22nd February, 2022

MEK4100

Mandatory assignment 1 of 2

Submission deadline

Thursday 3rd March 2022, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1 *Logan (2013) problem 20 page 169.*

Find a two-term perturbation solution of

$$u' + u = \frac{1}{1+\epsilon u}, \qquad u(0) = 0, \qquad 0 < \epsilon \ll 1.$$

Problem 2 *Inspired from Logan (2013) problem 15 page 168.*

a) Find the exact solution to the initial value problem

$$\epsilon \frac{\mathrm{d}y}{\mathrm{d}t} = y - 1, \qquad y(0) = 0, \qquad 0 < \epsilon \ll 1.$$

b) Show that regular perturbation fails to find an approximate solution.

c) Do the substitution $\tau = t/\epsilon$. Now show that a regular perturbation expansion is capable of reproducing the exact solution.

Problem 3 *Inspired from Stephens & Dunbar (1993)*

IT IS NOT NECESSARY TO READ THAT PAPER!

(however, the paper may help you appreciate the biological context)

In order to determine the optimal size of the territory of an animal of prey, we consider the energy gained by the animal eating prey and the energy consumed by the animal fending off intruders.

We thus distinguish between the "benefit function" B and the "cost function" C, both measured by a unit of power W = J/s (Watt = Joule per second), respectively, the power gained by eating prey and the power consumed by fighting intruders.

The benefit function B (J/s) is assumed to depend on the territorial area A (m^2), the prey density ρ (prey/m^2), the speed s (m/s) of the animal searching for its prey, the prey energy e (J/prey), and the prey handling time h (s/prey).

The cost function C (J/s) is assumed to depend on the territorial area A (m^2), the intrusion rate ξ (intruder/(m s)), and the intrusion cost c_i (J/intruder).

Notice that "prey" and "intruder" are considered to be independent units. The prey is considered to be distributed over the area while the intruders are considered to enter through the perimeter of the area.

a) Considering only the problem for the benefit function (B, A, ρ, s, e, h), argue why we can reduce these six variables down to only two dimensionless variables, π_a and π_b, that should be related.

Let π_a be linear in ρ and independent of B. Let π_b be linear in B and independent of ρ.

Show that the benefit function can be expressed as

$$B = \frac{e}{h} g(h\rho s\sqrt{A})$$

where $g()$ is some function.

b) Argue why the ratio e/h is the maximum achievable benefit, thus $g() \leq 1$.

c) Considering only the problem for the cost function (C, A, ξ, c_i), argue why we can reduce these four variables down to only one dimensionless variable π_c which should be equal to a constant k. Find an expression for C proportional to k.

d) The net benefit is the difference $N = B - C$. Suppose we normalize this expression by the maximum achievable benefit, $\{\hat{B}, \hat{C}, \hat{N}\}e/h = \{B, C, N\}$, show that the normalized net benefit can be written as

$$\hat{N} = g(\pi_a) - r\pi_a$$

where r is a dimensionless parameter proportional to the constant k.

e) There are good biological reasons for the shape of the function $g()$ to be a sigmoid (an S-shaped curve). In the interest that the remaining calculations are doable we take

$$g(\pi_a) = \frac{\pi_a^2}{1 + \pi_a^2}$$

Sketch the curve and argue that the constraint $g(\pi_a) \leq 1$ is satisfied. Show that the survival of the animal ($\hat{N} > 0$) requires $r < 1/2$.

f) Show that the optimal territorial size, corresponding to the solution π_a^* maximizing $\hat{N}$, is given by

$$\frac{2\pi_a}{(1 + \pi_a^2)^2} = r.$$

This equation can be solved by singular perturbation expansion assuming r is small. Show how dominant balance suggests $\pi_a^* \approx (2/r)^{-1/3}$. Reproduce some of the terms

$$\pi_a^* \approx \sqrt[3]{\frac{2}{r}} - \frac{2}{3}\sqrt[3]{\frac{r}{2}} - \frac{r}{6}$$

and sketch the curve $\pi_a^*(r)$.

Bibliography

Logan, J. D. 2013 Applied mathematics, 4th edition. Wiley.

Stephens, D. W. & Dunbar, S. R. 1993 Dimensional analysis in behavioral ecology. *Behavioral Ecology* **4**, 172–183.