Lecture on: Acoustic (secondary) streaming. MEK4300 Viscous flow and turbulence, Univ. of Oslo.

by John Grue, September 19, 2017

The non-stationary boundary layer equations read:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}, \tag{1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

where the first equation expresses conservation of momentum in the x-direction, and the second conservation of mass. The motion is assumed to be two-dimensional. The velocities are function of x, y, t as follows: $u = (x, y, t)$ and $v = (x, y, t)$. The velocity field $U = U(x, t)$ outside the boundary layer is given and drives the motion of the boundary layer.

The boundary conditions read $u = v = 0$ at $y = 0$ and $u = U$ for $y \to \infty$.

We note, however, that a function of the boundary layer is to set up an additional, constant horizontal velocity outside the boundary layer. This means that $u = U + const.$ outside the boundary layer where the time-independent velocity is a steady, secondary streaming. This varies according to the x-coordinate. The purpose in this lecture is to calculate this streaming.

We assume that the motion outside the boundary layer is represented by the velocity field $U(x, t)$. The pressure gradient within the boundary layer is then given by

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}. \tag{3}$$

Combining (3) with (1) gives

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x}. \tag{4}$$

The method of successive approximations is used to solve the boundary layer equations. We assume that the velocity vector (u, v) may be obtained by

$$u = u_0 + u_1 + u_2 + ... +, \tag{5}$$

$$v = v_0 + v_1 + v_2 + ... +, \tag{6}$$

where $u_0 >> u_1$, $u_1 >> u_2$, ..., $v_0 >> v_1$, $v_1 >> v_2$. The first approximation to the equation of motion reads

$$\frac{\partial u_0}{\partial t} - \nu\frac{\partial^2 u_0}{\partial y^2} = \frac{\partial U}{\partial t}, \tag{7}$$

which together with the continuity equation determines the leading approxiation of the velocity field. The boundary conditions read: $u_0 = v_0 = 0$ at $y = 0$, and $u_0 = U$ for $y \to \infty$.

The next approximation gives

$$\frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial y^2} = U\frac{\partial U}{\partial x} - u_0\frac{\partial u_0}{\partial x} - v_0\frac{\partial u_0}{\partial y}, \tag{8}$$

which together with the continuity equation determines u_1 and v_1. The boundary conditions read: $u_1 = v_1 = 0$ at $y = 0$. Further, u_1 is bounded outside the boundary layer.

The latter equation (8) may be rewritten using conservation of mass where $\partial u_0/\partial x + \partial v_0/\partial y = 0$ giving,

$$\frac{\partial u_1}{\partial t} - \nu \frac{\partial^2 u_1}{\partial y^2} = U\frac{\partial U}{\partial x} - \frac{\partial}{\partial x}(u_0 u_0) - \frac{\partial}{\partial y}(u_0 v_0). \tag{9}$$

The method is valid provided that

$$\frac{\partial U}{\partial t} >> U\frac{\partial U}{\partial x}. \tag{10}$$

Periodic boundary layer

We assume that $U(x,t) = Re\Big(U_0(x)e^{\mathrm{i}\omega t}\Big) = \frac{1}{2}U_0(x)e^{\mathrm{i}\omega t} + c.c.$ where $c.c.$ means complex conjugate. We introduce the dimensionless coordinate $\eta = y/\delta$ where $\delta = \sqrt{2\nu/\omega}$ and assume that

$$u_0(x,y,t) = Re\Big(U_0(x)\zeta_0'(\eta)e^{\mathrm{i}\omega t}\Big). \tag{11}$$

We further introduce the stream function such that $u_0 = \partial\psi_0/\partial y$ and $v_0 = -\partial\psi_0/\partial x$. We obtain

$$\psi_0 = Re(\delta U_0(x)\zeta_0(\eta)e^{\mathrm{i}\omega t}), \tag{12}$$

$$v_0 = Re\Big(-\delta\frac{dU_0}{dx}\zeta_0(\eta)e^{\mathrm{i}\omega t}\Big). \tag{13}$$

The function ζ_0 is determined by the following equation

$$\zeta_0''' - 2\mathrm{i}\zeta_0' = -2\mathrm{i}. \tag{14}$$

The solution of this equation, satisfying the boundary conditions $\zeta_0' = 0$ for $\eta = 0$ and $\zeta_1' = 1$ for $\eta \to \infty$, reads

$$\zeta_0' = 1 - e^{-(1+\mathrm{i})\eta}. \tag{15}$$

Another integration determines ζ_0 by

$$\zeta_0 = \eta - \frac{\zeta_0'}{\kappa}, \tag{16}$$

where $\zeta_0' = 1 - e^{-\kappa\eta}$, $\kappa = 1 + \mathrm{i}$ and we have used the boundary condition $v_0 = 0$ at $y = 0$ giving $\zeta_0 = 0$ at $\eta = 0$. Use of (15) gives

$$u_0(x, y, t) = Re\Big(U_0(x)[1 - e^{-(1+\mathrm{i})\eta}]e^{\mathrm{i}\omega t}\Big). \tag{17}$$

In the case with $U_0(x)$ real we obtain

$$u_0(x, y, t) = U_0(x)\Big(\cos\omega t - e^{-\eta}\cos(\omega t - \eta)\Big). \tag{18}$$

Acoustic streaming. Steady, secondary streaming.

An important effect of the oscillatory boundary layer is a steady, secondary streaming that becomes introduced outside the boundary layer. The streaming is a consequence of the oscillatory boundary layer at the wall. The streaming is obtained by evaluating the next term u_1 of the velocity expansion, see eq. (9). We first evaluate the r.h.s. of (9).

With $U(x, t) = Re\Big(U_0(x)e^{\mathrm{i}\omega t}\Big)$ we obtain for the term $U\partial U/\partial x$ in (9):

$$U\frac{\partial U}{\partial x} = \tfrac{1}{4}\Big(U_0U_0'^* + c.c.\Big) + \tfrac{1}{4}\Big(U_0U_0'e^{\mathrm{i}2\omega t} + c.c.\Big), \tag{19}$$

where a star and $c.c.$ denote complex conjugate.

The product u_0u_0 in (9), with u_0 given in (11), gives

$$u_0u_0 = \tfrac{1}{4}\Big(U_0U_0^*\zeta_0'\zeta_0'^* + c.c.\Big) + \tfrac{1}{4}\Big(U_0^2\zeta_0'\zeta_0'e^{\mathrm{i}2\omega t} + c.c.\Big). \tag{20}$$

The second term on the r.h.s. of (9) becomes

$$-\frac{\partial}{\partial x}(u_0u_0) = -\tfrac{1}{2}\Big(U_0U_0'^*\zeta_0'\zeta_0'^* + c.c.\Big) - \tfrac{1}{2}\Big(U_0U_0'\zeta_0'\zeta_0'e^{\mathrm{i}2\omega t} + c.c.\Big). \tag{21}$$

The product u_0v_0 in (9), with u_0 given in (11) and v_0 in (13), gives

$$u_0v_0 = -\tfrac{1}{4}\Big(\delta U_0U_0'^*\zeta_0'\zeta_0^* + c.c.\Big) - \tfrac{1}{4}\Big(\delta U_0U_0'\zeta_0'\zeta_0e^{\mathrm{i}2\omega t} + c.c.\Big). \tag{22}$$

Evaluating the derivative $-\partial/\partial y = -(1/\delta)(d/d\eta)$ we obtain

$$-\frac{\partial}{\partial y}(u_0v_0) = \tfrac{1}{4}\Big(U_0U_0'^*(\zeta_0'\zeta_0^*)' + c.c.\Big) - \tfrac{1}{4}\Big(U_0U_0'(\zeta_0'\zeta_0)'e^{\mathrm{i}2\omega t} + c.c.\Big). \tag{23}$$

Our primary interest is to calculate the steady part of the horizontal velocity component u_1. Because of the form of the terms (19), (21) and (23) it is tempting to write the steady part of u_1 on the form

$$u_{1,steady} = u_{1s} = \frac{1}{2}\Big(\frac{U_0U_0'^*}{\omega}\,\zeta_{1b}' + c.c.\Big). \tag{24}$$

Putting this into (9), noting that

$$-\nu\frac{\partial^2 u_{1s}}{\partial y^2} = \frac{\nu}{\delta^2}\frac{\partial^2 u_{1s}}{\partial \eta^2} = -\frac{\nu\omega}{2\nu}\frac{1}{2}\Big(\frac{U_0 U_0^{'*}}{\omega}\zeta_{1b}^{'''} + c.c.\Big), \tag{25}$$

we obtain the following equation for $\zeta_{1b}^{'''}$:

$$0 = U_0 U_0^{'*}\Big[\tfrac{1}{4}\zeta_{1b}^{'''} + \tfrac{1}{4} - \tfrac{1}{2}\zeta_0'\zeta_0^{'*} + \tfrac{1}{4}(\zeta_0'\zeta_0^*)'\Big] + c.c., \tag{26}$$

where the paranthesis must be zero, giving for $\zeta_{1b}^{'''}$:

$$\zeta_{1b}^{'''} = -1 + 2\zeta_0'\zeta_0^{'*} - (\zeta_0'\zeta_0^*)'. \tag{27}$$

Boundary conditions for u_{1s} and $\zeta_{1b}^{'}$

The boundary conditions for u_{1s} and $\zeta_{1b}^{'}$ are:

$$u_{1s}(y=0) = \zeta_{1b}^{'}(\eta=0) = 0, \tag{28}$$

at the wall. Further the velocity u_{1s} and its variant $\zeta_{1b}^{'}$ must be finite for $y \to \infty$ (or $\eta \to \infty$), i.e.

$$|u_{1s}| < \infty, \quad y \to \infty; \qquad |\zeta_{1b}^{'}| < \infty, \quad \eta \to \infty. \tag{29}$$

Integration of (27)

Integration of the first term on the r.h.s. of (27) gives

$$-\tfrac{1}{2}\eta^2 + a\eta + b, \tag{30}$$

where a and b are constants. Integration of the second term on the r.h.s. of (27), using that $\zeta_0' = 1 - e^{-\kappa\eta}$, and that $\zeta_0'\zeta_0^{'*} = 1 - e^{-\kappa\eta} - e^{-\kappa^*\eta} + e^{-2\eta}$, gives

$$2\int \zeta_0'\zeta_0^{'*} d\eta = 2\eta + \frac{2e^{-\kappa\eta}}{\kappa} + \frac{2e^{-\kappa^*\eta}}{\kappa^*} - e^{-2\eta} + const., \tag{31}$$

where $\kappa = 1 + \mathrm{i}$, $\kappa^* = 1 - \mathrm{i}$ and $\kappa + \kappa^* = 2$. Another integration gives

$$\eta^2 - \frac{2e^{-\kappa\eta}}{\kappa^2} - \frac{2e^{-\kappa^*\eta}}{\kappa^{*2}} + \frac{e^{-2\eta}}{2} + const. \times \eta. \tag{32}$$

Integration of the third term on the r.h.s. of (27), using that $\zeta_0^* = \eta - \zeta_0^{'*}/\kappa^*$, and $\zeta_0'\zeta_0^* = \zeta_0'\eta - \zeta_0'\zeta_0^{'*}/\kappa^*$, gives

$$-\int \zeta_0'\zeta_0^* d\eta = -\zeta_0\eta + \frac{1}{2}\eta^2 - \frac{\zeta_0}{\kappa} + \frac{1}{\kappa^*}\Big(\eta + \frac{e^{-\kappa\eta}}{\kappa} + \frac{e^{-\kappa^*\eta}}{\kappa^*} - \frac{e^{-2\eta}}{2}\Big) + const. \tag{33}$$

Collecting the terms in (30), (32) and (33) we obtain

$$\zeta'_{1b} = \eta^2 - \Big(\eta + \frac{1}{\kappa}\Big)\Big(\eta - \frac{\zeta'_0}{\kappa}\Big) + \eta\Big(a + \frac{1}{\kappa^*}\Big) + b$$
$$- \frac{2e^{-\kappa\eta}}{\kappa^2} - \frac{2e^{-\kappa^*\eta}}{\kappa^{*2}} + \frac{e^{-2\eta}}{2} + \frac{1}{\kappa^*}\Big(\frac{e^{-\kappa\eta}}{\kappa} + \frac{e^{-\kappa^*\eta}}{\kappa^*} - \frac{e^{-2\eta}}{2}\Big). \quad (34)$$

The boundary condition (28) is then used to determine the constant b. Evaluating (34) at $\eta = 0$ obtains

$$0 = \zeta'_{1b}(\eta = 0) = b - \frac{2}{\kappa^2} - \frac{2}{\kappa^{*2}} + \frac{1}{2} + \frac{1}{\kappa^*}\Big(\frac{1}{\kappa} + \frac{1}{\kappa^*} - \frac{1}{2}\Big) = b + \frac{3}{4} + \frac{\mathrm{i}}{4}, \quad (35)$$

giving that

$$b = -\frac{3}{4} - \frac{\mathrm{i}}{4}. \quad (36)$$

Letting $\eta \to \infty$ in (34) we obtain

$$\zeta'_{1b} \sim \eta\Big(a + \frac{1}{\kappa^*}\Big) + b + \frac{1}{\kappa^2}. \quad (37)$$

The integration constant a is chosen such that $a + 1/\kappa^* = 0$, giving

$$\zeta'_{1b} \to b + \frac{1}{\kappa^2} = -\frac{3}{4}(1 + \mathrm{i}), \quad \eta \to \infty. \quad (38)$$

The resulting streaming velocity obtains the following form:

$$u_1(x, \infty) = -\frac{3}{4}\, Re\Big(\frac{U_0}{\omega}\frac{dU_0^*}{dx}(1 + \mathrm{i})\Big). \quad (39)$$

In the case when $U_0(x)$ is real, the result is

$$u_1(x, \infty) = -\frac{3}{4}\,\frac{U_0}{\omega}\frac{dU_0}{dx}. \quad (40)$$

Example 1. Steady, secondary streaming and flow cells at a cylinder in lateral oscillation.

Consider a long circular cylinder of radius R. The cylinder is oscillating with lateral motion of frequency ω and velocity amplitude U_∞ in otherwise calm fluid. The section-wise velocity along the boundary is given by $U_0(x) = U_\infty \sin\theta = U_\infty \sin(x/R)$ where θ is the angle between the oscillation direction at time $t = 0$ and the position along the boundary. The steady, secondary streaming becomes

$$\frac{u_1(x, \infty)}{U_\infty^2/\omega R} = -\frac{3}{8}\sin 2\theta, \quad (41)$$

where we have used that $\sin\theta\cos\theta = (1/2)\sin 2\theta$. The streaming velocity at the wall is negative in the first and fourth quadrants, and positive in the second and third. The boundary layer induced streaming velocity sets up closed flow cells as illustrated in figure 1. The cells become unstable to the Honji instability, a particular mushroom shaped pattern, if the motion amplitude gets sufficiently large, more precisely, for $\beta = (2R)^2/(\nu(2\pi/\omega)) \sim 200 - 300$ and $KC = U_\infty(2\pi/\omega)/(2R) \sim 2$. This kind of instability was first obtained experimentally by Honji (1981) and later calculated by Rashid, Vartdal and Grue (2011), see figure 2.

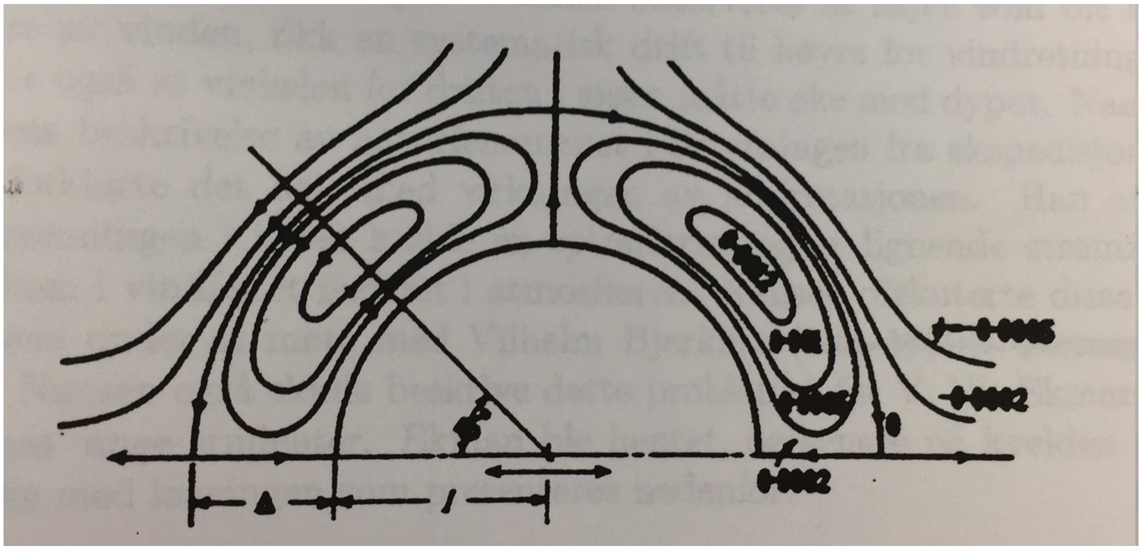

Figure 1: Oscillating cylinder. Convective cells of steady, secondary streaming.

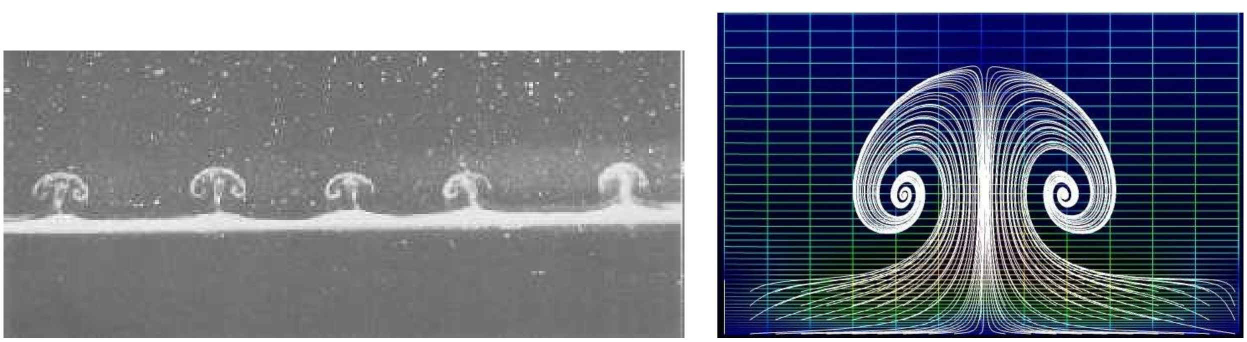

Figure 2: Honji instability in the boundary layer of an oscillating cylinder. Experiments by Honji (1981) (left) and calculation by Rashid et al. (2011) (right).

Example 2. Steady, secondary streaming and flow cells at the sea bed below a standing water wave.

Consider the boundary layer at a sea bed below a standing wave of elevation $\eta = 2A \sin kx \sin \omega t$ where k is the wavenumber and ω the frequency. The horizontal velocity at the bottom is given by

$$U(x,t) = U_0(x) \cos \omega t, \quad \text{where} \quad U_0 = U_\infty \cos kx. \tag{42}$$

The bottom boundary layer below the standing wave induces a streaming velocity above the bottom given by

$$\frac{u_1(x,\infty)}{U_\infty^2/(\omega/k)} = \frac{3}{8} \sin 2kx. \tag{43}$$

The flow pattern is illustrated in figure 3 where particles at the sea bed are collected at $2x/R = \pm\pi/2, \pm 3\pi/2, ...$ This explains a mechanism for systematic erosion of the sea bed, as well as formation of period sand bars below the waves.

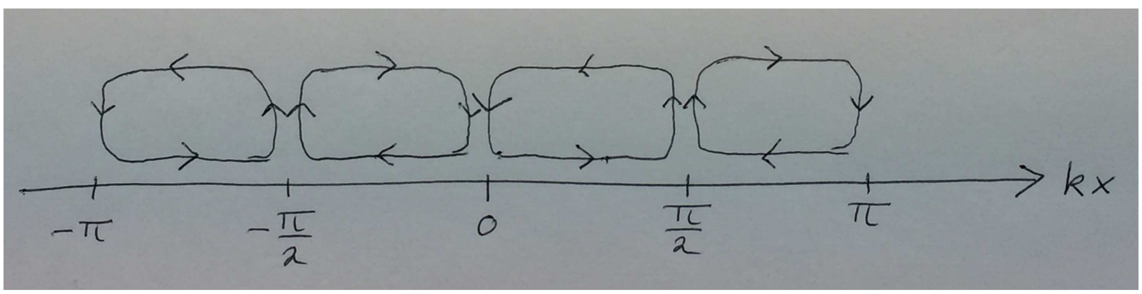

Figure 3: Convective cells of secondary streaming at the sea bed below a standing wave.

References

H. Honji (1981). Streaked flow around an oscillating circular cylinder. J. Fluid Mech. 107:509-520.

F. Rashid, M. Vartdal and J. Grue (2011). Oscillating cylinder in viscous fluid: Calculation of flow patterns and forces. J. Eng. Math. 70:281-295.