Lecture on: Darcy's law.
MEK4300 Viscous flow and turbulence, Univ. of Oslo.

by John Grue, September 20, 2017

A derivation of Darcy's law is given. Darcy's law forms the basis for flow in porous media at low Reynolds number. It represents an important generalisation and simplification of Stokes flow. The formulation has wide applications including simulations of reservoir modeling (oil, gas), simulation of the flow of ground water and water supply, and flow and evaporation in the blades (biology). Darcy's law is also widely used as model for flow of disolutions in the brain and in tissues as well. Darcy's law has formed the basis for quite extensive analyses in ocean engineering applications where breakwaters composed by rocks or man-made slotted structures are modelled. Modeling of the flow at and forces on fish cages for applications in aquaculture is another example. In the latter examples the flow does not take place at small Reynolds number. Thus a quadratic law of the pressure drop across the thin structures, including flow separation effects should be taken into account.

The derivation is based on the outline of E. Palm and J. E. Weber of 1971, published in Preprint Series, Dept. of Mathematics, University of Oslo.

In this derivation the flow and pressure gradient are linear. The Newtonian fluid motion takes place within a porous material. We assume that this material consists of small spheres of the same (small) radius. Consider a part of the porous material included in a cube of volume V and edge L'. This length is much greater than the diameter d of the of the spheres. However, the length L' is much smaller than the length scale L of the gradients of the average fluid flow, i.e.,

$$d << L' << L. \tag{1}$$

We assume that the fluid is Newtonian and incompressible. The mass and momentum equations read:

$$\nabla \cdot \mathbf{v} = 0, \tag{2}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mu \nabla^2 \mathbf{v}. \tag{3}$$

We shall now perform a spacial average of these equations over the box of volume V. The averaged velocity and pressure are defined by

$$nV\bar{\mathbf{v}} = \int_V \mathbf{v} dV, \tag{4}$$

$$nV\bar{p} = \int_V p dV, \tag{5}$$

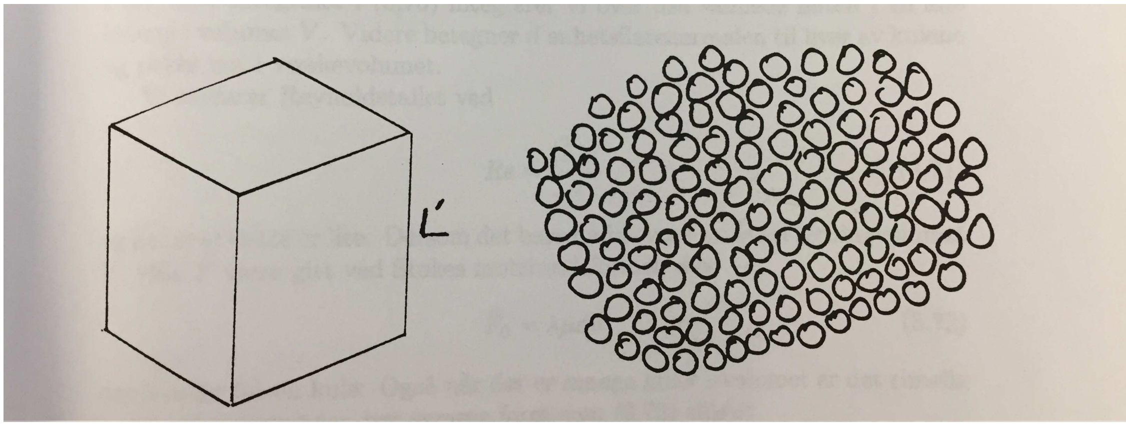

Figure 1: Illustration of a porous medium (right) and averaging cube of volume V and edge L' (left)

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where a bar denotes spatial average and n the volume porosity, defined by the ratio between the volume occupied by the fluid and the volume occupied by the porous material. We may then write

$$\mathbf{v} = \bar{\mathbf{v}} + \mathbf{v}', \quad p = \bar{p} + p', \tag{6}$$

where $\bar{\mathbf{v}'} = 0$ and $\bar{p'} = 0$ by definition.

We first perform the average of the r.h.s. of the momentum equation obtaining

$$\int_V \Big(- \nabla p + \mu \nabla^2 \mathbf{v} \Big) dV = nV(-\nabla \bar{p} + \mu \nabla^2 \bar{\mathbf{v}}) + \int_V \Big(- \nabla p' + \mu \nabla^2 \mathbf{v}' \Big). \tag{7}$$

The integral on the r.h.s. of (7) may be rewritten using Gauss' theorem, obtaining

$$-\mathbf{F} = \int_V \Big(- \nabla p' + \mu \nabla^2 \mathbf{v}' \Big) = - \int_i \Big(- p'\mathbf{n} + \mu \frac{\partial \mathbf{v}'}{\partial n} \Big) dS, \tag{8}$$

where in the latter integration is over the collective surface i of all of the spheres in the volume V, and $\mathbf{n}$ denotes the unit surface normal vector of the spheres pointing into the fluid volume.

The Reynolds number of the flow defined by

$$Re = \frac{|\bar{\mathbf{v}}| d}{\nu}, \tag{9}$$

is assumed small. If only one sphere is included in the box volume, the force in (8) is simply given by Stoke's resistance formula, i.e.,

$$\mathbf{F}_0 = \lambda \mu d \bar{\mathbf{v}}, \tag{10}$$

where $\lambda = 3\pi$ for the single sphere. In the case when several spheres exist in the volume, we may assume a resistance taking the similar form as (10), obtaining

$$\mathbf{F} = N\mathbf{F}_0 = N\lambda\mu d\bar{\mathbf{v}}, \tag{11}$$

where N is the number of spheres of the cube. The permeability is then introduced by

$$k = \frac{nV}{N\lambda d}, \tag{12}$$

obtaining for $\mathbf{F}$:

$$\mathbf{F} = \frac{nV\mu}{k}\bar{\mathbf{v}}. \tag{13}$$

We note that while $\lambda = 3\pi$ for a single sphere, λ is much greater for a densely packed cube of small spheres, where typical values of n and k may be $n = 0.37$ and $k = 1.8 \cdot 10^{-5}d^2$.

The l.h.s. of the momentum equation is then averaged. Dividing by a factor nV we obtain

$$\rho\frac{\partial\bar{\mathbf{v}}}{\partial t} + \rho\bar{\mathbf{v}} \cdot \nabla\bar{\mathbf{v}} = -\nabla\bar{p} + \mu\nabla^2\bar{\mathbf{v}} - \frac{\mu}{k}\bar{\mathbf{v}}. \tag{14}$$

In the next step the magnitude of the individual terms in (14) are compared. The analysis employs that the diameter d of the spheres is much less than the length scale L of the averaged fluid flow. We obtain

$$|\nabla^2\bar{\mathbf{v}}| \sim \frac{|\bar{\mathbf{v}}|}{L^2} << \frac{|\bar{\mathbf{v}}|}{k}, \tag{15}$$

where we have used that $k \sim d^2 \times factor$ where "$factor$" is a very small number. Further we find

$$\rho|\bar{\mathbf{v}} \cdot \nabla\bar{\mathbf{v}}| \sim Re\,\frac{\mu|\bar{\mathbf{v}}|}{Ld} << \frac{\mu|\bar{\mathbf{v}}|}{k}. \tag{16}$$

A similar justification applies to the term $\rho\partial\bar{\mathbf{v}}/\partial t$. The momentum equation (14) then simplifies to

$$\nabla\bar{p} + \frac{\mu}{k}\bar{\mathbf{v}} = 0, \tag{17}$$

which is Darcy's law from 1856, initially formulated on empirical reasoning following a set of experimental measurements.

In the case when the Reynolds number is moderate or large, the force will assume a quadratic relation in the fluid velocity, i.e.

$$\mathbf{F} \sim |\bar{\mathbf{v}}|\bar{\mathbf{v}}. \tag{18}$$

We then consider the averaged mass conservation equation obtaining

$$\nabla \cdot \bar{\mathbf{v}} = 0. \tag{19}$$

Introducing (17) one obtains

$$\nabla \cdot (k \nabla \bar{p}) = 0, \tag{20}$$

where the permability k in general is function of the spatial coordinates, while the dynamic viscosity of the fluid is constant in space. The pressure field satisfies the Laplace equation, in the case of $k = constant$. The velocity field is obtained from (17) (when the pressure $\bar{p}$ has been obtained).

Example. Water flow through a lock.

The water flow through a porous lock may illustrate Darcy's law. The porous lock is filled with spheres of common diameter d. On the right hand side of the lock, the water depth is Δh higher than on the left hand side. The gravity force acts in along the vertical. The pressure on each side of the lock is obtained by

$$p^+ = p_0 + \rho g(\Delta h - y), \quad x > x^+, \tag{21}$$

$$p^+ = p_0 - \rho g y, \quad x < x^-, \tag{22}$$

obtaining the pressure gradient by $dp/dx = (p^+ - p^-)/(x^+ - x^-)$, where ρ is the density, g acceleration of gravity, y vertical coordinate and p_0 the atmospheric pressure. The lock and flow configuration are illustrated in figure 2.

The average velocity of the flow through the lock is obtained by Darcy's law giving

$$\frac{dp}{dx}\mathbf{i} + \frac{\mu}{k} w_n \mathbf{i} = 0, \tag{23}$$

where $\mathbf{i} = \nabla x$ and $w_n \mathbf{i}$ the flow through velocity, obtaining

$$w_n = -\frac{k}{\mu}\frac{dp}{dx} = -\frac{k\, g\, \Delta h}{\nu\, (x^+ - x^-)}. \tag{24}$$

Using the value of $k = 1.8 \cdot 10^{-5} d^2$ given below eq. (13) and $\nu = 10^{-6}$ $\mathrm{m^2 s^{-1}}$ (fresh water at 20°C) we obtain

$$w_n \simeq -1.8 \left(\frac{10d}{\mathrm{m}}\right)^2 \frac{\Delta h}{x^+ - x^-} \;\; \mathrm{ms^{-1}}\,. \tag{25}$$

We put $\Delta h/(x^+ - x^-) = 1$. Eq. (25) then predicts a flow-through velocity of 1.8 $\mathrm{cm\,s^{-1}}$ for spheres of diameter 1 cm, and 0.18 $\mathrm{mm\,s^{-1}}$ for spheres of diameter 1 mm.

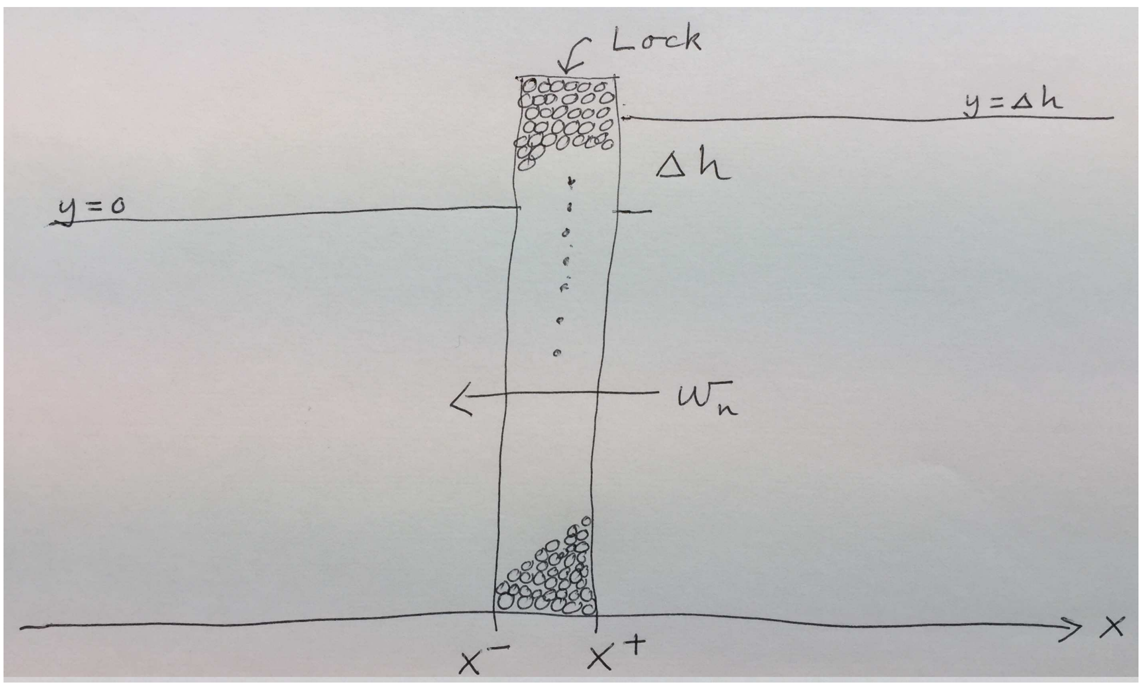

Figure 2: Illustration of a vertical porous lock filled with small spheres of diameter d.