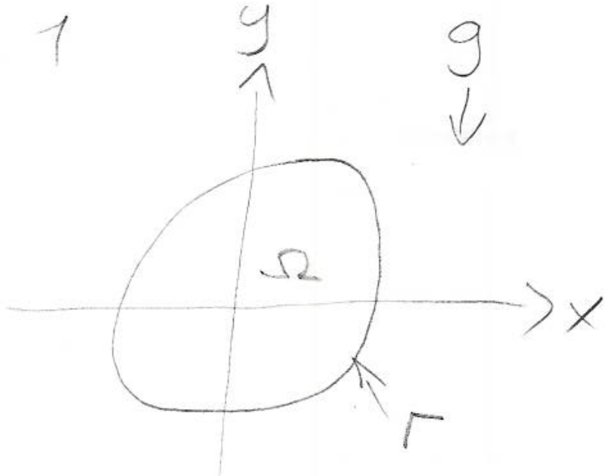
Eksamen MEK 4300 15/6-2011

Problem 1



a) $\mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} - g\mathbf{k}$

$$\nabla\cdot\mathbf{u} = 0$$

$$\mathbf{u} = 0 \text{ on } \Gamma$$

b) $\mathbf{u} = u_z\mathbf{k} = (0, 0, u_z)$

Momentum eq. z-dir:

$$\cancel{\mathbf{u}\cdot\nabla u_z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 u_z$$

Uniform in z

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = \nu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right)$$

Momentum eq. x-direction:

$$0 = \frac{\partial p}{\partial x}$$

Momentum eq. y-direction:

$$0 = -\frac{\partial p}{\partial y} - \rho g$$

$$\Rightarrow \quad p = -\rho g y + p_0(z) \quad , \; p_0 \text{ const}$$

$$\frac{\partial p}{\partial z} = \frac{\partial p_0}{\partial z}$$

$$\frac{\partial P_0}{\partial z} = \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right)$$

Left hand side function of z only and right hand side function of x and y.

$$\Rightarrow \quad \frac{\partial P_0}{\partial z} = \text{constant}$$

We have

$$\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} = -\beta$$

$$\Rightarrow -\frac{1}{\mu}\frac{\partial P_0}{\partial z} = \beta$$

c) $\tau_w = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot \mathbf{n} \quad (= \mu \nabla \mathbf{u} \cdot \mathbf{n})$

$$\nabla \mathbf{u} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \cancel{\frac{\partial u_z}{\partial z}} \end{pmatrix}, \quad \nabla \mathbf{u}^T = \begin{pmatrix} 0 & 0 & \frac{\partial u_z}{\partial x} \\ 0 & 0 & \frac{\partial u_z}{\partial y} \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{n} = n_x \mathbf{i} + n_y \mathbf{j} = (n_x, n_y, 0)$$

$$\tau_w = \mu \begin{pmatrix} 0 & 0 & \frac{\partial u_z}{\partial x} \\ 0 & 0 & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & 0 \end{pmatrix} \cdot \begin{pmatrix} n_x \\ n_y \\ 0 \end{pmatrix} = \underline{\left(\mu \frac{\partial u_z}{\partial x} n_x + \mu \frac{\partial u_z}{\partial y} n_y\right)}$$

Show:

$$\phi = \mu \left(\left(\frac{\partial u_z}{\partial x} \right)^2 + \left(\frac{\partial u_z}{\partial y} \right)^2 \right)$$

Definition:

$$\Phi = \mu \nabla u : (\nabla u + \nabla u^T)$$

$$= \mu \frac{\partial u_i}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$= \mu \cdot \begin{pmatrix} 0 & 0 & \frac{\partial u_z}{\partial x} \\ 0 & 0 & \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & 0 \end{pmatrix}$$

$$= \mu \left(\left(\frac{\partial u_z}{\partial x} \right)^2 + \left(\frac{\partial u_z}{\partial y} \right)^2 \right)$$

d)

$$F(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

$$\nabla^2 F = 2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right) = \text{constant}$$

and $F(x,y) = 0$ on boundary Γ.

$\Rightarrow$ F satisfies both equation and boundary condition

$u_z = F \cdot C$, where C is constant

We have

$$\nabla^2 u_z = -\beta$$

$$2C\left(\frac{1}{a^2}+\frac{1}{b^2}\right) = -\beta$$

$$C = -\frac{\beta}{2}\frac{1}{\left(\frac{1}{a^2}+\frac{1}{b^2}\right)} = -\frac{\beta}{2}\frac{a^2b^2}{b^2+a^2}$$

$$u_z = -\frac{\beta}{2}\frac{a^2b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

e)

$$u_z = -\frac{\beta}{2}\frac{a^{\cancel{2}4}}{2\cancel{a^2}}\left(\frac{x^2+y^2}{a^2}-1\right)$$

$$= -\frac{\beta}{2}a^2\left(\frac{x^2+y^2}{a^2}-1\right)$$

$$= -\frac{\beta}{4}\left(x^2+y^2-a^2\right)$$

Cylindrical coordinates

$$u_z = -\frac{\beta}{4}\left(r^2-a^2\right)$$

7e)

$$Q = \int_{\Omega} u_z \, dA$$

$$= \int_0^{2\pi} \int_0^{a} -\frac{3}{4}(r^2 - a^2)\, r \, d\theta \, dr$$

$$= -\frac{3}{4} 2\pi \int_0^{a} (r^3 - a^2 r)\, dr$$

$$= -\frac{3\pi}{2} \left(\frac{1}{4} r^4 - \frac{a^2}{2} r^2 \right)_0^a$$

$$= -\frac{3\pi}{2} \left(\frac{a^4}{4} - \frac{a^4}{2} \right)$$

$$\underline{\underline{Q = \frac{3\pi a^4}{8}}}$$

$$\tau_w = \mu \nabla u \cdot n$$

Cylinder coordinates

$$\tau_w = \mu \frac{\partial u_z}{\partial r} k = -\mu \frac{3}{4}(2r)\Big|_{r=a} = \underline{\underline{-\mu \frac{3a}{2}}}$$

1 f) $\nabla^2 u_z = -\beta \left(= \frac{1}{\mu} \frac{dP_0}{dz} \right)$

Multiply by u_z and integrate over A

$$\int u_z \nabla^2 u_z \, dA = \int -\beta u_z \, dA$$

Gauss theorem

$$\int \left(\nabla \cdot (u_z \nabla u_z) - \nabla u_z \cdot \nabla u_z \right) dA = -\beta \int u_z \, dA$$

$$\oint u_z \nabla u_z \, d\Gamma - \int \nabla u_z \cdot \nabla u_z \, dA = -\beta Q$$

$$-\int \nabla u_z \cdot \nabla u_z \, dA = \frac{1}{\mu} \frac{dP_0}{dz} Q$$

Use $\quad \Phi = \mu \nabla u_z \cdot \nabla u_z$

$$\Rightarrow -\int \Phi \, dA = \frac{dP_0}{dz} Q$$

Interpretation - Effect of pressure work goes to dissipation.

Problem 2

a) Navier - Stokes (neglecting body forces)

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u}$$

$$\nabla\cdot\mathbf{u} = 0$$

Reynolds decomposition

$$\mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}'$$
$$p = \bar{p} + p'$$

where overline represents an ensamble average and prime is fluctuation about the average.

Take average of momentum eq + div. eq. and use commutation of averaging and differentiation:

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + \overline{(\mathbf{u}\cdot\nabla)\mathbf{u}} = -\frac{1}{\rho}\nabla\bar{p} + \nu\nabla^2\overline{\mathbf{u}}$$

$$\nabla\cdot\overline{\mathbf{u}} = 0$$

Rewrite convection term

$$\overline{(\mathbf{u}\cdot\nabla)\mathbf{u}} = \overline{\nabla\cdot\mathbf{u}\mathbf{u}} - \cancel{\overline{\mathbf{u}\,\nabla\cdot\mathbf{u}}}$$

where $\mathbf{u}\mathbf{u}$ is the outerproduct of $\mathbf{u}$ times $\mathbf{u}$

Index notation:

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \overline{\frac{\partial u_i u_j}{\partial x_j}} - \overline{u_i \cancel{\frac{\partial u_j}{\partial x_j}}}$$

Introduce Reynolds decomposition:

$$\overline{\nabla \cdot \mathbf{u}\mathbf{u}} = \overline{\nabla \cdot (\bar{\mathbf{u}} + \mathbf{u}')(\bar{\mathbf{u}} + \mathbf{u}')}$$

$$= \nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}} + \overline{\mathbf{u}'\mathbf{u}'})$$

$$= (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} + \nabla \cdot \overline{\mathbf{u}'\mathbf{u}'})$$

$\Rightarrow$ Momentum eq:

$$\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} = -\frac{1}{\rho}\nabla \bar{p} + \nu \nabla^2 \bar{\mathbf{u}} - \nabla \cdot \overline{\mathbf{u}'\mathbf{u}'}$$

where $\overline{\mathbf{u}'\mathbf{u}'}$ is the Reynolds stress tensor.

b) Convection is zero since $\mathbf{u}$ is aligned with the walls. Stationary $\Rightarrow \frac{\partial \bar{u}}{\partial t} = 0$

Momentum eq:

only change in y

v' = vel. fluct in y-dir

x-dir

$$0 = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial x} + \nu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \overline{u'v'}}{\partial y}$$

$$0 = -\frac{\partial \bar{P}}{\partial x} + \mu \frac{\partial}{\partial y}\left(\frac{\partial \bar{u}}{\partial y} - \rho \overline{u'v'}\right)$$

y-dir

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial y} - \cancel{\frac{\partial \overline{v'u'}}{\partial x}} - \frac{\partial \overline{v'v'}}{\partial y} - \cancel{\frac{\partial \overline{v'w'}}{\partial z}}$$

$$\frac{\partial \bar{P}}{\partial y} = \rho \frac{\partial \overline{v'v'}}{\partial y}$$

z-dir

$$0 = -\frac{1}{\rho}\frac{\partial \bar{P}}{\partial z} \underbrace{- \frac{\partial \overline{w'u'}}{\partial x} - \frac{\partial \overline{w'v'}}{\partial y} - \frac{\partial \overline{w'w'}}{\partial z}}_{\text{all zero}}$$

$\overline{w'v'}$ zero because statistics are homogeneous in x and z-dir

$$\frac{\partial \bar{P}}{\partial z} = 0$$

$$\bar{P} = P(x,y)$$

$$\frac{\partial \bar{P}}{\partial y} = \rho \frac{\partial \overline{v'v'}}{\partial y}$$

$$\bar{P} = \rho \int_{-h}^{y} \frac{\partial \overline{v'v'}}{\partial y} dy + P_0(x)$$

$$\bar{P} = \rho\, \overline{v'v'}(y) + P_0(x)$$

From x-dir.

$$\frac{d\bar{P}}{\partial x} = \underbrace{\mu \frac{\partial}{\partial y}\left(\frac{\partial \bar{u}}{\partial y} - \rho\, \overline{u'v'}\right)}_{\text{depends only on } y}$$

$\Rightarrow \dfrac{\partial \bar{P}}{\partial x}$ is constant

$$\frac{\partial \bar{P}}{\partial x} = \frac{\partial P_0}{\partial x} = \text{constant}$$

$$P_0 = \beta \cdot x + d$$

$$\underline{\bar{P}(x,y) = \rho\, \overline{v'v'} + \beta x + d}$$

c) Very close to the wall the only non-zero term in the momentum equation is the Laplacian:

$$\frac{d\bar{P}}{dx} = \mu \frac{\partial^2 \bar{u}}{\partial y^2}$$

$$\frac{d^2 \bar{u}}{dy^2} = -\beta \left(= -\frac{1}{\mu}\frac{d\bar{P}}{dx}\right) = \text{constant}$$

Consider y as the distance to the wall located at $y = 0$.

Hence $\bar{u}(y) = -\frac{1}{2} 3 y^2 + c_1 y + c_2$

As $y \to 0$ $\quad \bar{u}(y) \simeq c_1 y$

The wall shear stress $\tau_w = \mu \frac{d\bar{u}}{dy}$

$$\bar{u} = \frac{1}{\mu}\int_0^y \tau_w \, dy$$

$$\underline{\underline{\bar{u} = \frac{\tau_w y}{\mu}}} \quad \cancel{+ u_0}$$

Problem 3

a) $$u \frac{\partial u}{\partial X} = \frac{u^*}{LB} \frac{\partial \frac{u^*}{LB}}{\partial \frac{X^*}{L}} = \frac{L u^*}{L^2 B^2} \frac{\partial u^*}{\partial X^*} = \frac{u^*}{LB^2} \frac{\partial u^*}{\partial X^*}$$

$\Rightarrow$ X-momentum has been multiplied by $\frac{1}{LB^2}$

$$\Rightarrow \frac{1}{\rho} \frac{\partial P^*}{\partial X^*} \cdot \frac{1}{LB^2} = \frac{1}{\rho} \frac{\partial P^*}{\partial \frac{X^*}{L}} L^2 B^2 = \frac{\partial \frac{P^*}{\rho L^2 B^2}}{\partial \frac{X^*}{L}} = \frac{\partial P}{\partial X}$$

$$\Rightarrow \quad \underline{P = \frac{P^*}{\rho L^2 B^2} \quad \text{and} \quad \hat{P} = \rho L^2 B^2}$$

b) $v = -F(y)$

Continuity:

$$\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(y)}{\partial y} = 0$$

$$\frac{\partial u}{\partial x} = F'$$

Integrate

$$u(x,y) = F' \int_0^x dx$$

$$u(x,y) = F'x + C_0(y)$$

Symmetry:
$$u(0,y) = F'(y)\cdot 0 + C_0(y) = 0$$

$$\Rightarrow C_0 = 0$$

$$\underline{u(x,y) = F'(y)\,x}$$

Boundary conditions

$$y = 0 \begin{cases} u = 0 & \Rightarrow F'(0) = 0 \\ v = 0 & \Rightarrow F(0) = 0 \end{cases}$$

$$y \to \infty \quad u \to u_{outer}$$

$$u^*_{outer} = Bx^* = BLx$$

$$u_{outer} = \frac{BLx}{BL} = x$$

$$u = F'(\infty)\, x = x$$

$$\Rightarrow F'(\infty) = 1$$

3c) Substitute $u = F'x$

$v = -F$

into y-eq

$$\gamma\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \gamma\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial x} = 0 \ , \ \frac{\partial v}{\partial y} = -F' \ , \ \frac{\partial^2 v}{\partial x^2} = 0 \ , \ \frac{\partial^2 v}{\partial y^2} = -F''$$

$$\gamma\left(F'x \cdot 0 + (-F)(-F')\right) = -\frac{\partial p}{\partial y} + \gamma\left(0 + (-F'')\right)$$

$$\gamma FF' = -\frac{\partial p}{\partial y} - \gamma F''$$

$$\frac{\partial p}{\partial y} = -\gamma\left(F'' + FF'\right)$$

3c) Integrate

$$P = -\gamma \int_0^y F'' + FF' \, dy + P_0$$

$$P = -\gamma \left(F'(y) - \cancel{F'(0)} + \int_0^y FF' dy \right) + P_0$$

$$\int_0^y FF' dy = F^2 \Big|_0^y - \int_0^y F'F dy$$

$$\Rightarrow \int_0^y FF' dy = \frac{1}{2} F^2$$

$$\underline{\underline{P = -\gamma \left(F'(y) + \frac{1}{2} F^2(y) \right) + P_0}}$$

$\uparrow$

1/2 missing in assigment ?

3d)

x - component

$$u \frac{du}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \gamma \frac{\partial^2 u}{\partial x^2}$$

$$F' x F' + (-F) F'' x = -\frac{\partial p}{\partial x} + F''' x + \gamma \cdot 0$$

$$\frac{dp}{\partial x} = x \left(F''' + F F'' - F'^2 \right)$$

$\Rightarrow$

$$F''' + F F'' - F'^2 = \alpha = \text{constant}$$

As $y \to \infty$ $\quad F'(\infty) = 1$ (constant)

and $F''(\infty) = F'''(\infty) = 0$

$$\Rightarrow \alpha = -1$$