UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MEK4300/9300 — Viscous flow og turbulence

Day of examination: Friday 15. June 2012

Examination hours: 9.00 – 13.00

This problem set consists of 5 pages.

Appendices: None

Permitted aids: Rottmann: Matematische Formelsamlung, certified calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Turbulence

Splitting of $\mathbf{v}$ and p

$$\mathbf{v} = \overline{\mathbf{v}} + \mathbf{v}', \quad p = \overline{p} + p'.$$

The continuity and Navier-Stokes equations

$$\nabla \cdot \mathbf{v} = 0,$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{v}$$

Averaging then gives

$$\nabla \cdot \overline{\mathbf{v}} = 0,$$

$$\frac{\partial \overline{\mathbf{v}}}{\partial t} + \overline{\mathbf{v} \cdot \nabla \mathbf{v}} = \frac{1}{\rho} \nabla \overline{p} + \nu \nabla^2 \overline{\mathbf{v}}.$$

We need to work more on the convective term only:

$$\overline{\mathbf{v} \cdot \nabla \mathbf{v}} = \nabla \cdot \overline{\mathbf{v}\mathbf{v}},$$

and

$$\overline{\mathbf{v}\mathbf{v}} = \overline{\overline{\mathbf{v}}\overline{\mathbf{v}}} + \overline{\overline{\mathbf{v}}\mathbf{v}'} + \overline{\mathbf{v}'\overline{\mathbf{v}}} + \overline{\mathbf{v}'\mathbf{v}'} = \overline{\mathbf{v}}\overline{\mathbf{v}} + \overline{\mathbf{v}'\mathbf{v}'},$$

since the terms which are linear in the fluctuations are nihilated by the averaging. Insertion in the averaged momentum equation then gives

$$\frac{\partial \overline{\mathbf{v}}}{\partial t} + \overline{\mathbf{v}} \cdot \nabla \overline{\mathbf{v}} = \frac{1}{\rho} \nabla \overline{p} + \frac{1}{\rho} \nabla \cdot \tau,$$

where

$$\tau = \mu(\nabla \mathbf{v} + \nabla \mathbf{v}^*) + \mathcal{S}, \quad \mathcal{S} = -\rho \overline{\mathbf{v}'\mathbf{v}'}.$$

The term $\mathcal{S} = \{S_{ij}\} = \{-\rho \overline{u_i' u_j'}\}$, which stems from the convective term, is re-interpreted as the Reynolds stress tensor.

(Continued on page 2.)

Problem 2 Gravity driven viscous flow (weight xx%)

2a (weight xx%)

We align the z-axis vertically downwards and employ cylindrical coordinates. Due to the symmetry and informaton given in the text we assume

$$\mathbf{v} = v_r(r)\mathbf{i}_r + v_z(r)\mathbf{k}.$$

From the formula sheets we then find the continuity equation

$$\frac{1}{r}\frac{\partial(rv_r)}{\partial r} = 0.$$

while the components of the Navier-Stokes equation become

$$\mathbf{i}_r: \quad v_r\frac{\partial v_r}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2}\right\},$$
$$\mathbf{k}: \quad v_r\frac{\partial v_z}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right).$$

At the cylinder we have the no-slip condition

$$v_z(a) = v_r(a) = 0.$$

On the free surface, $r = b$ we have the dynamic condition for the stress

$$\vec{p} = \mathbf{i}_r \cdot \mathcal{P} = -p_0\mathbf{i}_r,$$

which means zero shear stress.

2b (weight xx%)

The continuity equation gives

$$v_r = \frac{A}{r},$$

which corresponds to a line source at $r = 0$. The boundary condition $v_r(a) = 0$ the implies that v_r is zero throughout the fluid. To find an expression for the stress at the surface we start with

$$\nabla\mathbf{v} = \left(\mathbf{i}_r\frac{\partial}{\partial r} + \mathbf{k}\frac{\partial}{\partial z}\right)v_z(r)\mathbf{k} = \frac{\mathrm{d}v_z}{\mathrm{d}r}\mathbf{i}_r\mathbf{k}.$$

The stress tensor then becomes

$$\mathcal{P} = -pI + \mu(\nabla\mathbf{v} + \nabla\mathbf{v}^*) = -pI + \mu\frac{\mathrm{d}v_z}{\mathrm{d}r}(\mathbf{i}_r\mathbf{k} + \mathbf{k}\mathbf{i}_r),$$

and the condition at $r = b$

$$-p_0\mathbf{i}_r = \mathbf{i}_r \cdot \mathcal{P} = -p\mathbf{i}_r + \mu\frac{\mathrm{d}v_z}{\mathrm{d}r}\mathbf{k},$$

which imply $p = p_0$ and $\frac{\mathrm{d}v_z}{\mathrm{d}r} = 0$ at the surface.

(Continued on page 3.)

The $\mathbf{i}_r$ component of the momentum equation now simply states

$$\frac{\partial p}{\partial r} = 0,$$

which imply $p = p(z)$. Combined with $p = p_0$ at $r = b$ this gives $p = p_0$ throughout the fluid. Then the equation set for v_r becomes

$$0 = g + \frac{\nu}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}v_z}{\mathrm{d}r}\right),$$
$$v_z(a) = 0, \quad \tfrac{\mathrm{d}v_z(b)}{\mathrm{d}r} = 0.$$

Integration of the momentum equation

$$r\frac{\mathrm{d}v_z}{\mathrm{d}r} = -\frac{g}{2\nu}r^2 + C,$$
$$v_z = -\frac{g}{4\nu}r^2 + C\ln(r/a) + D,$$

where C and D are integration constants. The boundary conditions give

$$-\frac{g}{4\nu}a^2 + D = 0,$$
$$-\frac{g}{2\nu}b + \frac{C}{b} = 0.$$

The expresssion for v_z then becomes

$$v_z = \frac{g}{4\nu}\left(a^2 - r^2 + 2b^2\ln(r/a)\right).$$

2c (weight xx%)

Since the downward acceleration is zero the drag per height must equal the weight of the fluid per height. Also, since the kinetec energy is constant the rate of loss of potential energy per height must equal the dissipation per height.

Problem 3 Boundary layer (weight xx%)

3a (weight xx%)

We assume a constant density. Then conservation of mass implies conservation of volume. There is no transport of volume through either (ii) or (iii). For boundary (iv) we have a transport into the volume (per width)

$$Q_{(iv)} = HU.$$

At (ii) we have

$$Q_{(ii)} = -\int_0^Y u(x,y)dy,$$

where the minus sign appears because fluid is leaving the volume through (ii) when $u > 0$. Since $y = H$ is outside the boundary layer the flux $Q_{(ii)}$ may be rewritten

$$Q_{(ii)} \approx -\int_0^H u(x,y)dy - \int_H^Y u\,dy = -\int_0^H u\,dy - U\delta^*.$$

(Continued on page 4.)

Zero net influx then implies

$$0 = Q_{(ii)} + Q_{(ii)} = QH - \int_0^H u(x,y)\,dy - U\delta^* = \int_0^H (U-u)\,dy - U\delta^*.$$

Rearrangement then gives

$$\delta^* = \frac{1}{U}\int_0^H (U-u)\,dy \approx \int_0^\infty \left(1 - \frac{u}{U}\right)\,dy.$$

Comment: This calculation is asymptotically valid for large H, as indicated by the occasional $\approx$.

3b (weight xx%)

The boundary layer equations in this case read

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}.$$

At the plate we have a no-slip slip condition

$$u(x,0) = 0, \quad v(x,0) = 0, \quad \text{for} \quad x \geq 0,$$

while the boundary layer flow is matched to the outer flow through

$$\lim_{y\to\infty} u = U.$$

There is no matching condition on v. The above set is simplified in relation to the Navier-Stokes equations in the following manners

- The pressure is adapted from the outer solution and is treated as known and constant through the boundary layer. For the Blasius profile the pressure is constant in the outer solution, hence no pressure gradient in the equation.
- Since the pressure is not unknown the y component of the momentum equation may be disregarded.
- The $\nu\partial^2 u/\partial^2 x$ term is neglected, since variations along the boundary layer are small than the transverse ones.

3c (weight xx%)

To find an expression for v it is slightly simpler to introduce the stream function, ψ, than to employ the continuity equation directly

$$\frac{\partial \psi}{\partial y} = u \Rightarrow \psi = \int U f'(\eta)\,dy = \Delta U f(\eta) + \psi_0(x).$$

(Continued on page 5.)

We may choose $\psi = 0$ at the streamline $y = 0$ and an f such that $f(0) = 0$. Then $\psi = \Delta U f$ and

$$v = -\frac{\partial \psi}{\partial x} = \Delta' U \left(\eta f' - f \right).$$

substitution into the momentum equation implies

$$Uf' \left(-\frac{U\Delta' y}{\Delta^2} f'' \right) + \Delta' U \left(\eta f' - f \right) \frac{U}{\Delta} f'' = \frac{\nu U}{\Delta^2} f''',$$

which is re-arranged into

$$-\frac{U\Delta\Delta'}{\nu} f f'' = f'''.$$

Since the first factor on the left hand side dependens only on x (while the

rest depends only on η) it must be a constant. This implies $\Delta\Delta'$ =const. and $\Delta = B\sqrt{x}$. B is arbitrary and conveniently chosen such that $\Delta = \sqrt{2\nu x/U}$. The equation for f then becomes

$$f''' + f f'' = f'''.$$

Boundary conditions are

$$f(0) = 0, \quad f'(0) = 0, \quad f'(\infty) = 1.$$

The End

3d

From point (a) the boundary layer thickness is given by

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{U} \right) dy = \Delta \int_0^\infty \left(1 - f'(\eta) \right) d\eta = \Delta \lim_{\eta \to \infty} (\eta - f(\eta)).$$

From the graph and table in figure 2 $\eta = 7.475$ appears to be way outside the boundary layer and $\eta - f(\eta) = 1.217$ for this η value. Hence

$$\delta^* = 1.217\sqrt{\frac{2\nu x}{U}} = 1.721\sqrt{\frac{\nu x}{U}}.$$

The shear stress is

$$\tau_w = \mu \frac{\partial u(0)}{\partial y} = \frac{\mu U}{\Delta} f''(0) = 0.33\sqrt{\frac{\rho \mu U}{x}}.$$

The drag then is

$$\int_0^D \tau_w \, dx = 0.66\sqrt{\rho \mu U x}.$$

Results are not valid near the leading edge (δ^* is singular, for instance) and the solution becomes unstable for large $Re_x = Ux/\nu$.