Ex. 1 . Using the P.L. method; $\tau = \omega t$.

$$\omega^2 y'' + y = \epsilon y \left(1 - \omega^2 \left(y'\right)^2\right), \quad y(0) = 1, \quad \frac{\mathrm{d}y(0)}{\mathrm{d}t} = 0,$$

where $y' = \frac{\mathrm{d}y}{\mathrm{d}\tau}$ etc. Perturbation series

$$y = y_0 + \epsilon y_1 + ..., \quad \omega = \omega_0 + \epsilon\omega_1 + ...$$

Requirement:
$*$ All y_j has period 2π in τ.

$$\begin{aligned} O(1): \quad & \omega_0^2 y_0'' + y_0 = 0, & y_0(0) = 1, \quad y_0'(0) = 0, \\ O(\epsilon): \quad & \omega_0^2 y_1'' + y_1 = -2\omega_1\omega_0 y_0'' + y_0(1 - (y_0')^2), & y_1(0) = 0, \quad y_1'(0) = 0. \end{aligned}$$

Solution $O(1)$
ODE and B.L: $y_0 = \cos(\tau/\omega_0)$. The requirement $*$ yields

$$\omega_0 = 1, \quad y_0 = \cos\tau.$$

Solution $O(\epsilon)$

$$y_1'' + y_1 = 2\omega_1 \cos\tau + \cos^3\tau = (2\omega_1 + \frac{3}{4})\cos\tau + \frac{1}{4}\cos(3\tau).$$

Fulfillment of $*$ (avoid secular terms that cause linear growth in y_1)

$$\omega_1 = -\frac{3}{8},$$

and solution of ODE + initial conditions:

$$y_1 = \frac{1}{32}\left(\cos\tau - \cos(3\tau)\right).$$

Ex. 2 .
a) Position is $\vec{r} = \ell\vec{\imath}_r$. Then

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \ell\dot{\theta}\vec{\imath}_\theta, \quad \Rightarrow T = \frac{1}{2}m(\vec{v})^2 = \frac{1}{2}m\ell^2\dot{\theta}^2.$$

Potential energy

$$V = mgy = -mg\ell\cos\theta.$$

Lagrangian

$$L = T - V = \frac{1}{2}m\ell^2\dot{\theta}^2 + mg\ell\cos\theta.$$

Lagrange equation

$$0 = \frac{\partial L}{\partial\theta} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = -mg\sin\theta - m\ell^2\ddot{\theta}.$$

b) Yes, since $\frac{\partial L}{\partial t} = 0$ we have the first integral

$$\text{const.} = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L.$$

Moreover

$$\dot{\theta}\frac{\partial L}{\partial \dot{\theta}} = \dot{\theta}m\ell^2\dot{\theta} = 2T.$$

Hence

$$\text{const.} = \dot{\theta}\frac{\partial L}{\partial \dot{\theta}} - L = 2T - (T - V) = T + V.$$

Conservation of energy!

c) Generalized momentum

$$p = \frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \quad \Rightarrow \quad \dot{\theta} = \frac{p}{m\ell^2}.$$

Hamiltonian

$$H = \dot{\theta}p - L = \frac{p^2}{2m\ell^2} - mg\ell\cos\theta.$$

(We also see that $H =$ const.=energy.)

Ex. 3 .

a) The transformation is

$$y = e^{\int k(x)dx}.$$

Substitution into the ODE yields

$$\Big(\epsilon(k' + k^2 + qk) + W\Big)y = 0.$$

Since y is nonzero this implies the Ricatti equation

$$\begin{array}{ccccccccc} \epsilon k' & + & \epsilon k^2 & + & \epsilon qk & + & W & = & 0 \\ (1) & + & (2) & + & (3) & + & (4) & = & 0 \end{array} \tag{1}$$

When $\epsilon \to 0$ we must expect that $|k| \to \infty$ is required to obtain a dominant balance. Then (2) $\gg$ (3) and we also expect that (2) $\gg$ (1). Then (2) and (4) dominates, and

$$\epsilon k_0^2 = -W \quad \Rightarrow k_0 = \pm i\epsilon^{-\frac{1}{2}}W^{\frac{1}{2}}.$$

This will give two independent solutions for y and substitution shows that the terms (1) and (3) are indeed sub-dominant.

Second order term is introduced as $k = k_0 + k_1$, $k_1 \ll k_0$. Substitution in (1)

$$\epsilon\Big(k_0' + k_1' + k_0^2 + 2k_0k_1 + k_1^2 + q(k_0 + k_1)\Big) + W = 0$$

Canceling of leading order $\epsilon k_0^2 + W = 0$ and use of $k_1 \ll k_0 \Rightarrow$

$$k_0' + 2k_0k_1 + qk_0 = 0,$$

with solution

$$k_1 = -\frac{1}{2}\frac{k_0'}{k_0} - \frac{1}{2}q = -\frac{W'}{4W} - \frac{1}{2}q$$

. Since $k_1 \sim 1$ we do have $k_1 \ll k_0$ and the ignored terms are small.

b) The transformation

$$y = e^{\int k dx} = e^{-\frac{1}{4}\ln W + \int \left(\pm i\epsilon^{-\frac{1}{2}} W^{\frac{1}{2}} - \frac{1}{2}q\right) dx + C},$$

where C is a constant. This may be written

$$y = AW^{-\frac{1}{4}} e^{\int \left(\pm i\epsilon^{-\frac{1}{2}} W^{\frac{1}{2}} - \frac{1}{2}q\right) dx}.$$

Combining the solutions corresponding to $\pm$:

$$y = W^{-\frac{1}{4}} e^{-\frac{1}{2}\int q dx} \left(A_+ e^{i\epsilon^{-\frac{1}{2}} \int W^{\frac{1}{2}} dx} + A_- e^{-i\epsilon^{-\frac{1}{2}} \int W^{\frac{1}{2}} dx} \right),$$

where A_+ and A_- are independent constants.

Ex. 4 .

a) First we eliminate x and y.

$$\frac{\mathrm{d}z^*}{\mathrm{d}t^*} = k \frac{(C_y - \frac{1}{2}z^*)(C_x - \frac{1}{2}z^*)^{\frac{3}{2}}}{C_x + (m - \frac{1}{2})z^*}$$

Guided by $0 \le z \le 2C_x$ we attempt the scaling

$$z^* = C_y z, \quad t^* = t_c t.$$

where z then is between 0 and 2 and t_c is still undetermined. The ODE becomes

$$\frac{\mathrm{d}z}{\mathrm{d}t} = t_c k C_x^{\frac{1}{2}} \frac{(1 - \frac{1}{2}z)(1 - \frac{1}{2}\frac{C_y}{C_x}z)^{\frac{3}{2}}}{1 + \frac{C_y}{C_x}(m - \frac{1}{2})z}.$$

Then $\epsilon = C_y/C_x$, which is small, and $t_c = 1/(kC_x^{\frac{1}{2}})$. By the way, k must have dimension concentration$^{-\frac{1}{2}}$, divided by time, while m is without dimension.
From the statement that there is no hydrogen bromide initially, $z(0) = 0$ follows.

b) Naive approach is attempted

$$z = z_0 + \epsilon z_1 + ...$$

First the right hand side of the ODE is expanded in powers in ϵ

$$(1 - \frac{1}{2}\epsilon z)^{\frac{3}{2}} = 1 - \frac{3}{4}\epsilon z + O(\epsilon^2), \qquad \frac{1}{1 + (m - \frac{1}{2})\epsilon z} = 1 - (m - \frac{1}{2})\epsilon z + O(\epsilon^2),$$

and

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{(1 - \frac{1}{2}z)(1 - \frac{1}{2}\epsilon z)^{\frac{3}{2}}}{1 + (m - \frac{1}{2})\epsilon z} = (1 - \frac{1}{2}z)(1 - \frac{3}{4}\epsilon z)(1 - (m - \frac{1}{2})\epsilon z) + O(\epsilon^2) = 1 - \frac{1}{2}z - \epsilon(m + \frac{1}{4})z(1 - \frac{1}{2}z) + O(\epsilon^2).$$

Inserting the power series for z we obtain

$$\begin{aligned} O(1): &\quad \tfrac{\mathrm{d}z_0}{\mathrm{d}t} + \tfrac{1}{2}z_0 = 1, & z_0(0) = 0, \\ O(\epsilon): &\quad \tfrac{\mathrm{d}z_1}{\mathrm{d}t} + \tfrac{1}{2}z_1 = -(m + \tfrac{1}{4})z_0(1 - \tfrac{1}{2}z_0), & z_1(0) = 0. \end{aligned}$$

Solution $O(1)$
The problem is standard.

$$z_0 = 2 - 2e^{-\frac{1}{2}t}.$$

z_0 remains in the interval $[0, 2)$. *Solution* $O(\epsilon)$

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} + \frac{1}{2}z_1 = -(m + \frac{1}{4})z_0(1 - \frac{1}{2}z_0) = (2m + \frac{1}{2})\left(e^{-t} - e^{-\frac{1}{2}t}\right).$$

A particular solution is assumed on the form

$$z_1^{(p)} = Ate^{-\frac{1}{2}t} + Be^{-t},$$

where the factor t is needed since $e^{-\frac{1}{2}t}$ is an homogeneous solution. Formula or integrating factor may also be used. Result

$$z_1^{(p)} = -(2m + \frac{1}{2})\left(te^{-\frac{1}{2}t} + 2e^{-t}\right).$$

Homogeneous solution and initial condition

$$z_1 = (2m + \frac{1}{2})\left(2e^{-\frac{1}{2}t} - te^{-\frac{1}{2}t} - 2e^{-t}\right).$$