UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MEK4300/9300 — Viscous flow og turbulence

Day of examination: Wednesday 15. June 2011

Examination hours: 9.00 – 13.00

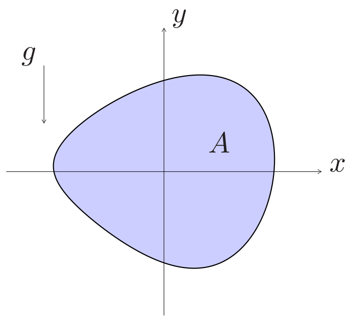
This problem set consists of 4 pages.

Appendices: None

Permitted aids: Rottmann: Matematische Formelsamlung, certified calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Poiseuille flow (weight 36%)



A viscous flow in a duct of arbitrary cross-section (see figure) is driven by a pressure change along the duct. The cross section is parallel to the xy plane and is denoted by A, while the z axis is aligned along the duct. The fluid is incompressible, the dynamic viscosity coefficient is μ, gravity is directed in the negative y direction and the duct wall is impenetrable. Moreover, we assume that the flow is stationary and uniform in the z direction.

1a (weight 6%)

Write down the equations and the boundary conditions for this problem.

1b (weight 6%)

Assume a velocity on the form

$$\mathbf{v} = w(x, y)\mathbf{k},$$

and find an expression for the pressure. Furthermore, show that

$$\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = -\beta \tag{1}$$

and state how the constant β relates to the pressure.

(Continued on page 2.)

1c (weight 6%)

Find the wall stress τ_w expressed by derivatives of w. Show also that the dissipation in this case becomes

$$\Phi = \mu \left\{ \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right\}.$$

1d (weight 6%)

Find w, expressed by β, when the duct perimeter is elliptical and defined as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Hint: observe that the Laplacian of the left hand side of this expression is constant.

1e (weight 6%)

In this sub-problem we assume $a = b$. Find the total volume flux, Q, along the duct. Do also determine the wall stress τ_w expressed by the pressure gradient and other parameters of the problem.

1f (weight 6%)

We again assume a duct of general shape. Use (1) to show the energy equation on the form

$$Q \frac{\partial p}{\partial z} = - \iint_A \Phi \, dx \, dy,$$

where Q is the volume flux in the duct, and explain the physical content of this relation. Hint: you need to manipulate the integrals to obtain the right hand side.

Problem 2 Turbulence (weight 28%)

2a (weight 10%)

Derive the Reynolds averaged Navier-Stokes (RANS) equations for an incompressible, Newtonian fluid and identify the Reynolds stress tensor. You may use, without proof, that $\mathbf{v} \cdot \nabla \mathbf{v} = \nabla \cdot (\mathbf{v}\mathbf{v})$ for any divergence-free velocity field $\mathbf{v}$.

2b Turbulent duct flow (weight 10%)

We assume fully developed pressure-driven duct flow between two planes at $y = -h$ and $y = h$. The mean velocity will then be reduced to $\overline{\mathbf{v}} = \overline{u}(y)\mathbf{i}$. Find the pressure variation across the duct from the RANS equations. Show that

$$0 = -\frac{1}{\rho} \frac{\partial \overline{p}}{\partial x} + \frac{1}{\rho} \frac{\partial \tau}{\partial y},$$

where the shear stress is

$$\tau = \mu \frac{\mathrm{d}\overline{u}}{\mathrm{d}y} - \rho \overline{u'v'},$$

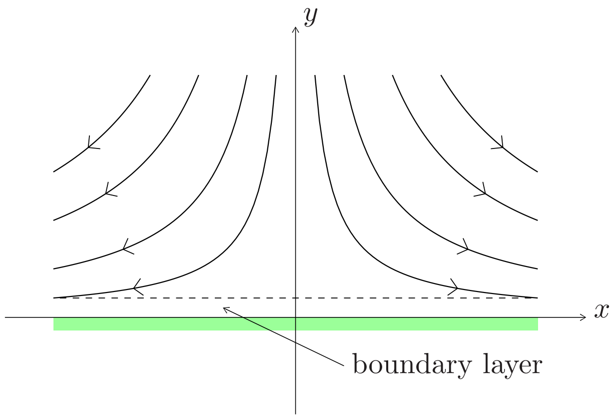
(Continued on page 3.)

ρ is the density and $\frac{\partial \overline{p}}{\partial x}$ is a constant.

2c (weight 8%)

Why must there be a viscous sublayer close to a no-slip boundary? The velocity profile in the sublayer may be approximated by a linear function. Find this expressed by the wall stress τ_w and the other relevant parameters.

Problem 3 Stagnation flow (weight 36%)



An inviscid solution for stagnation flow at a wall (see figure) is

$$\psi^* = Bx^*y^*, \quad u^* = Bx^*, \quad v^* = -By^*,$$

where ψ^* is the stream function, and u^* and v^* are the velocity components. The star indicates quantities with dimension. We seek the modification of this flow due to the no-slip condition at the wall and viscosity.

3a (weight 9%)

We denote the density by ρ and the dynamic viscosity coefficient by μ. Dimensionless variables are then introduced according to

$$\begin{array}{lll} x^* = Lx, & y^* = Hx, & t^* = Bt, \\ u^* = LBu, & v^* = HBv, & p^* = \hat{p}p, \end{array}$$

where L is some length scale in the x direction and $H = \sqrt{\frac{\mu}{\rho B}}$ is a scale related to the boundary layer. Determine the pressure scale, $\hat{p}$, such that the non-dimensional equations become

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \gamma\frac{\partial^2 u}{\partial x^2}, \\ \gamma\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) &= -\frac{\partial p}{\partial y} + \gamma\left(\frac{\partial^2 v}{\partial y^2} + \gamma\frac{\partial^2 v}{\partial x^2}\right), \end{aligned}$$

where $\gamma = H^2/L^2$. For any sensible choice of L the parameter γ is small. However, we seek exact solutions of the equations and will not discard terms of order γ as such.

(Continued on page 4.)

3b (weight 9%)

We assume that v is independent of x

$$v = -F(y),$$

where F is some function that must be determined. Show that this implies that the other velocity component is on the form

$$u = xF'(y),$$

and that F must fulfill the boundary conditions

$$F(0) = 0, \quad F'(0) = 0, \quad \lim_{y\to\infty} F'(y) = 1.$$

3c (weight 9%)

Show that the pressure must be on the form

$$p = -\gamma(F' + F^2) + p_0(x).$$

3d (weight 9%)

Show that F must solve the equation

$$F''' + FF'' - \left(F'\right)^2 = -1.$$

The End