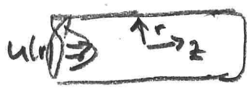
(Eks. 2.2) 04/01/2016 ①

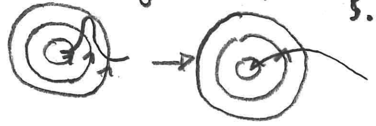
Regn ut selv; Hagen-Poisuelle strømning/rørstrømning



- konstant trykkfall
- aksesymmetrisk
- stasjonært

Virvlings diffusjon

s. 46/47



- 1-D diffusjon.
- similaritetsløsning (hvorfor?)

Så langt;

- statikk
- ideellt/friksjonsfritt fluid (inviskøst, Euler's lig)
- enkle tilfeller av viskøs strømning (laminært)
  - * 1d. $u(x,t)$

Veldig viskøs strømning - kryp/Stoke's-strømning ($Re=0$)

$$Re = \frac{|\rho u \cdot \nabla u|}{|\mu \nabla^2 u|} = \frac{\rho U L}{\mu} \ll 1;$$

Viktig for
- teknologi; lab-on-chip, kosmetikk ect
- micro-organismer
- geologiske strømninger teks. lava strøm.

→ Dropper den ulinjære delen i N.S. i) $\nabla \cdot \underline{u} = 0$

$$\text{ii)} \quad \frac{\partial \underline{u}}{\partial t} = -\nabla P + \nu \nabla^2 \underline{u} \qquad St = \frac{|\partial u/\partial t|}{|\nu \nabla^2 u|}$$

skalering; $\sim U/T \qquad \sim \nu\, U/L^2 \Rightarrow$ Stokes tallet, $St = \dfrac{L^2/\nu}{T}$

(tids effekter/avhengighet) ~ Strouhal nr.

Eks. Bakterie stopper svømme → glide tid/lengde?

$U \sim 10^{-6}\,m/s$, $L \sim 10^{-6}\,m$, $\nu = 10^{-6}\,m^2/s$ (vann) $\Rightarrow Re \approx 10^{-6}$

$$\frac{\partial u}{\partial t} \sim \nu \frac{\partial^2 u}{\partial y^2} \Rightarrow \frac{\cancel{U}}{T} \sim \nu \cdot \frac{\cancel{U}}{L^2} \Rightarrow T = L^2/\nu = 10^{-12}\,m^2/10^{-6}\,m^2/s = \underline{10^{-6}\,s}$$

"glide tid"

Distanse før den stopper; $\ell \approx U \cdot T \approx 10^{-6}\,m/s \cdot 10^{-6}\,s = \underline{10^{-12}\,m} < \underbrace{1\,\text{Å}}_{\text{molekyler lengde}}$

$\nabla \times$ (i), ii)) $\Rightarrow$ i*) $\nabla \cdot (\nabla \times \underline{u}) = \underline{\nabla \cdot \underline{\omega}}$

$$\text{ii*)} \quad \underline{\frac{\partial \underline{\omega}}{\partial t} = \nu \nabla^2 \underline{\omega}}$$

$$\hat{\underline{\omega}} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\underline{i} - \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right)\underline{j} + \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\underline{k}$$

i), ii) → i*) ii*) Matematikk, ingen ny fysikk

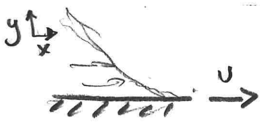
$(2D: \underline{\omega} = \omega \underline{e}_z)$

Klart at dersom; $\nu \to \infty$ (veldig høy viskositet)

$\frac{\partial \underline{\omega}}{\partial t} \to 0$ dvs. virvlingen diffuseres umiddelbart!

⇒ Analo

Analogt med Stoke's 1. problem
- umiddelbar diff.:
- generert ved veggen $\left(\frac{\partial u}{\partial y}\right)$.



↑ diffusjon virvling /momentum $\sim (\nu t)^{1/2}$

Dersom $Re \wedge St \ll 1 \rightarrow$ Stoke's strømning

$$\nabla p = \nu \nabla^2 \underline{u} \qquad \left[\text{virvling; } \nabla^2 \underline{\omega} = 0\right]$$

Konsekvenser:

1) Ingen transient /akselerasjon $\frac{\partial \underline{u}}{\partial t} = 0$, endringer i strømningen merkes umiddelbart i hele feltet!

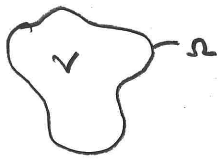
$$Re = \frac{UL}{\nu} = \frac{L^2/\nu}{L/u} = \frac{\text{diffusjons tid}}{\text{adveksjons tid}} \ll 1$$

2) Lineære ligninger

Hvorfor? Ingen produkter av variablene $\underline{u}, P.$ $(\neq Re \gg)$

Unik løsning?

$\boxed{\text{Bevis}}$; Anta $\{\hat{\underline{u}}, \hat{p}\}$ løsning i V med $\hat{u}_B\big|_\Omega = 0$
$\{\underline{u}^*, p^*\}$ også en løsning i V med $u_B^*\big|_\Omega = 0$



"Differensial" hastighet; $\underline{u} = \hat{\underline{u}} - \underline{u}^*$, $p = \hat{p} - p^*$

$$\underline{u}\cdot\Big|\ 0 = -\nabla p + \mu \nabla^2 \underline{u} \quad \Rightarrow 0 = -\underline{u}\cdot\nabla p + \mu\, \underline{u}\cdot\nabla^2 \underline{u}$$

$$0 = \int -\nabla\cdot(p\underline{u})\, dV + \mu \int \underline{u}\cdot\nabla^2\underline{u}\, dV \quad \wedge \quad \nabla\cdot\underline{u} = 0$$

$$0 = \underbrace{-\int p\underline{u}\cdot\underline{n}\, d\Omega}_{\underline{u}\cdot\underline{n}|_\Omega = 0} + \mu\int \underline{u}\cdot\nabla^2\underline{u}\, dV$$

$$0 = \int \underline{u}\cdot\nabla^2\underline{u}\, dV = -\int\left(\nabla\cdot(\underline{u}\cdot\nabla\underline{u}) + (\nabla\underline{u})^2\right) dV$$

$$= \underbrace{-\int(\underline{u}\cdot\nabla\underline{u})\cdot\underline{n}\, d\Omega}_{=0} + \int(\nabla\underline{u})^2 dV = 0$$

QED

- siden $u|_\Omega = 0$ må $u = 0$ i hele V → ved gitt grensebetingelse $\underline{u}$ = unik
- superposisjonsprinsipp, $a\underline{u}_1 + b\underline{u}_2 \wedge aP_1 + bP_2 \wedge$ løsning på Stoke's lig.
- kraft/dreiemoment (Torque) $\sim \int (p, \tau)\, dA \sim \propto u^1$

3.) Reversibel

- Tidsreversibel dersom grenseverdiene er reversert i tid
  (video G.I. Taylor)
- $\{u, p\} \to \{-u, -p\} \Rightarrow$ endring retning hastighet
  — " — kraft

$$(\underset{\sim U^2}{\underline{u}\cdot\nabla\underline{u}}) = -\nabla p + \underset{\sim U}{\mu\nabla^2\underline{u}}$$

ødelegger reversibilitet

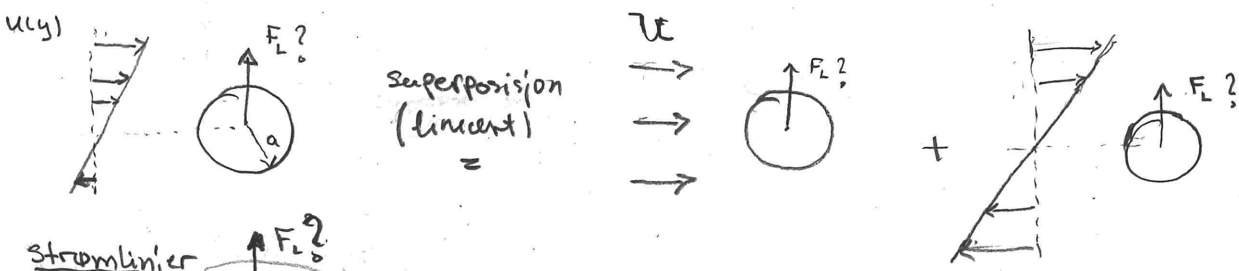
mixing? → som å blande deig ↗ ~~tøye~~ – brette – ~~tøye~~

4.) Symmetri

- Hastigheten rettet langs strømlinjer, ψ. $\underline{u} = [u, v, 0] = [\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0]$
- Stasjonært, $\frac{\partial\psi}{\partial t} = 0 \wedge \underline{u}\cdot\nabla\psi = 0$ → Partikkel baner / strømlinjer / streak lines er like!

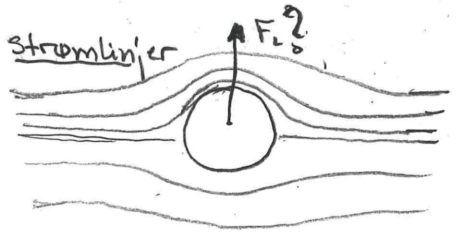
$u\frac{\partial\psi}{\partial x} + v\frac{\partial\psi}{\partial y} = 0$ → $\psi = \text{konst.} \equiv$ strømlinje

Eks. Sylinder i skjærstrøm, lift?



symmetri + reversibilitet $= F_L \equiv 0!$

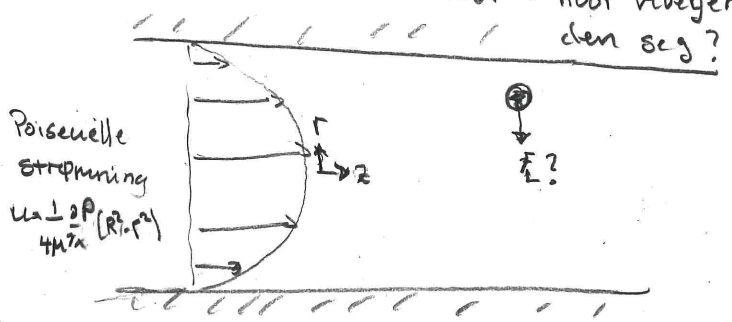
(også sant ved rotasjon)



¡Retning på strømmen?, symmetri!

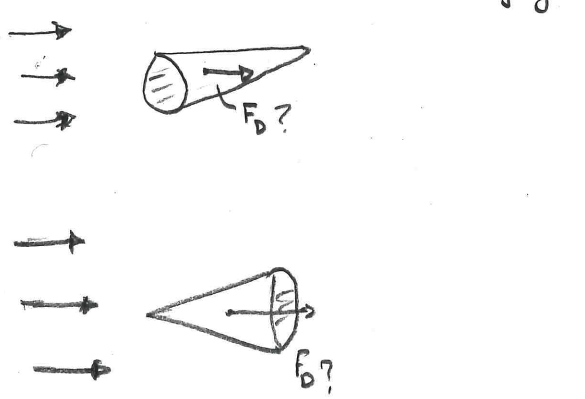
Tenk på to problemstillinger;

1) Partikkler i en kanal → hvor beveger den seg?



$\rho_{fluid} \cdot \rho_{partikkel}$

2) Motstandskraft / drag på en kjegle



Interessant konsekvens av linearitet.

→ forenkler fluid dynamikken til linear algebra

<u>Translasjon</u> (kraften) : $\underline{F} = \underline{\underline{A}}\,\underline{u}$; $\underline{u} = \underline{\underline{M}}\,\underline{F}$; $\underline{\underline{A}} = \begin{bmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix}$ (kan være komplisert å finne)

funksjon av geometrien ← motstands matrise $\xleftrightarrow{(\,)^{-1}}$ mobilitets matrise

<u>Rotasjon</u> (dreiemoment) ; $T = \frac{\mu U}{R} \underset{\text{areal}}{R^2} \cdot \underset{\text{arm}}{R} = \mu R^3 \Omega$

$u = \Omega R$

$\underline{T} = \underline{\underline{D}}\,\underline{\Omega}$

Generelt har vi :

$$\left.\begin{aligned} \underline{F} &= \underline{\underline{A}}\,\underline{u} + \underline{\underline{B}}\,\underline{\Omega} \\ \underline{T} &= \underline{\underline{C}}\,\underline{u} + \underline{\underline{D}}\,\underline{\Omega} \end{aligned}\right\}$$

$\underline{\underline{A}}, \underline{\underline{B}}, \underline{\underline{C}}, \underline{\underline{D}}$ uavhengig av strømning, kun funksjon av geometri

$\underline{\underline{A}} = \underline{\underline{A}}^T$, $\underline{\underline{D}} = \underline{\underline{D}}^T$, $\underline{\underline{B}} = \underline{\underline{C}}^T$ (linear forbindelse)

Eks. kule (sphærisk sym.) $\underline{\underline{A}} = a\underline{\underline{I}}$, $\underline{\underline{B}} = \underline{\underline{C}} = 0$, $D = d\underline{\underline{I}}$ ⇒ Translasjon , $\underline{F} \neq 0 \wedge \underline{T} = 0$

Spiral $\underline{\underline{C}} \neq \underline{\underline{B}}$

$\underline{u}$ genererer $\underline{\Omega}$
eller
$\underline{\Omega}$ genererer $\underline{u}$

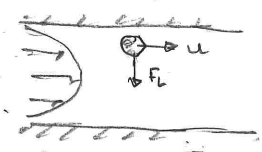
→ translasjon + rotasjon

⇓ bryter kiral symmetri → $\underline{u} \wedge \underline{\Omega}$ er koblet

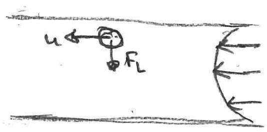
Hvordan <u>unngå</u> rotasjon ?

$\underline{\Omega} = 0$, $\underbrace{B = C^T = 0}_{\text{geometri !}}$

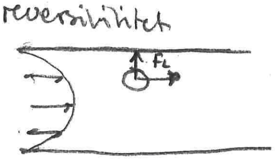
<u>Partikkel</u> ;



symmetri (rotasjon)

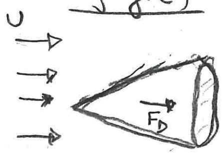


reversibilitet

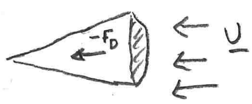


⇒ $\underline{F_L \equiv 0}$

<u>kjegle</u> ;



reversibilitet



⇓

Samme geometri, $\underline{\underline{A}}$

$|F_D|$ uavhengig av retningen til $\underline{u}$!

For $Re = 0$ (ikke intuitivt)

⇒
- drag kan <u>ikke</u> minimeres ved å strømlinjeforme objekter
  ⇒ motsatt for store Re, teks. flyvinge, biler ect.
- geometrien kan generere trans / rot.
- partikler / organismer følger strømlinjer ! $\underline{u} \cdot \nabla \psi = 0$

  ↳ Bevegelser på tvers av strømlinjer ⟺ svømming

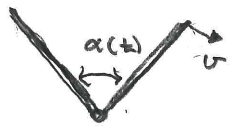
$F_D = \cancel{M}\, A e_x = 4\pi\mu U$ (regneoppgave)

Svømme ved $Re = 0$? (les artikkel; E.M. Purcell, "Life at low Reynolds nr", 1973, Am. J. Phys.)
Nobel pris fysikk)

Scallop (Kamskjell) theorem

- Hva er den enkleste sykliske bevegelsen for å svømme/translasjon?

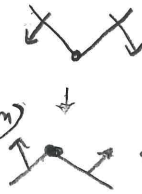
Kamskjell
(forenklet)
2-link svømmer



1 - frihetsgrad

Bevegelse ved $Re = 0$?

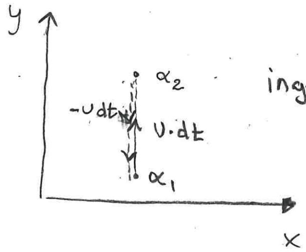
$Re \gg 1$ (utbreloen viktig)



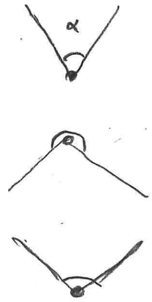
netto bevegelse
(uklart a-priori)

Siden $\frac{\partial u}{\partial t} \equiv 0$, raten uviktig!

grafisk



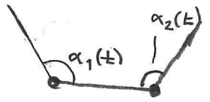
ingen netto bevegelse!



reversibel i tid

2 - frihetsgrader

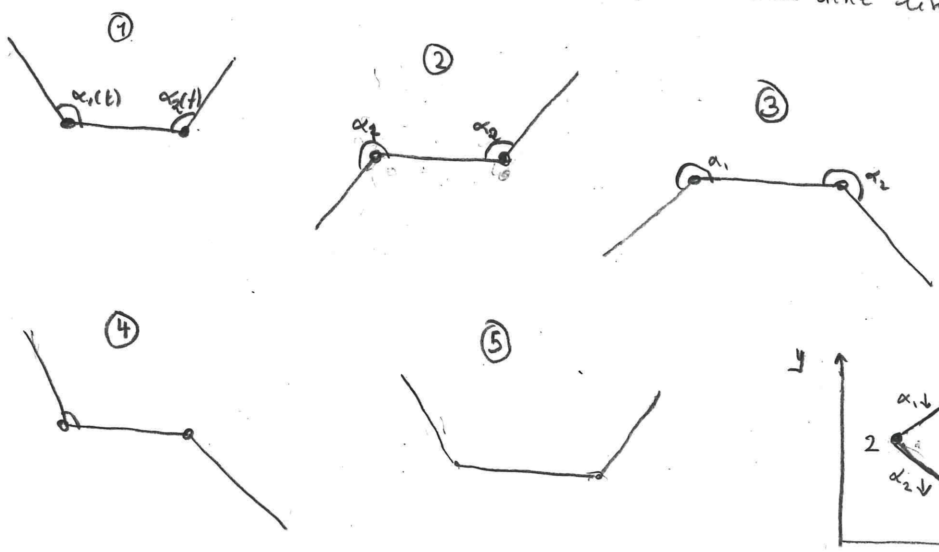
3 - link svømmer
(biologisk irrelevant

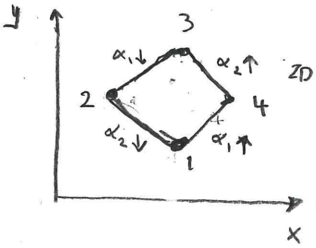


Eksperiment, (Tam og Hosoi, PoF, 2007)

Bryter reversibilitets prinsippet (= reversibel bevegelsen i tid ikke lik den cykliske bev,)

$t = 0$

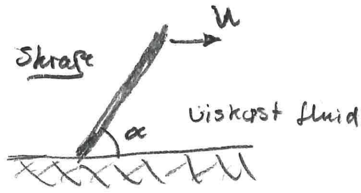




2D-Svømmer i en sirkel!

3D - helix

Matematisk beskrivelse av krypstrømning $Re \equiv 0$



2D-strømning

Stoke's strømfunksjon, $\boxed{u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}}$

Kontinuitet, $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$, $\frac{\partial \psi}{\partial t} = 0$, $\underline{u} \cdot \nabla \psi = 0$

$\psi = \text{konst}$, $\equiv$ strømlinje

$\left(\frac{\partial \psi}{\partial n}, \frac{\partial \psi}{\partial s}\right)$, (norm, tang. hastighet)

ψ-verdi uviktig → endringen.

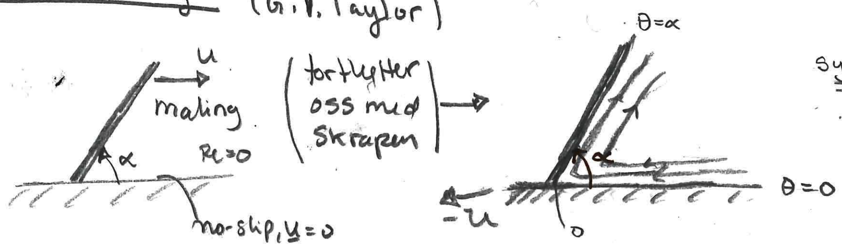
Virvlingen, $\underline{\omega} = \nabla \times \underline{u} = \left(0, 0, -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \underline{(0, 0, -\nabla^2 \psi)}$

$\nabla \times$ (Stoke's lig.), $\mu \nabla \times (\nabla^2 \underline{u}) = \nabla \times (\nabla P)$

$$-\mu \boxed{\nabla^2 \underline{\omega} = 0}$$ Harmonisk ligning

$$\Rightarrow \boxed{\nabla^4 \psi = 0}$$ Bi-harmonisk ligning

Skrape maling (G.I. Taylor)



sylindriske koordinater;

u_r, u_θ, $\underline{e}_z$ inn i tavla ~ $u_z = 0$.

$$\underline{u} = \nabla \times \psi = \frac{1}{r}\begin{vmatrix} \underline{e}_r & r\underline{e}_\theta & \underline{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \psi & \psi & \psi \end{vmatrix} = \underline{\frac{1}{r}\frac{\partial \psi}{\partial \theta}\underline{e}_r - \frac{\partial \psi}{\partial r}\underline{e}_\theta}, \quad u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \quad u_\theta = -\frac{\partial \psi}{\partial r}$$

$\psi = \psi(r, \theta)$

Grensebetingelser; $\theta = 0, \alpha$ → strømlinje, $u_\theta = 0$, $\psi = \text{konst.} = 0$ ⇒ veggen er en strømlinje

$\theta = 0$, $u_r = -U$

$\theta = \alpha$, $u_r = 0$

Løser de harmoniske ligningene (ω, $\nabla^4\psi$) ved variabel separasjon

$$[\psi] = \frac{L^2}{T} = [u]\cdot[r]$$, bare en kombinasjon av (u, r) gir denne dimensjonen

$\underline{\psi = u \cdot r\, f(\theta)}$ + grensebetingelser

$$\nabla^2\psi = \left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}\right)\psi = \frac{u}{r}\underbrace{\left(f(\theta) + f''(\theta)\right)}_{F = F(\theta)}$$, $f = f(\theta)$, $f' = \frac{\partial f}{\partial \theta}$

$$\text{i)} \quad = \underline{\frac{u}{r}F} = -\omega$$

Momentum lig. $\nabla^2\omega = 0$

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial^2\theta}\right)\frac{u}{r}F = \frac{uF}{r}\cdot\left(-\frac{1}{r^2}\right) + \frac{u}{r^3}F'' = \underline{\frac{u}{r^3}\left[-F + F''\right]} \quad \text{ii)}$$

Kjente løsninger av ii) $F = A'\cos\theta + B'\sin\theta$

i) $F = f + f''$ $\wedge$ $f = \underline{A\cos\theta + B\sin\theta + C\theta\cos\theta + D\theta\cdot\sin\theta}$

Bestemmer A, B, C, D ved hjelp av grenseverdier;

langs veggen, $\theta = 0$, $\psi = f = 0 \Rightarrow \underline{A = 0}$ (I)

$\theta = \alpha$, $f = \underline{B\sin\alpha + C\alpha\cos\alpha + D\alpha\sin\alpha = 0}$ (I)

$$u_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = uf' = \left(B\cos\theta + C\left(\cos\theta - \theta\sin\theta\right) + D\left(-\sin\theta + \theta\cos\theta\right)\right)\cdot u$$

$\theta = 0$, $u_r = 0$ $\Rightarrow$ $\underline{B + C = 0}$ (III)
$f' = 0$

$\theta = \alpha$, $u_r = 0$ $\Rightarrow$ $\underline{B\cos\alpha + C(\cos\alpha - \alpha\sin\alpha) + D(\alpha\cos\alpha - \sin\alpha) = 0}$ (IV)
$f' = 0$

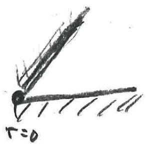
Løser (I-IV) $\Rightarrow$ $f(\theta) = \dfrac{\alpha(\alpha-\theta)\sin\theta - \theta\sin(\alpha-\theta)\sin\alpha}{\sin^2\alpha - \alpha^2}$

$\underline{\psi = r\,u\,f(\theta)}$

(Acheson A.10) 12/01/2015 ⑤

Hvordan finne trykket? $\nabla P = \mu \nabla^2 u = \mu\left(-\nabla\times(\nabla\times \underline{u}) + \nabla(\nabla\cdot\underline{u})\right) = -\mu\nabla\times\underline{\omega}$

$$\rightarrow \quad \frac{\partial P}{\partial r} = \frac{\mu}{r}\frac{\partial}{\partial\theta}\nabla^2\psi \;, \quad \frac{1}{r}\frac{\partial P}{\partial\theta} = -\mu\frac{\partial}{\partial r}\nabla^2\psi$$



stresset divergerer når $r\to 0$ men over et areal $A\to 0$ og kraften blir endelig!

kraften holder skrapen til veggen.

Integrasjon gir trykket, $\underbrace{P = P_0 + \frac{2\mu U}{r}\cdot g(\theta,\alpha)}$

$r\to 0\,,\ P\to\infty$

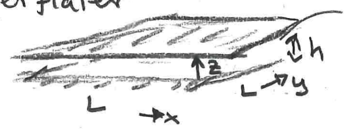
$r\to\infty\,,\ P\to P_0$ (atmosfærisk trykk)

[Følg Acheson og bruk samme teknikk for å beregne stoke's drag på en kule, ukens regneøving; $F_D = 6\pi\mu U\cdot R$]

Viskøs strømning via geometri (tynne væskefilmer)

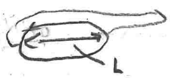
Lav dimensjons strømning mellom faste overflater

① Geologi (lava-strømning) $h\sim 1\text{m}$, $L\sim 1\text{km}$, $\frac{h}{L}\ll 1$

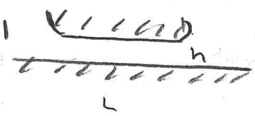


2d-"Poiseuille" strøm (Hele-Shaw strømning)

② Biologi (sugefisk)

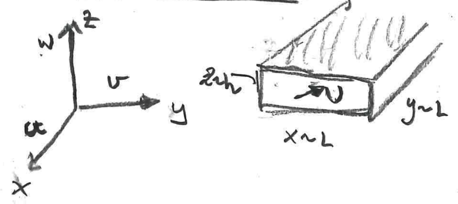


$L\sim 10\text{cm}$, $h\sim 1\text{mm}$, $h/L\ll 1$



③ μfluedikk (Lab-on-chip), $h\sim 1\mu\text{m} = 10^{-6}\text{m}$, $L\sim 1\text{cm}$, $h/L\ll 1$

Kartesiske koordinater;



Kontinuitets ligningen;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Skalering $\sim\frac{U}{L} \quad \sim\frac{U}{L} \quad \sim\frac{W}{h} = 0 \quad \Rightarrow \underline{W\sim \frac{h}{L}U} \;\wedge\; \frac{h}{L}\ll 1$

Momentum lign.

$$x-\quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\sim\frac{U^2}{L} \quad \sim\frac{U^2}{L} \quad \sim\frac{U^2}{L} \quad \sim\frac{U^2}{L}\right) = \sim\frac{P_0}{L} + \mu\left(U/L^2 + U/L^2 + U/h^2\right) \Rightarrow \underline{P_0 \sim \frac{\mu U}{h^2}L}$$

$$y-\quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\sim \rho U^2/L \qquad = P_0/L + \mu\left(U^2/L^2 + U^2/L^2 + U/h^2\right) \Rightarrow \underline{P_0 \sim \frac{\mu U}{h^2}L}$$

$$z-\quad \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \Rightarrow P_0 \sim \frac{\mu U}{L}$$

$$\sim \rho h^2/L^2 u \qquad \rho h U/L \qquad P_0/h + \mu\,(Uh/L^3) \qquad \mu U/hL$$

Vi ser at trykket skalerer i $\{x,y\}$ retning $P_0 \sim \frac{\mu U L}{h^2} \gg \frac{\mu U}{L}$ (langs z)

$\sim \mu U \left(\frac{L}{h}\right)^2 \gg \mu u \quad \wedge \; h/L \ll 1$

Introduserer skaleringen → lubrikasjons ligningene

$$x \quad - \frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2} + \mathcal{O}(h^2)$$

$$y \quad - \frac{\partial p}{\partial y} = \mu \frac{\partial^2 v}{\partial z^2} + \mathcal{O}(h^2)$$

$$\Rightarrow \quad u(z) = \frac{1}{2\mu}\frac{\partial p}{\partial x} \cdot z^2 + A \cdot z + B$$

$$v(z) = \frac{1}{2\mu}\frac{\partial p}{\partial y} \cdot z^2 + C \cdot z + D$$

$$z \quad - \frac{\partial p}{\partial z} = 0 + \mathcal{O}(h^3) \Rightarrow P = P(x,y)$$

$A, B, C, D \rightarrow$ grensebeting.

kont. $\nabla \cdot \underline{u} = 0$ (urørlige overflater) $\rightarrow w \equiv 0$

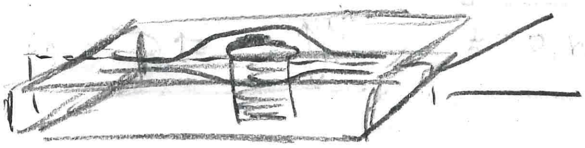
Film Reynolds tallet, $Re_f = \frac{|\rho \underline{u} \cdot \nabla \underline{u}|}{|\mu \nabla^2 \underline{u}|} = \frac{\rho U^2 / L}{\mu U / h^2} = \frac{\rho U L}{\mu}\left(\frac{h}{L}\right)^2 = Re \left(\frac{h}{L}\right)^2 \ll 1$

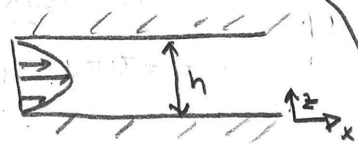
$Re \rightarrow \infty$

$\frac{h}{L} \rightarrow 0$

lub.t. forhatt gyldig!

Hele-Shaw problemet





grensebetingelser? $z = 0, h$; $u = v = 0$

$$\Rightarrow \quad u(z) = \frac{1}{2\mu}\frac{\partial p}{\partial x}(z^2 - hz)$$

$$v(z) = \frac{1}{2\mu}\frac{\partial p}{\partial y}(z^2 - hz)$$

$\frac{u}{v} \neq f(z) \Rightarrow$ strømlinjer uavhengig av z

Virvlingen?

$$\underline{\omega} = \nabla \times \underline{u} = \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x}\right)\underline{e}_z = \frac{(z^2 - hz)}{\mu}\left(\frac{\partial^2 p}{\partial x \partial y} - \frac{\partial^2 p}{\partial x \partial y}\right) \equiv 0 \; !$$

- Identisk med/modell for 2D rotasjonsfri invisкøs strømning (sjekk om sirkulasjonen også forsvinner, regneoppgave)

Vi ser også at ; $\nabla \cdot \underline{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}} = 0$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = \nabla^2 p = 0$$

[Video - G.I. Taylor]

Stresset?

$$\underline{\underline{\sigma}} = P\underline{\underline{I}} + \mu \frac{\partial u}{\partial z}$$

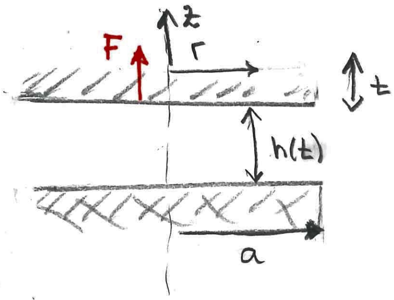
$$\sim \frac{\mu U L}{h^2} \gg \frac{\mu U}{h}$$

$\Rightarrow$ akkurat som inviskøs, $Re \gg 1$, strømning!

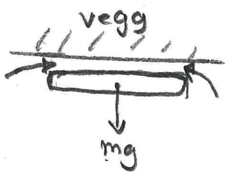
$\rightarrow \underline{\underline{\sigma}} \sim P\underline{\underline{I}}$, $P \equiv \phi$, hastighets potensial

$\underline{U} = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, 0\right)$ (2D)

Viskøst sug - fisk, insekt føtter og sugekopper! (Stefan adhesion)



eks. sugekopp



når løsner den?

- axi-symmetrisk $\rightarrow$ $\underline{u} = u_r \underline{e}_r + u_z \underline{e}_z$
- dynamisk vegg $\frac{\partial h}{\partial t} \neq 0$
- geometrien gir, $\frac{h}{a} \ll 1$

lubrikasjons lign.

r — $$\frac{\partial P}{\partial r} = \mu\left(\frac{\partial^2 u_r}{\partial z^2} + \frac{\partial^2 u_r}{\partial r^2} + \frac{u_r}{r^2}\right) = \mu \frac{\partial^2 u_r}{\partial z^2}$$

z — $$\frac{\partial P}{\partial z} = \mu\left(\frac{\partial^2 u_z}{\partial z^2} + \frac{\partial^2 u_z}{\partial r^2}\right) = 0 \implies P = P(r)$$

(Ser bort fra $(h/L)^2 \ll 1$ effekter)

$$\underline{u_r = \frac{1}{2\mu}\frac{\partial P}{\partial r}(z^2 - zh)}$$; $\underline{u_r(0) = u_r(h) = 0}$ (grensebetingelser)

Kont.

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z} = 0$$;

grensebetingelser; øvre veggen forflytter seg langs z
$u_z(0) = 0$
$u_z(h) = \frac{\partial h}{\partial t} = \dot{h}$

$$u_z = \int_0^h -\frac{1}{r}\frac{\partial}{\partial r}(r u_r)\, dz$$

$$u_z = \frac{-1}{2\mu r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)\int_0^h (z^2 - zh)\, dz = \frac{-1}{2\mu r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right)\left[\frac{z^3}{3} - \frac{hz^2}{2}\right]_0^h = \underline{\frac{h^3}{12\mu r}\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) = \dot{h}}$$

$$\implies \frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial r}\right) = \frac{12\dot{h}\cdot\mu\cdot r}{h^3}$$

$\dot{h} = \frac{\partial h}{\partial t}$

Integrasjon gir; $$\frac{\partial P}{\partial r} = \frac{6\dot{h}\mu r}{h^3} + \underbrace{\cancel{\frac{C(t)}{r}}}_{=0}$$

$=0$, ingen singularitet i volumet

$$\implies P = \frac{3\dot{h}\mu r^2}{h^3} + D(t)$$, atmosfærisk trykk ved $r = a$, $P = P_0$

trykket i filmen $$\underline{P - P_0 = \frac{3\dot{h}\mu}{h^3}(r^2 - a^2)}$$

Sugekraften fra fluidet på overflaten/sylinderen

$$F = \int_0^a\int_0^{2\pi} (p-p_0)\cdot r\,d\theta\,dr = \int_0^a\int_0^{2\pi} \frac{3\dot h\mu}{h^3}(r^2-a^2)\,r\,d\theta\,dr = \frac{6\pi\mu\dot h}{h^3}\left[\frac{r^4}{4}-\frac{a^2r^2}{2}\right]_0^a = \underline{-\frac{3\pi\mu\dot h a^4}{2h^3}}$$

Negativ kraft $F\downarrow$ dersom $\underset{\text{sug}}{\dot h>0}$, motsatt dersom $\underset{\text{repulsiv}}{\dot h<0}$ $F\uparrow$. Sett at $F(\dot h)$, effekt?

Er det vanskelig å trykke ut en viskøs film? Hvorfor? $F\sim h^{-3}$

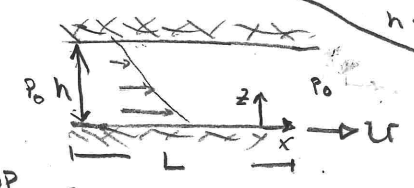
Antar $F=-mg \Rightarrow mg = \frac{3\pi\mu\dot h a^4}{2h^3}$

$$\frac{\dot h}{h^3} = \frac{2mg}{3\pi\mu} \quad \wedge \quad \dot h = \frac{\partial h}{\partial t}\sim\frac{h}{\tau} \quad \wedge\ \tau = \text{dreneringstid}$$

$\boxed{\tau\sim\frac{\mu a^4}{mgh^2}}$ → Veldig tregt når $h\to 0$, $\tau\uparrow$. Når aldri $h=0$ ved gitt F.
Sensitiv til radius a, $a\to 2a$, $\tau\to 16\tau$!

(Eks. sugekopp; $a=10\,cm$, $\mu=10^{-3}\,Pa\cdot s$ (vann), $\rho=m/ha^2=10^3\,kg$ (vann) $\Rightarrow \tau=\frac{\mu a^2}{\rho g h^3}\approx\frac{10^{-3}\cdot10^{-2}}{10^{-9}\cdot 10\cdot 10^{3}}$
$h\sim 1mm$
$\approx 1\,s$

Glidende lagring (Reynolds 1890's)
Uten tilt



Lubrikasjons teori; $u(z) = \frac{1}{2\mu}\frac{\partial p}{\partial x}\cdot z^2 + Az + B$, grensebetingelser $u(0)=U$, $u(h)=0$

$$u(z) = \left[\frac{U}{h} - \frac{z}{2\mu}\frac{\partial p}{\partial x}\right](h-z) \qquad B = U, \quad A = -\frac{1}{2\mu}\frac{\partial p}{\partial x}h - \frac{U}{h}$$

gjennomsnittlig hastighet — trykk drevet.

Masse fluxen; $Q = \int_0^h u(z)\,dz = \frac{Uh}{2} - \frac{h^3}{12\mu}\frac{\partial p}{\partial x} = \text{Konst.}$ • filmens høyde uendret ⇒ massebevarelse / inkompressibilitet

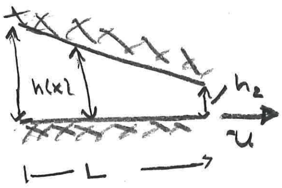
$$\Rightarrow \frac{\partial p}{\partial x} = -\frac{12\mu Q}{h^3} + \frac{6\mu U}{h^2} \quad\wedge$$

$$p-p_0 = -12\mu Q\int_0^s h^{-3}(s)\,ds + 6\mu U\int_0^s h^{-2}(s)\,ds \quad\wedge\ p=p_0 \wedge s=0,L$$

$$\underline{Q = \frac{U\int_0^L h^{-2}(s)\,ds}{2\int_0^L h^{-3}(s)\,ds}} \quad\wedge\ h=h_0 \Rightarrow \underline{Q=\frac{Uh_0}{2}} \Rightarrow \frac{p-p_0}{6\mu} = -\frac{2\,Uh_0}{2h_0^3}\int_0^s ds + \frac{U}{h_0^2}\int_0^s ds$$

(skjærstrømning) $\underline{\underline{=0}}$ ingen net kraft!

Hva skjer dersom kanalen heller?

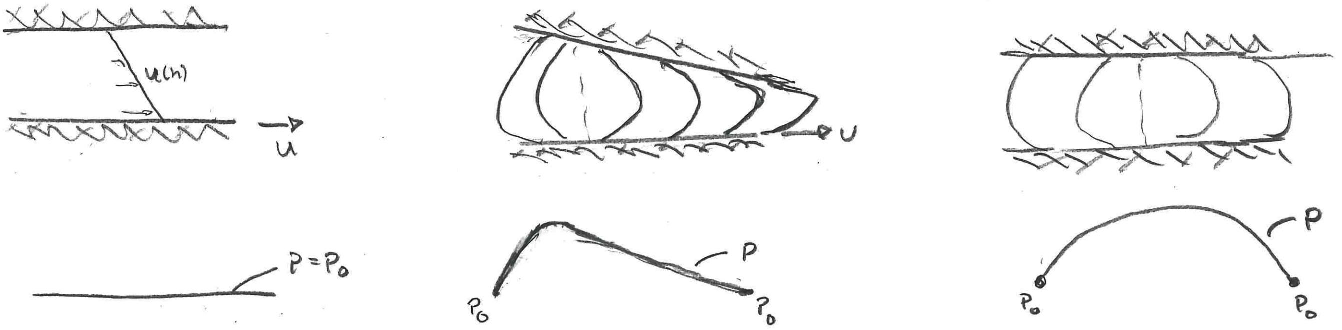


$$h(s) = h_1 - (h_1 - h_2)\frac{s}{L} \Rightarrow Q = \frac{u\int_0^L h^{-2}\,ds}{2\int_0^L h^{-3}\,ds} = \frac{u h_1 h_2}{(h_1+h_2)}$$

$$p - p_0 = -12\mu Q \int_0^s h^{-3}(s)\,ds + u\cdot 6\mu \int_0^s h^{-2}(s)\,ds$$

$$\frac{p-p_0}{6\mu} = -2Q\int_0^s \left(h_1 - (h_2-h_1)\frac{s}{L}\right)^{-3} ds + u\int_0^s \left(h_1-(h_2-h_1)\frac{s}{L}\right)^{-2} ds = \frac{6\mu u L\,(h_1-h)(h_2-h)}{(h_2^2-h_1^2)\,h^2}$$

Geometrien genererer net-kraft ↑



Så langt;

Bulk fluid strømning – Euler's ligninger (inviskøst)
– Stoke's ligninger (viskøst)

geometri – tynne filmer

Neste steg; Overflatedynamikk!

→ fri-overflate + fluid dynamikk

- grensebetingelser + overflatespenning
- "frie" væskefilmer
- bølger