Eksamen 2005, oppgave 3.

ⓐ * No slip at the bottom: $\boxed{\frac{\partial\phi}{\partial z}\Big|_{z=-H} = 0}$ (1)

* At the surface:

○ <u>kinematic boundary condition:</u> "a particle that is at the surface, stays at the surface":

The exact formula is: $$\frac{\partial\eta}{\partial t}\Big|_{z=\eta} + \underbrace{\cancel{\frac{\partial\phi}{\partial x}\Big|_{z=\eta}\cdot\frac{\partial\eta}{\partial x}\Big|_{z=\eta}}}_{\text{second order}} = \frac{\partial\phi}{\partial z}\Big|_{z=\eta}$$

assume small waves, this becomes:

(note: this is like doing a Taylor expansion: $\frac{\partial\eta}{\partial t}\Big|_{z=\eta} = \underbrace{\frac{\partial\eta}{\partial t}\Big|_{z=0}}_{\hookrightarrow \gg} + \underbrace{\frac{\partial^2\eta}{\partial t.\partial z}\Big|_{z=0}}_{\omega}\cdot\eta$) $\boxed{\frac{\partial\eta}{\partial t}\Big|_{z=0} = \frac{\partial\phi}{\partial z}\Big|_{z=0}}$ (2)

○ <u>dynamic boundary conditions:</u> use Bernoulli

$$\frac{P}{\rho} + \frac{\partial\phi}{\partial t} + \frac{1}{2}\cdot\left(\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right) + gz = cst(t)$$

use cst(t) to take away atmospheric pressure, and assume $\rho_{air} \ll \rho_{water}$:

$$\frac{P}{\rho} + \frac{\partial\phi}{\partial t} + \underbrace{\cancel{\frac{1}{2}\left(\left(\frac{\partial^2\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right)}}_{\text{second order}} + gz = 0$$

in particular at the surface, $P = 0$ and:

$$\frac{\partial\phi}{\partial t}\Big|_{t=\eta} + gz\Big|_{z=\eta} = 0,$$ assume small waves:

(This is like doing a Taylor expansion on the $\frac{\partial\phi}{\partial t}$ term). $\boxed{\frac{\partial\phi}{\partial z}\Big|_{z=0} + g\eta = 0}$ (3)

- The basin has <u>finite length</u>, no water penetration at the walls:

$$\boxed{\left.\frac{\partial \phi}{\partial x}\right|_{x=0} = \left.\frac{\partial \phi}{\partial x}\right|_{x=L} = 0} \quad (4)$$

ⓑ. * First, let us find an equation for $\hat{\phi}$; for this, write the Laplace equation:

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0}$$

$$\Leftrightarrow \quad -\hat{\phi}(z).\,k^2 \cos(kx) \sin(\omega t) + \hat{\phi}''(z).\cos(kx)\sin(\omega t) = 0$$

This must be true for all x and t, therefore:

$$\underline{\hat{\phi}''(z) = \hat{\phi}(z).\,k^2}$$

which can be solved into:

$$\underline{\hat{\phi} = A.\cosh[k(z+H)] + B.\sinh[k(z+H)]}$$

(note: you could also write $\hat{\phi} = A.e^{kz} + B.e^{-kz}$, this is strictly equivalent to using cosh and sinh, but the calculations are heavier).

* Next, we can use the boundary conditions to ~~simplify~~ ~~our~~ compute the integration constants.

$$* \; (1) \Leftarrow \left.\frac{\partial \phi}{\partial z}\right|_{z=-H} = 0 \Leftrightarrow A.k\sinh\underbrace{[k.0]}_{0} + B.k.\sinh\underbrace{[k.0]}_{1} = 0$$

so that $\boxed{B = 0}$

we are now reduced to:

$$\phi = A.\cosh[k(z+H)].\cos(kx).\sin(\omega t)$$

* Now use (2) to further simplify by writing A is a function of a:

$$(2)\ \frac{\partial \eta}{\partial t}\Big|_{z=0} = \frac{\partial \phi}{\partial z}\Big|_{z=0} \iff -a\omega\cos(kx)\sin(\omega t) = A\cos(kx)\sin(\omega t) . k\sinh\Big[k(\underset{\underset{0}{\uparrow}}{z}+H)\Big]$$

true $\forall(x,t)$, i.e.: $\boxed{A = -\dfrac{a\omega}{k\sinh[kH]}}$

We still have boundary conditions to use, but this is for the next questions; our expression for $\hat{\phi}$ is:

$$\boxed{\hat{\phi} = -\frac{a\omega}{k.\sinh[kH]} . \cosh\big[k(z+H)\big]} \quad (5)$$

ⓒ To get the dispersion relation, we use (3):

$$(3) \iff -\frac{a\omega}{k\sinh[kH]} . \cosh\Big[k(\underset{\underset{0}{\uparrow}}{z}+H)\Big] . \cos(kx) . \omega . \cos(\omega t)$$

$$+ ga\cos(kx)\cos(\omega t) = 0$$

i.e.: (true $\forall\ (x;t)$):

$$\boxed{\omega^2 = g.k.\tanh\big[kH\big]} \quad (6) \text{ (dispersion relation).}$$

ⓓ. To find which values of k can be used, need to use condition (4):

$$* \ \frac{\partial \phi}{\partial x}\Big|_{x=0} = 0 \iff -\hat{\phi}(z).k\sin(k\overset{\overset{x=0}{\downarrow}}{x})\sin(\omega t) = 0,$$

always true $(\sin(0) = 0)$

* $\left.\frac{\partial \phi}{\partial x}\right|_{x=L} = 0 \iff$

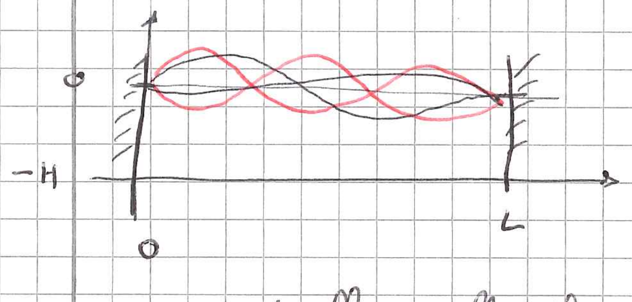
$$-\hat{\Phi}(z)\cdot k\cdot \sin(k\overset{\overset{x=L}{\downarrow}}{x})\cdot \sin(\omega t) = 0$$

true $\forall (x;t) \iff$ $\sin(kL) = 0$

i.e. $\boxed{k = \frac{N\cdot\pi}{L} \, , \, N \in \mathbb{N}}$

A short comment about this question: the condition on k is imposed by the finite length of the basin. You can see that the ~~condi~~ formula for η and ϕ are a bit different compared to what is usually assumed ($\cos(kx)\cdot\cos(\omega t)$ instead of $\cos(kx-\omega t)$ generally).

What is computed here is a standing wave solution, i.e. a wave that oscillates with nodes at both walls.



example modes $N = 2$ and $N = 3$

actually, all the derivations we did are very similar to usual progressive waves, as a standing wave is the sum of 2 propagating waves, going in opposite directions; this arises from the fact that:

$$\frac{1}{2}\cdot\left\{\cos(kx-\omega t) + \cos((-k)x-\omega t)\right\}$$

$$= \cos(\omega t)\cdot\cos(kx).$$

In the case of propagating waves, k real is a "usual"

Ⓥ

harmonic propagating wave, and k with a real and imaginary part is a wave that is either exponentially damped (evanescent mode), or exponentially amplified (unphysical).

ⓔ. We now use the Bernoulli formula:
(without 2nd order terms)

$$\frac{P}{\rho} + \frac{\partial \phi}{\partial t} + g\underbrace{\overset{\uparrow}{z}}_{\underbrace{-H}_{cst}} = 0 \qquad (z = -H\,;\, x = 0)$$

so that fluctuations are:

$$P_{fluctuations} = -\rho \cdot \frac{\partial \phi}{\partial t}\bigg|_{z=-H}$$

Using (5) and (6): $$\hat{\phi} = -\frac{a\cdot\omega}{\cancel{\sinh[kH]}} \cdot \frac{g \cdot \frac{\cancel{\sinh[kH]}}{\cosh[kH)}}{\omega^2} \cdot \cosh[k(z+H)]$$

$$= -\frac{ag}{\omega\cdot\cosh[kH)} \cdot \cosh[k(z+H)]$$

i.e.: $$P_{fluctuations} = \frac{\rho a g}{\cancel{\omega}\cdot\cosh[kH]} \cdot \underbrace{\cosh[k(\underbrace{z+H}_{0})]}_{1} \cdot \underbrace{\cos(k\overset{\uparrow}{x})}_{1} \cdot \cancel{\omega}\cdot\cos(\omega t)$$

$$= \frac{\rho a g}{\cosh[kH]} \cos(\omega t)$$

i.e.: $$\frac{\rho a g}{\cosh[kH]} = b \iff \boxed{a = \frac{b\cdot\cosh[kH]}{\rho g}}$$