12$^{\text{th}}$ April, 2023

MEK4100

Mandatory assignment 2 of 2

Submission deadline

Thursday 4$^{\text{th}}$ May 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1 *Problem 82 in leaflet:*

Consider the nonlinear boundary value problem

$$\epsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}+y^2=0,\qquad y(0)=0,\qquad y(1)=\frac{1}{2}$$

where $\epsilon\to 0$. *Find a uniform solution by combining an outer solution with a boundary layer approximation.*

Plot the inner, outer and uniform solutions for $\epsilon=0.1$ *and* $\epsilon=0.01$.

Problem 2 *Approximately problem 76 in leaflet:*

Consider the ordinary differential equation

$$\epsilon y'''' + W(x)y = 0$$

where $W(x)>0$ *and* $\epsilon\to 0$.

Find the first two terms in WKB expansions for all the solutions.

Problem 3 *Approximately problem 77 in leaflet:*

Consider the ordinary differential equation

$$\epsilon\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+W(x)y=0$$

where $W(x)$ *is continuous and* $\epsilon\to 0$. *We have* $W(x)>0$ *for* $x<x_t$, $W(x_t)=0$, *and* $W(x)<0$ *for* $x>x_t$.

a) Describe the behaviour of the WKB expansions on each side of x_t *and explain why the expantions are not valid at* $x=x_t$. *Such a point is called a turning point and demands special treatment.*

b) Introduce a local coordinate $z=(x-x_t)/\delta$ *in a boundary layer around the turning point and show that the behaviour in this boundary layer is approximated by the equation*

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2}=zy$$

This equation is known as the Airy equation.

In the old days we used to consult Abramowitz & Stegun, a "telephone catalogue" full of mathematical formulas and tables, in order to deal the the Airy equation. These days we rather consult the Digital Library of Mathematical Functions, which is essentially a digital revision of Abramowitz & Stegun, https://dlmf.nist.gov.

Consult https://dlmf.nist.gov/9.2. Look at the figures that show the two fundamental solutions Ai(z) and Bi(z) here https://dlmf.nist.gov/9.3. In

Matlab these functions are available as `airy(0,z)` *and* `airy(2,z)`*. We will need expressions for the asymptotic behaviour for large arguments, those can be found here https://dlmf.nist.gov/9.7.*

As z becomes large and positive, $Ai(z)$ approaches zero exponentially while $Bi(z)$ approaches infinity exponentially.

c) Find a uniform solution employing WKB expansions as outer solutions on each side of the turning point, and an Airy function as inner solution. You should impose that $y(x)$ is finite (0) in the limit $x \to \infty$ and that $y(0) = 1$.

Do this in particular for $W(x) = -\tanh x$ and plot the result for $\epsilon = 0.1$ and $\epsilon = 0.01$.