(1) Løsningsforslag Eksamen MEK4300 14/6-1

Oppgave 1

a) Massekonservering og Newton's 2 lov
Det er i tillegg gjort antagelse om
inkompressibilitet og Newtonsk væske.

b) $u_i = \bar{u}_i + u_i'$
$p = \bar{p} + p'$

Kontinuitet:
siden midling
og derivering
er kommutative.

$$\overline{\frac{\partial u_i}{\partial x_i}} = \frac{\partial \bar{u}_i}{\partial x_i} = 0$$

Momentum likning midlet. Alle lineære termer er kommutative

$$\overline{\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + f_i}$$

$$\frac{\partial \bar{u}_i}{\partial t} + \overline{u_j \frac{\partial u_i}{\partial x_j}} = -\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} + \bar{f}_i$$

Videre:

$$\overline{u_j \frac{\partial u_i}{\partial x_j}} = \frac{\partial \overline{u_i u_j}}{\partial x_j} - \cancel{\overline{u_i \frac{\partial u_j}{\partial x_j}}}$$

Kon
inu

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$$\overline{\frac{\partial u_i u_j}{\partial x_j}} = \frac{\partial \overline{u_i u_j}}{\partial x_j}$$

$$= \frac{\partial}{\partial x_j} \overline{(\bar{u}_i + u_i')(\bar{u}_j + u_j')}$$

$$= \frac{\partial}{\partial x_j} \bar{u}_i \bar{u}_j + \frac{\partial}{\partial x_j} \overline{u_i' u_j'}$$

$$= \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial}{\partial x_j} \overline{u_i' u_j'}$$

Satt inn i RANS likning

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j} - \frac{\partial}{\partial x_j} \overline{u_i' u_j'} + \dots$$

c) Se lecture notes eller White

d) En likning for $\overline{u_i u_j}$ kan finnes ved å multiplisere momentumlikningen for komponent u_j med komponent u_i og så midle likningen og forenkle.

(3) Utledning kreves ikke!

d) ① $\overline{u_i \frac{\partial u_j}{\partial t} + u_i u_k \frac{\partial u_j}{\partial x_k} = -\frac{1}{\rho} u_i \frac{\partial p}{\partial x_j} + \nu u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k} + u_i f_j}$

Vi har også den transponerte:

② $\overline{u_j \frac{\partial u_i}{\partial t} + u_j u_k \frac{\partial u_i}{\partial x_k} = -\frac{1}{\rho} u_j \frac{\partial p}{\partial x_i} + \nu u_j \frac{\partial^2 u_i}{\partial x_k \partial x_k} + u_j f_i}$

Summerer ① og ② og får

$$\overline{\frac{\partial u_i u_j}{\partial t} + \frac{\partial}{\partial x_k} u_i u_j u_k} = \overline{-\frac{1}{\rho}\left(u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i}\right)}$$
$$\overline{+\nu\left(u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k} + u_j \frac{\partial^2 u_i}{\partial x_k \partial x_k}\right)}$$
$$+ \overline{u_i f_j} + \overline{u_j f_i}$$

Venstresiden:

$$\overline{\frac{\partial u_i u_j}{\partial t}} = \frac{\partial \overline{u_i u_j}}{\partial t} \qquad \text{Kommutativ}$$

$$\overline{\frac{\partial}{\partial x_k} u_i u_j u_k} = \frac{\partial \overline{u_i u_j u_k}}{\partial x_k} = \frac{\partial}{\partial x_k} \overline{u_i u_j (\bar{u}_k + u_k')}$$

$$= \frac{\partial (\overline{u_i u_j}\, \bar{u}_k)}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u_i u_j u_k'}$$

$$= \bar{u}_k \frac{\partial \overline{u_i u_j}}{\partial x_k} + \frac{\partial}{\partial x_k} \overline{u_i u_j u_k'}$$

+ videre forenklinger

④ 1) a)

Modell for $\overline{u_i' u_j'}$

Eddy viscosity:

$$\overline{u_i' u_j'} = -2 \nu_T S_{ij} + \frac{2}{3} k \delta_{ij}$$

der $S_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right)$

ν_T = Turbulent viskositet

$k = \frac{1}{2} \overline{u_i' u_i'}$

δ_{ij} = Kroneckers delta funksjon

To ukjente ν_T og k. Kan løses ved for eksempel en $k - \varepsilon$ modell el. l.

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Oppgave 2

a) $\mathbb{u} = (u(y,t), 0, 0)$

Rettlinjet $\Rightarrow$ konveksjon utgår

X-komponent av Navier-Stokes

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Ingen påsatt tryktgradient og ingen endring i x-retning ($u(y,t)$)

$$\frac{\partial u}{\partial t} - \nu \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0,t) = U \quad , \quad \text{for } t > 0$$

$$u(h,t) = 0$$

$$u(y,0) = 0$$

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b) $v(y,t) = u(y,t) - U\left(1 - \frac{y}{h}\right)$

$$\frac{\partial v}{\partial t} - \nu \frac{\partial^2 v}{\partial y^2} = 0$$

$$v(0,t) = v(h,t) = 0 \ , \ t > 0$$

$$v(y,0) = -U\left(1 - \frac{y}{h}\right)$$

Løses ved separasjon av variable

$$v(y,t) = V(y)T(t)$$

$$V\dot{T} - \nu V''T = 0$$

$$\frac{\dot{T}}{\nu T} = \frac{V''}{V} = -\lambda^2 \ (= \text{konstant})$$

der $\dot{T} = \frac{dT}{dt}$ og $V' = \frac{dV}{dy}$

Generelle løsninger:

$$\frac{dT}{dt} + \lambda^2 \nu T = 0$$

$$T(t) = e^{-\lambda^2 \nu t}$$

$$V(y) = A\sin(\lambda y) + B\cos(\lambda y)$$

der A og B er konstanter

1) b) Bestemmer konstanter mha grensebet.

$$V(0) = 0 \quad \Rightarrow \quad B = 0$$

$$V(h) = 0 \quad \Rightarrow \quad \lambda_n = \frac{n\pi}{h}, \quad n = 1, 2, 3 \ldots$$

Superposisjon av løsninger $V_n(y) = A_n \sin(\lambda_n$

$$u(y,t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n y)\, e^{-\lambda_n^2 \nu t}$$

Konstantene A_n finnes ved å benytte initialbetingelsen og ortogonalitet.
Multipliserer med $\sin(\lambda_m y)$ og integrerer fra 0 til h:

$$\int_0^h u(y,0)\sin(\lambda_m y)\,dy = \int_0^h \sum_{n=1}^{\infty} A_n \sin(\lambda_m y)\sin(\lambda_n y)\, e^{-\lambda_n^2 \nu \cdot 0}\,dy$$

$$\int_0^h -U\left(1 - \frac{y}{h}\right)\sin(\lambda_m y)\,dy = A_m \int_0^h \sin^2(\lambda_m y)\,dy$$

Bruker

$$\int_0^h \sin^2\left(\frac{m\pi y}{h}\right) dy = \frac{h}{2}, \quad m = 1, 2, 3, \ldots$$

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$$\int_0^h \sin\left(\frac{m\pi y}{h}\right) dy = -\frac{h}{m\pi}\cos\left(\frac{m\pi y}{h}\right)\Big|_0^h$$

$$= -\frac{h}{m\pi}\cos(m\pi) + \frac{h}{m\pi}$$

$$\int_0^h \frac{y}{h}\sin\left(\frac{m\pi y}{h}\right) dy = -\frac{y}{h}\frac{h}{m\pi}\cos\left(\frac{m\pi y}{h}\right)\Big|_0^h - \int_0^h \frac{1}{\cancel{h}}\frac{\cancel{h}}{m\pi}\cos\left(\frac{m\pi y}{h}\right)dy$$

$$= -\frac{h}{m\pi}\cos(m\pi) - \frac{1}{m\pi}\frac{h}{m\pi}\cancel{\sin\left(\frac{m\pi y}{h}\right)}\Big|_0^h$$

$$= \underline{-\frac{h}{m\pi}\cos(m\pi)}$$

Og får innsatt:

$$-U\left(-\cancel{\frac{h}{m\pi}\cos(m\pi)} + \frac{h}{m\pi} + \cancel{\frac{h}{m\pi}\cos(m\pi)}\right) = A_m\frac{h}{2}$$

$$\underline{A_m = \frac{-2U}{m\pi}}$$

Og dermed

$$\underline{\underline{u(y,t) = U\left(1-\frac{y}{h}\right) - \frac{2U}{\pi}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n y)}{n}e^{-\lambda_n^2 \nu t}}}$$

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Oppgave 3

a) δ^* = displacement thickness

θ = momentum thickness

δ^* følger fra konservering av masse

θ følger fra konservering av momentum i x-retning.

b) $u(y,\delta) = U \sin\left(\frac{\pi y}{2\delta}\right)$

Friksjonskoeffisient $C_f = \frac{\tau_w}{0{,}5 \rho U^2} = \frac{2\, d\theta}{dx}$

$$\tau_w = \mu \left.\frac{du}{dy}\right|_{y=0}$$

$$\frac{du}{dy} = U \cos\left(\frac{\pi y}{2\delta}\right) \frac{\pi}{2\delta}$$

$$\underline{\tau_w = \mu \frac{U \pi}{2\delta}}$$

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$$\theta = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy$$

$$= \int_0^\delta \sin\left(\frac{\pi y}{2\delta}\right)\left(1 - \sin\left(\frac{\pi y}{2\delta}\right)\right) dy$$

$$\eta = \frac{y}{\delta}, \quad d\eta = \frac{dy}{\delta}$$

$$\theta = \int_0^1 \sin\left(\frac{\pi \eta}{2}\right)\left(1 - \sin\left(\frac{\pi \eta}{2}\right)\right) d\eta \; \delta$$

$$\int_0^1 \sin\left(\frac{\pi \eta}{2}\right) d\eta = -\frac{\cos\left(\frac{\pi\eta}{2}\right)}{\pi/2}\Bigg|_0^1 = \underline{\frac{2}{\pi}}$$

$$\int_0^1 \sin^2\left(\frac{\pi\eta}{2}\right) d\eta = \cancel{-\frac{\sin\left(\frac{\pi\eta}{2}\right)\cos\left(\frac{\pi\eta}{2}\right)}{\pi/2}}\Bigg|_0^1 + \int_0^1 \cos^2\left(\frac{\pi\eta}{2}\right) d\eta$$

$$\int_0^1 \sin^2\left(\frac{\pi\eta}{2}\right) d\eta = \int_0^1 1 - \sin^2\left(\frac{\pi\eta}{2}\right) d\eta$$

$$\Rightarrow \int_0^1 \sin^2\left(\frac{\pi\eta}{2}\right) d\eta = \underline{\frac{1}{2}}$$

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$$\Theta = \delta\left(\frac{2}{\pi} - \frac{1}{2}\right)$$

$$\Theta = \delta\left(\frac{4-\pi}{2\pi}\right)$$

$$\frac{d\Theta}{dx} = \frac{4-\pi}{2\pi}\,\frac{d\delta}{dx}$$

Vi har nå 2 uttrykk for C_f. Setter inn

$$\frac{\mu U \pi}{2\delta \cdot \frac{1}{2}\rho U^2} = 2\left(\frac{4-\pi}{2\pi}\right)\cdot\frac{d\delta}{dx}$$

$$\frac{\mu \pi^2\, dx}{\rho(4-\pi)U} = \delta\, d\delta$$

Integrerer og bruker $\delta(0) = 0$

$$\frac{\mu \pi^2 x}{\rho(4-\pi)U} = \frac{1}{2}\delta^2$$

Setter inn for $Re_x = \frac{\rho U x}{\mu}$

$$\Rightarrow \frac{2\pi^2}{(4-\pi)Re_x} = \left(\frac{\delta}{x}\right)^2$$

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$$\frac{\delta}{x} \simeq \frac{4{,}80}{\sqrt{Re_x}} \quad \left(= \sqrt{\frac{2\pi^2}{4-\pi}} \; \frac{1}{\sqrt{Re_x}} \right.$$

$$\Theta = \frac{4{,}80 \cdot x}{\sqrt{Re_x}} \left(\frac{4-\pi}{2\pi} \right)$$

$$\Theta \approx \frac{0{,}655\, x}{\sqrt{Re_x}}$$

$$\delta^* = \int_0^1 \left(1 - \sin\left(\frac{\pi}{2}\eta\right)\right) d\eta \; \delta$$

$$= \delta \left(1 - \frac{2}{\pi} \right) = \delta \left(\frac{\pi - 2}{\pi} \right)$$

$$\simeq \frac{4{,}80\, x}{\sqrt{Re_x}} \left(\frac{\pi - 2}{\pi} \right)$$

$$\delta^* \approx \frac{1{,}74\, x}{\sqrt{Re_x}}$$

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Oppgave 3

c) Multipliserer hastighetslikning i x-retning med $\frac{L}{U^2}$ og tar grensen $Re \to \infty$
$(\mathbf{u} = (u, v, 0))$

$$\frac{L}{U^2}\frac{\partial u}{\partial t} = \frac{\partial \, u/U}{\partial \frac{tU}{L}} = \frac{\partial \bar{u}}{\partial \bar{t}}$$

$$\frac{L}{U^2} u \frac{\partial u}{\partial x} = \frac{u}{U}\frac{\partial \, u/U}{\partial \, x/L} = \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}}$$

$$\frac{L}{U^2} v \frac{\partial u}{\partial y} = \frac{v}{U}\sqrt{Re}\,\frac{\partial \, u/U}{\partial \frac{y}{L}\sqrt{Re}} = \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}}$$

$$\frac{L}{U^2}\nu\frac{\partial^2 u}{\partial x^2} = \frac{\nu}{UL}\frac{\partial^2 \, u/U}{\partial \frac{x}{L}\partial \frac{x}{L}} = \frac{1}{Re}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}$$

$$\frac{L}{U^2}\nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \, u/U}{\partial\left(\frac{y}{L}\sqrt{Re}\right)\partial\left(\frac{y}{L}\sqrt{Re}\right)} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2}$$

$$\frac{L}{U^2}\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial \frac{p}{\rho U^2}}{\partial \frac{x}{L}} = \frac{\partial \bar{p}}{\partial \bar{x}}$$

Ved $Re \to \infty$ forsvinner $\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}$ leddet.
Vi har da

$$\frac{\partial \bar{u}}{\partial \bar{t}} + \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v}\frac{\partial \bar{u}}{\partial \bar{y}} = \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \frac{\partial \bar{p}}{\partial \bar{x}}$$

(14) Kontinuitet

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\right) \cdot \frac{L}{U}$$

$$\frac{\partial\, u/U}{\partial \frac{x}{L}} + \frac{\partial \frac{v}{U}\sqrt{Re}}{\partial\left(\frac{y}{L}\sqrt{Re}\right)} = 0$$

$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0$$

Hastighet i veggnormal-retning:

$$\frac{L}{U^2}\frac{\partial v}{\partial t} = \frac{\partial \frac{v}{U}\sqrt{Re}}{\partial \frac{tU}{L}}\frac{1}{\sqrt{Re}} = \frac{1}{\sqrt{Re}}\frac{\partial \bar{v}}{\partial \bar{t}}$$

$$\frac{L}{U^2} u \frac{\partial v}{\partial x} = \frac{u}{U}\frac{\partial \frac{v}{U}\sqrt{Re}}{\partial \frac{x}{L}}\frac{1}{\sqrt{Re}} = \bar{u}\frac{\partial \bar{v}}{\partial \bar{x}}\frac{1}{\sqrt{Re}}$$

$$\frac{L}{U^2} v \frac{\partial v}{\partial y} = \frac{v}{U}\sqrt{Re}\,\frac{\partial \frac{v}{U}\sqrt{Re}}{\partial \frac{y}{L}\sqrt{Re}}\frac{1}{\sqrt{Re}} = \bar{v}\frac{\partial \bar{v}}{\partial \bar{y}}\frac{1}{\sqrt{Re}}$$

$$\frac{L}{U^2}\nu\frac{\partial^2 v}{\partial x^2} = \frac{1}{Re}\frac{\partial \frac{v}{U}\sqrt{Re}}{\partial\left(\frac{x}{L}\right)\partial\left(\frac{x}{L}\right)}\frac{1}{\sqrt{Re}} = \frac{1}{Re^{3/2}}\frac{\partial \bar{v}}{\partial \bar{x}^2}$$

$$\frac{L}{U^2}\nu\frac{\partial^2 v}{\partial y^2} = \frac{1}{Re}\frac{\partial^2 \frac{v}{U}\sqrt{Re}}{\partial\left(\frac{y}{L}\sqrt{Re}\right)\partial\left(\frac{y}{L}\sqrt{Re}\right)} = \frac{1}{\sqrt{Re}}\frac{\partial^2 \bar{v}}{\partial \bar{y}^2}$$

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$$\frac{L}{U^2}\frac{1}{\rho}\frac{\partial p}{\partial y} = \sqrt{Re}\,\frac{\partial \frac{p}{\rho U^2}}{\partial \frac{y}{L}\sqrt{Re}} = \sqrt{Re}\,\frac{\partial \bar{p}}{\partial \bar{y}}$$

Deler med $\sqrt{Re}$ og tar grensen $Re \to \infty$

Det eneste leddet igjen da er

$$\underline{\frac{\partial \bar{p}}{\partial \bar{y}} = 0}$$

d) $\psi = \sqrt{2\nu U x}\, f(\eta)$, $\eta = y\sqrt{\frac{U}{2\nu x}}$

$$u = \frac{\partial \psi}{\partial y} = \sqrt{2\nu U x}\,\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$u = \sqrt{\frac{\cancel{2}\cancel{\nu}U\cancel{x}U}{\cancel{2}\cancel{\nu}\cancel{x}}}\, f' = \underline{U f'}$$

$$\frac{\partial \eta}{\partial x} = y\sqrt{\frac{U}{2\nu}}\left(-\frac{1}{2}\right)x^{-3/2} = -\frac{y}{2}\sqrt{\frac{U}{2\nu x}}\,\frac{1}{x}$$

$$\underline{\frac{\partial \eta}{\partial x} = -\frac{\eta}{2x}}$$

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$$v = -\frac{\partial \psi}{\partial x} = -\left(\sqrt{2\nu U x}\,\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + f\sqrt{2\nu U}\,\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= -\left(-\sqrt{2\nu U x}\,\frac{\eta}{2x}f' + \frac{f}{2}\sqrt{\frac{2\nu U}{x}}\right)$$

$$= -\left(-\sqrt{\frac{2\nu U}{x}}\,\frac{\eta}{2}f' + \frac{f}{2}\sqrt{\frac{2\nu U}{x}}\right)$$

$$\underline{v = \sqrt{\frac{\nu U}{2x}}\left(\eta f' - f\right)}$$

Tar partiell deriverte og setter inn i grenseskiktlikningen.

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = U f''\left(-\frac{\eta}{2x}\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = U f''\sqrt{\frac{U}{2\nu x}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \eta}\left(U f''\sqrt{\frac{U}{2\nu x}}\right)\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = U f''' \frac{U}{2\nu x}$$

Setter inn

$$-U f'' \frac{\eta}{2x} U f' + \sqrt{\frac{\nu U}{2x}} (\eta f' - f) U f'' \sqrt{\frac{U}{2\nu x}}$$

$$= \nu U f''' \frac{U}{2\nu x}$$

Multipliser med $\frac{2x}{U^2}$

$$-f''\cancel{f'\eta} + (\cancel{\eta f'} - f) f'' = f'''$$

$$\underline{\underline{f''' + f f'' = 0}}$$

Grensebetingelser:

$$f(0) = f'(0) = 0$$

$$f'(\infty) = 1$$