\Gamma y}{2\pi(x^2+y^2)} \iff x^2+y^2+\dfrac{\Gamma y}{2\pi\psi} = 0$

$$\underline{x^2 + \left(y+\frac{\Gamma}{4\pi\psi}\right)^2 = \left(\frac{\Gamma}{4\pi\psi}\right)^2}$$

gitt strømlinje $\psi = \psi_0 =$ konst.
$\Rightarrow$ sirkel med senter $x=0$, $y = \dfrac{\Gamma}{4\psi_0\pi}$

radius; $r = \dfrac{\Gamma}{4\psi_0\pi}$

<u>Dipol</u>



Bruke prinsippet for superposisjon til å finne kompliserte strømnings felt;
(kilde, dipol ect)

<u>Strømning rundt en sylinder</u> (uniform hastighet + dipol) → vegg/sylinder

Virvlingsfri strømning med $w(z) = Uz + \dfrac{\Gamma}{2\pi z} = U\left(z+\dfrac{a^2}{z}\right)$ $\quad a =$ sylinder radius $= \sqrt{\Gamma/2\pi U}$

$$w(z) = U\left(re^{i\theta} + \frac{a^2}{r}e^{-i\theta}\right)$$

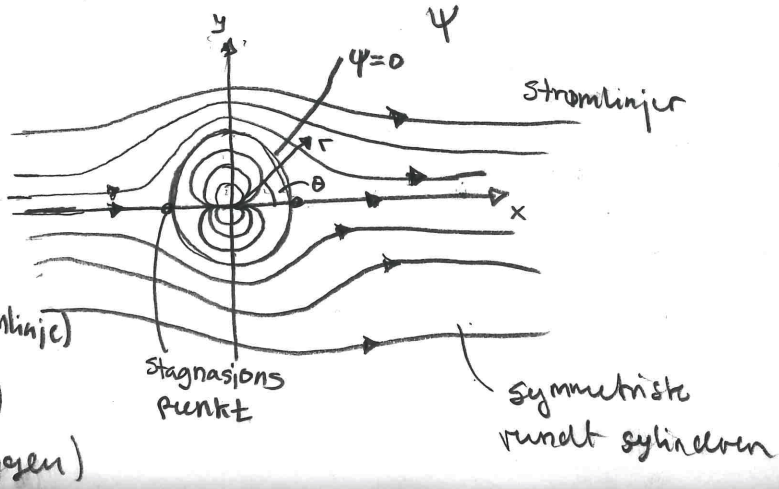
$$= Ur(\cos\theta + i\sin\theta) + \frac{Ua^2}{r}(\cos\theta - i\sin\theta)$$

$$w(z) = \underbrace{Ur\cos\theta\left(1+\frac{a^2}{r^2}\right)}_{\phi} + i\underbrace{Ur\sin\theta\left(1-\frac{a^2}{r^2}\right)}_{\psi}$$

Hastighetene;

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\theta\left(1-\frac{a^2}{r^2}\right)$$

$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta\left(1+\frac{a^2}{r^2}\right)$$



Ved sylinderoverflaten;

$\psi(r=a) = Ua\sin\theta\left(1-\dfrac{a^2}{a^2}\right) = 0$ (vegg = strømlinje)

$u_r(r=a) = 0$ (ingen fluks gjennom veggen)

$u_\theta(r=a) = \underline{-U\sin\theta \cdot 2}$ (slip langs veggen)

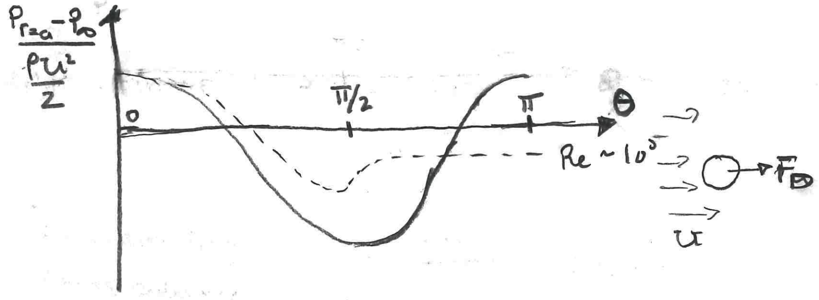
Siden sylinderoverflaten er en strømlinje, $\psi(r=a)=0$, virvelfri, stasjonær
$\omega=0$, $\frac{\partial}{\partial t}=0$

Stagnasjonspunkt på sylinderen $\theta=0,\pi$. $u_r=u_\theta=0$, $r=a$.

Finner trykket på sylinderen v.h.a. Bernoulli's thr.

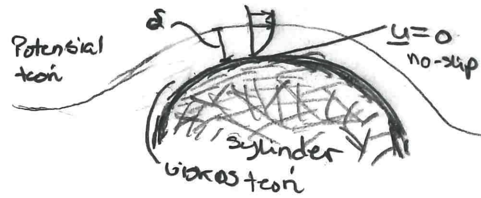
$$\underbrace{P_\infty+\frac{\rho U^2}{2}}_{r\gg a,\ \theta=\pi\ \text{(stag.trykk)}}=\underbrace{\left[p+\frac{\rho u^2}{2}\right]_{r=a}}_{\text{sylinder}}=P_{r=a}+\frac{\rho}{2}u_\theta^2=P_{r=a}+\frac{\rho}{2}U^2\cdot 4\sin^2\theta$$

$$\frac{P_{r=a}-P_\infty}{\frac{\rho U^2}{2}}=1-4\sin^2\theta$$

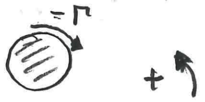


Trykket symmetrisk om sylinderen → net-kraft = 0? (per lengde z)
ingen drag

Ikke troverdig → "virkeligheten" gir motstandskraft/drag → d'Alembert's paradoks (1752) → grensesjikt $Re\to\infty$, $\delta\to 0$
− viskøse krefter (drag)
− Prandtl (1904)



Hva med en roterende sylinder?



$$w(z)=U\left(z+\frac{a^2}{z}\right)+\underbrace{\frac{i\Gamma}{2\pi}\ln\left(\frac{z}{a}\right)}_{\text{linjevirvling}}$$

linjevirvling, $\phi=\frac{\Gamma\theta}{2\pi}$, $\psi=\frac{-\Gamma}{2\pi}\ln(r)$

$$u_r=\frac{\partial\phi}{\partial r}=U\cos\theta\left(1-\frac{a^2}{r^2}\right)\quad\text{(uendret)}$$

$$u_\theta=\frac{1}{r}\frac{\partial\phi}{\partial\theta}=-U\sin\theta\left(1+\frac{a^2}{r^2}\right)+\frac{\Gamma}{2\pi r}$$

Ved sylinderoverflaten $r=a$; $u_r(r=a)=0$

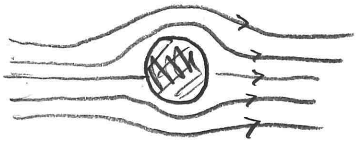
$$u_\theta(r=a)=-2U\sin\theta+\frac{\Gamma}{2\pi a}$$

Stagnasjonspunkt dvs. $u_r(r=a)=u_\theta(r=a)=0$.

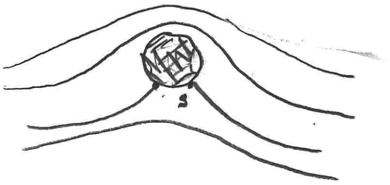
$$\sin\theta_s=\frac{\Gamma}{4\pi Ua}$$

Kraft på sylinderen?

$\frac{\Gamma}{4\pi Ua}=0$



$\frac{-\Gamma}{4\pi Ua}<1$



$\frac{-\Gamma}{4\pi Ua}=1$

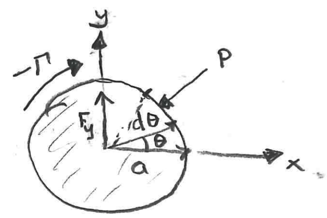
$\frac{-\Gamma}{4\pi Ua}>1$

Bernoullie gir trykket ; $p_\infty + \frac{\rho U^2}{2} = \left[p + \frac{\rho u_\theta^2}{2}\right]_{r=a}$

$$\underbrace{p_\infty + \frac{\rho U^2}{2}}_{\text{konst.}} = p_{r=a} + \frac{\rho}{2}\left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2$$

$p_{r=a} = p_\infty + \frac{\rho U^2}{2} - \frac{\rho}{2}\left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2$ $\Rightarrow$ $p_{r=a}$ er symmetrisk om y-aksen.

Drag, $F_D = F_x = 0$



Kraften på sylinderen i y-retning? oppdrift!

$$F_y = \int_0^{2\pi} -p(r=a)\sin\theta \cdot a\, d\theta = \int_0^{2\pi}\left[\underbrace{\cancel{\left(p_\infty + \frac{\rho U^2}{2}\right)\sin\theta\, a}}_{0} + \frac{\rho a}{2}\left(-2U\sin\theta + \frac{\Gamma}{2\pi a}\right)^2 \sin\theta\right] d\theta$$

$\left(\int_0^{2\pi}\sin\theta\, d\theta = \int_0^{2\pi}\sin^3\theta\, d\theta = 0\right)$

$$= \int_0^{2\pi} -\frac{\rho \cancel{a}}{\cancel{2}} \cdot \cancel{2}U\sin^2\theta \cdot \frac{\Gamma}{\pi \cancel{a}}\, d\theta$$

$$= -\frac{U\rho\Gamma}{\pi}\left[\frac{1}{2}(\theta - \sin\theta\cos\theta)\right]_0^{2\pi} = \underline{-\rho U \Gamma}$$

Rotasjon med klokken $\circlearrowright$ $\Gamma < 0$, $F_y > 0 \uparrow$ $\wedge$ $\circlearrowleft$ $\Gamma > 0$, $F_y < 0 \downarrow$

Rotasjon = oppdrift $\rightarrow$ Magnus effekt

- Fotball
- Baseball
- Skip.

} eksempler + artikkler

Hvordan beskrive strømning der $Re < \infty$?

- viskøse krefter
- no-slip langs veggen
- Andre forenkliger mulig $\rightarrow$ $\frac{\partial \underline{u}}{\partial t} \approx 0$ (stasjonært)

$\underline{u} \cdot \nabla \underline{u} \approx 0$ (lineært)

$Re = 0$ (lineært)

tynne filmer (geometri)