UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in	MEK 3100/4100 — Mathematical methods in mechanics
Day of examination:	Friday 9. June 2006.
Examination hours:	14:30 – 17:30.
This problem set consists of 5 pages.	
Appendices:	Formula sheet.
Permitted aids:	Mathematical handbook, by K. Rottmann.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

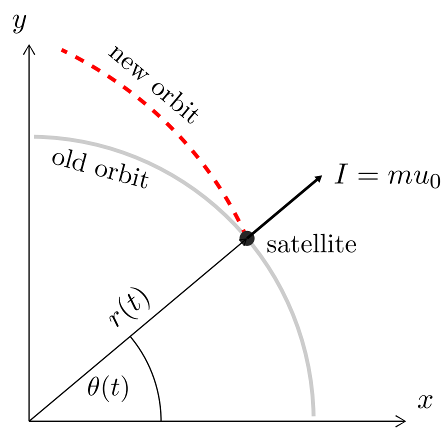
This set includes three exercises that will first be given in English and then in Norwegian. In addition a formula sheet is added at the end.

Exercise 1. Boundary layer
A boundary value problem is defined through

$$\epsilon y'' + x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1, \tag{1}$$

where $\epsilon \to 0^+$.
Find an approximate solution valid in $0 \le x \le 1$ and sketch it.

Exercise 2. Dimension analysis, scaling



This exercise comes with some text. Don't be in intimidated by this; the problems that you must solve are mostly quite standard. A satellite of mass m is orbiting the Earth. The force from the Earth on the satellite

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is $F(r) = -Gm/r^2$ (potential $V(r) = Gm/r$), where r is the distance from the centre of the Earth. When r and the angle θ (see figure) are used as coordinates the motion of the satellite is governed by

$$mr^2\dot{\theta} = S = \text{constant}, \tag{2}$$

$$m\ddot{r} - mr(\dot{\theta})^2 = -\frac{Gm}{r^2}, \tag{3}$$

that correspond to preservation of angular momentum and the radial component of Newtons second law, respectively. The dot denotes temporal differentiation. These equations shall *not* be derived.

a) A circular orbit corresponds to $r = r_0$ =constant and $\dot{\theta} = \omega_0$ =constant. Show that the relation $G = r_0^3\omega_0^2$ must be fulfilled. The orbit is then shifted by a short blast from a rocket engine that yields an radial impulse (change of linear momentum) $I = mu_0$, while the angular momentum remains unaltered. Moreover, we ignore the change of the satellite position during the impulse. Presumably, the new orbit will be closed, but not circular. The characteristics of the motion depend on the parameters m, r_0, ω_0, u_0 and t (time). Find a complete set of non-dimensional numbers.

b) The new orbit may be found by solving (2) and (3) with the initial conditions

$$r(0) = r_0, \quad \dot{r}(0) = u_0$$

In the following we assume that I is a weak impulse, implying that the orbit becomes only mildly perturbed, and wish to scale and make equations dimensionless accordingly. Start with eliminating $\dot{\theta}$ between (2) and (3) and remove dimensions to obtain the differential equation with initial conditions

$$(1+\epsilon z)^3\frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + z = 0, \quad z(0) = 0, \quad \frac{\mathrm{d}z(0)}{\mathrm{d}\tau} = 1, \tag{4}$$

where $r = r_0(1+\epsilon z)$, τ is a time variable and ϵ is a small dimensionless number. ϵ and τ must be chosen as part of this process and should be related to the findings in point a).

Exercise 3. The perturbed satellite orbit.
We seek an approximate solution to (4) by a perturbation expansion.

a) Find the first two terms in a series for z.

b) Explain why the expansion breaks down at order ϵ^2 (for z). What is the remedy ? To answer these questions you do not need to work out the full solutions to order ϵ^2.

— End of exercises in English —

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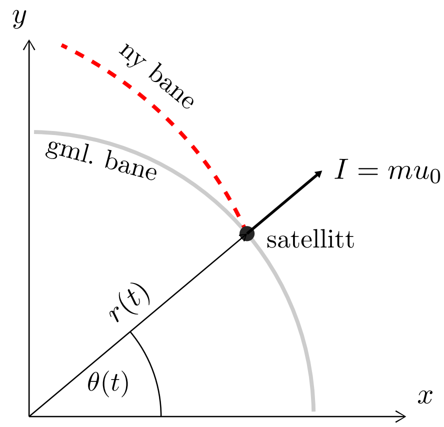
Oppgave 1. Grensesjikt
Et randverdiproblem er gitt ved

$$\epsilon y'' + x^2 y' - y = 0 \quad ; \quad y(0) = y(1) = 1, \tag{5}$$

der $\epsilon \to 0^+$.
Finn en tilnærmet løsning i $0 \leq x \leq 1$ og skisser den.

Oppgave 2. Dimensjonsanalyse, skalering



Denne oppgaven inneholder en del tekst. Ikke mist motet av den grunn – problemene du skal løse er for det meste av standard type. En satellitt med masse m går i bane om Jorda. Kraften fra Jorda på satellitten er $F(r) = -Gm/r^2$ (potensial $V(r) = Gm/r$), der r er distansen fra Jordas sentrum. Dersom r og vinkelen θ (se figur) brukes som koordinater, styres sattelittbevegelsen av

$$mr^2\dot{\theta} = S = \text{konstant}, \tag{6}$$

$$m\ddot{r} - mr(\dot{\theta})^2 = -\frac{Gm}{r^2}, \tag{7}$$

som svarer til hhv. bevaring av spinn og den radielle komponenten av Newtons andre lov. Prikken markerer tidsderivasjon. Disse likningene skal *ikke* utledes.

a) En sirkulær bane tilsvarer $r = r_0$ =konst. og $\dot{\theta} = \omega_0$ =konst. Vis at relasjonen $G = r_0^3\omega_0^2$ da må være oppfylt. Banen blir så endret ved en kort utblåsning fra en rakettmotor som gir en radiell impuls (endring av bevegelsesmengde) $I = mu_0$, mens spinnet forblir uendret. Videre ser vi bort fra endring av satellittposisjonen under impulsen. Den nye banen vil antagelig være lukket, men ingen sirkel. Bevegelsens egenskaper avhenger av parameterene m, r_0, ω_0, u_0 og t (tid). Finn et komplett sett av dimensjonsløse tall.

b) Den nye banen kan bestemmes fra (6) and (7) med initialbetingelsene

$$r(0) = r_0, \quad \dot{r}(0) = u_0$$

I det følgende antar vi at I er en svak impuls, med den følge at banen blir bare svakt forstyrret, og ønsker å skalere og gjøre likningene dimensjonsløse i tråd

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med dette. Start med å eliminere $\dot{\theta}$ mellom (6) og (7) og fjern dimensjoner slik at du ender med en enkelt differensiallikning med initialbetingelser

$$(1+\epsilon z)^3 \frac{\mathrm{d}^2 z}{\mathrm{d}\tau^2} + z = 0, \quad z(0) = 0, \quad \frac{\mathrm{d}z(0)}{\mathrm{d}\tau} = 1, \tag{8}$$

der $r = r_0(1+\epsilon z)$, τ er en tidsvariabel og ϵ er et lite dimensjonsløst tall. ϵ og τ må velges som en del av prosessen og bør relateres til resultatene i punkt a).

Oppgave 3. Den perturberte satellittbanen.
Vi søker en tilnærmet løsning til (8) ved en pertubasjonsutvikling.

a) Finn de første to leddene i en rekke for z.

b) Forklar hvorfor utviklingen bryter sammen til orden ϵ^2 (i z). Hva er botemidlet? Det er ikke nødvendig å finne hele løsningen til orden ϵ^2 for å svare på disse spørsmålene.

— Slutt på norske oppgaver —

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Formulas for Mek3100/4100

Trigonometric formulas

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$, $\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$,

$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$

$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta$, $\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$,

$\cos^2\theta\sin\theta = \frac{1}{4}\sin\theta + \frac{1}{4}\sin 3\theta$, $\cos\theta\sin^2\theta = \frac{1}{4}\cos\theta - \frac{1}{4}\cos 3\theta$,

$\cos\theta\cos 2\theta = \frac{1}{2}(\cos\theta + cos 3\theta)$, $\sin\theta\sin 2\theta = \frac{1}{2}(\cos\theta - \cos 3\theta)$,

$\cos\theta\sin 2\theta = \frac{1}{2}(\sin\theta + \sin 3\theta)$, $\sin\theta\cos 2\theta = \frac{1}{2}(-\sin\theta + 3\sin\theta)$,

$e^{i\theta} = \cos\theta + i\sin\theta$,

$\sinh(x) = \frac{1}{2}(e^x - e^{-x})$, $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$,

$\sin\theta = \theta - \frac{1}{6}\theta^3 + ... = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}\theta^{(2j+1)}$, $\cos\theta = \theta - \frac{1}{2}\theta^2 + ... = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j)!}\theta^{2j}$.

Taylors formula

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + ... + \frac{1}{n!}f^{(n)}(a)(x-a)^n + R_n,$$

where the residual is $R_n = \frac{1}{(n+1)!}f^{(n+1)}(c)(x-a)^{(n+1)}$ for some c between a and x.

First order differential equations

The equation set

$$\frac{dy}{dx} + f(x)y = g(x), \quad y(0) = a, \tag{9}$$

has the solution

$$y(x) = e^{-\int_0^x f(t)dt}\left(a + \int_0^x g(t)e^{\int_0^t f(s)ds}dt\right).$$

Particular solutions

Equation

$$\frac{d^2y}{dt^2} + \omega^2 y = F(t) \tag{10}$$

where ω is a nonzero constant. Selected inhomogeneous solutions

$F(t)$	Part. løsning
$\cos\sigma t$, $\sigma \neq \pm\omega$	$(\omega^2-\sigma^2)^{-1}\cos\sigma t$
$\sin\sigma t$, $\sigma \neq \pm\omega$	$(\omega^2-\sigma^2)^{-1}\sin\sigma t$
$\cos\omega t$	$\frac{1}{2\omega}t\sin\omega t$
$\sin\omega t$	$-\frac{1}{2\omega}t\cos\omega t$