MEK4300 Exam 10/6-2015

Suggested solution

Problem 1

a) Show that

$$\delta^* = \int_0^{y\to\infty} \left(1-\frac{u}{U}\right) dy$$

where δ^* is the <u>displacement thickness</u>!

Conservation of mass over CV

$$\int \nabla\cdot\rho\mathbf{u}\, dV = \oint \rho\mathbf{u}\cdot d\mathbf{A} = 0$$

Perform surface integral by summing over all faces (top and bottom zero)

$$\oint \rho\mathbf{u}\cdot d\mathbf{A} = \sum_{faces} \oint \rho\mathbf{u}\cdot d\mathbf{A} = \int_0^Y \rho u\, dy - \int_0^H \rho U\, dy$$

Constant density $\Rightarrow$ $\displaystyle\int_0^Y u\, dy - UH = 0$

Rearrange:

$$\int_0^Y U + u - U\, dy = UY + \int_0^Y u - U\, dy = UH$$

$$U\underbrace{(Y-H)}_{\delta^*} = \int_0^Y U - u\, dy$$

$$\delta^* = \int_0^Y 1 - \frac{u}{U}\, dy$$

b) Momentum equation in x-direction

$$\nabla \cdot \rho \mathbf{u} u = \nabla \cdot \tau_x \quad , \quad \tau_x = \mu \nabla u - p \delta_{xj}$$

Integrate over CV

$$\oint \rho u \, \mathbf{u} \cdot d\mathbf{A} = \oint \tau_x \cdot d\mathbf{A}$$

Convective term is zero for top and bottom faces. For bottom due to no-slip and top since this is a streamline.

Friction term is only nonzero for the bottom, where it represents drag

$$\oint \tau_x \cdot d\mathbf{A} = -\int_0^X \mu \left.\frac{\partial u}{\partial y}\right|_{y=0} dx = -\text{Drag}$$

$$\oint \rho u \, \mathbf{u} \cdot d\mathbf{A} = \int_0^Y \rho u^2 \, dy - \int_0^H \rho U^2 \, dy$$

$$= \int_0^Y \rho u^2 \, dy - \rho U^2 H$$

b) Now use conservation of mass:

$$UH = \int_0^Y u\,dy \Rightarrow H = \int_0^Y \frac{u}{U}\,dy$$

$$\Rightarrow \oint \rho u(\mathbf{u}\cdot d\mathbf{A}) = \int_0^Y \rho u^2\,dy - \rho U^2 \int_0^Y \frac{u}{U}\,dy$$

$$= \rho \int_0^Y u(u-U)\,dy$$

Finally set $\oint \rho u(\mathbf{u}\cdot d\mathbf{A}) = \oint \tau_x \cdot d\mathbf{A}$

$$-\text{Drag} = \rho \int_0^Y u(u-U)\,dy$$

and

$$\theta = \frac{\text{Drag}}{\rho U^2} = \int_0^Y \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

θ is the momentum thickness

1c) Derive RANS:

$$\overline{\rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) = \mu\nabla^2\mathbf{u} - \nabla p + \mathbf{f}}$$

Commuting:

$$\overline{\frac{\partial \mathbf{u}}{\partial t}} = \frac{\partial \bar{\mathbf{u}}}{\partial t}$$

$$\overline{\mu\nabla^2\mathbf{u}} = \mu\nabla^2\bar{\mathbf{u}}$$

$$\overline{\nabla p} = \nabla\bar{p}$$

Convection:

$$\overline{(\mathbf{u}\cdot\nabla)\mathbf{u}} = \overline{\nabla\cdot\mathbf{u}\otimes\mathbf{u}} = \nabla\cdot\overline{\mathbf{u}\otimes\mathbf{u}}$$

$$\overline{\mathbf{u}\otimes\mathbf{u}} = \overline{(\bar{\mathbf{u}}+\mathbf{u}')\otimes(\bar{\mathbf{u}}+\mathbf{u}')}$$

$$= \overline{\bar{\mathbf{u}}\otimes\bar{\mathbf{u}}} + \cancel{\overline{\mathbf{u}'\otimes\bar{\mathbf{u}}}} + \cancel{\overline{\bar{\mathbf{u}}\otimes\mathbf{u}'}} + \overline{\mathbf{u}'\otimes\mathbf{u}'}$$

$$\underline{= \bar{\mathbf{u}}\otimes\bar{\mathbf{u}} + \overline{\mathbf{u}'\otimes\mathbf{u}'}}$$

Continuity:

$$\underline{\overline{\nabla\cdot\mathbf{u}} = \nabla\cdot\bar{\mathbf{u}}} = 0 \text{ (commutes)}$$

1c) Convection simplified due to continuity:

$$\nabla \cdot \left(\bar{\mathbf{u}} \otimes \bar{\mathbf{u}} + \overline{\mathbf{u}' \otimes \mathbf{u}'} \right) = (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} + \nabla \cdot \overline{\mathbf{u}' \otimes \mathbf{u}'}$$

RANS:

$$\rho \left(\frac{\partial \bar{\mathbf{u}}}{\partial t} + (\bar{\mathbf{u}} \cdot \nabla)\bar{\mathbf{u}} \right) = \mu \nabla^2 \bar{\mathbf{u}} - \nabla \bar{p} - \rho \nabla \cdot \overline{\mathbf{u}' \otimes \mathbf{u}'}$$

$$\nabla \cdot \bar{\mathbf{u}} = 0$$

For a boundary layer there are only two components

$$\bar{\mathbf{u}} = (\bar{u}, \bar{v}, 0)$$

but $\bar{v} \ll \bar{u}$ and $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$

Continuity simplifies to

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

x-momentum

$$\rho \left(\cancel{\frac{\partial \bar{u}}{\partial t}} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) = \cancel{\mu \frac{\partial^2 \bar{u}}{\partial x^2}} + \mu \frac{\partial^2 \bar{u}}{\partial y^2} - \frac{\partial \bar{p}}{\partial x} \cancel{- \rho \frac{\partial}{\partial x} \overline{u'u'}} - \rho \frac{\partial}{\partial y} \overline{u'v'}$$

steady

small

1c) The pressure can also be neglected since it is given that $P = P_0 = \text{constant}$ outside the boundary layer

Final equations

$$\underline{\underline{\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0}}$$

$$\underline{\underline{\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = \mu\frac{\partial^2 \bar{u}}{\partial y^2} - \rho\frac{\partial \overline{u'v'}}{\partial y}}}$$

Y-momentum

$$\rho\left(\cancel{\frac{\partial \bar{v}}{\partial t}} + \cancel{\bar{u}\frac{\partial \bar{v}}{\partial x}} + \cancel{\bar{v}\frac{\partial \bar{v}}{\partial y}}\right) = \cancel{\mu\frac{\partial^2 \bar{v}}{\partial x^2}} + \cancel{\mu\frac{\partial^2 \bar{v}}{\partial y^2}} - \frac{\partial p}{\partial y}$$
$$- \cancel{\rho\frac{\partial}{\partial x}\overline{u'v'}} - \rho\frac{\partial}{\partial y}\overline{v'v'}$$

$$\frac{\partial p}{\partial y} = -\rho\frac{\partial}{\partial y}\overline{v'v'}$$

$$\Rightarrow \underline{\underline{P(y) = P_0 - \rho\overline{v'v'}}}$$

1c) Boundary conditions

$$\bar{u}(x,0) = \bar{v}(x,0) = 0 \quad , P(x,\delta) = P_0$$

Freestream $\quad \bar{u}(x,\delta) = U$

Only one condition on $\bar{v}$!

$\overline{v'v'}(x,0) = 0$, but further modeling required
$\overline{v'v'}(x,\delta) = 0$

1d) The Reynolds stress can be modelled using, e.g., an eddy-viscosity model

$$\overline{u'v'} = -\nu_T \frac{\partial \bar{u}}{\partial y}$$

where ν_T is an unknown turbulent viscosity. ν_T can be modelled using, e.g.,

$$\nu_T = u^* \cdot l$$

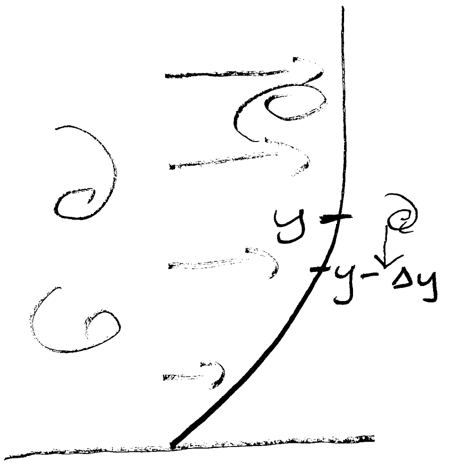
where u^* and l are velocity- and lengthscales, respectively. l is usually specified based on the problem, whereas a mixing-length model gives

$$u^* = \rho l \left| \frac{\partial \bar{u}}{\partial y} \right|$$

1d) A complete turbulence model is a model that computes both the velocity- and the lengthscale. For example k-epsilon model. No previous knowledge required about the system.

$\overline{u'v'} \leq 0$

Consider the boundary layer profile



Consider an eddy moving downwords, i.e., $v < 0$, $v' < 0$

This eddy comes from an area of high $\bar{u}$ and moves into a region of smaller $\bar{u}$. This means that on average the eddy will carry with it a positive contribution to the x-momentum $u' > 0$.

Hence $v' < 0$ and $u' > 0$ on average, and thus $\overline{u'v'} < 0$

Moves from y to $y - \Delta y$

$\bar{u}(y) > \bar{u}(y - \Delta y)$

Opposite for an eddy moving up with $v' > 0$. Here it will carry, on average, a negative contribution to x-momentum, $v' > 0$, $u' < 0$

$\Rightarrow \overline{u'v'} < 0$

Problem 2

a) Couette flows are driven by moving surfaces (walls).

Poiseuille flows are driven by applied pressure gradients.

Both are steady, fully developed, plane shear flows. Convection = 0.

b)
$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} + \beta$$

$u(1, t) = U$

$u(-1, t) = -U$

$u(y, 0) = Uy$ ← Linear Couette flow

c)
$$0 = \nu \frac{\partial^2 u_s}{\partial y^2} + \beta$$

Integrate to $u_s = -\frac{\beta}{2\nu} y^2 + ay + b$

$u_s(1) = -\frac{\beta}{2\nu} + a + b = U$

$u_s(-1) = -\frac{\beta}{2\nu} - a + b = -U$

$\Rightarrow \quad b = \frac{\beta}{2\nu}, \quad a = U$

$$\underline{\underline{u_s(y) = -\frac{\beta}{2\nu}(y^2 - 1) + Uy}}$$

2d) $u(y,t) = v(y,t) + u_s(y)$

Insert into governing equation

$$\frac{\partial v}{\partial t} + \cancel{\frac{\partial u_s}{\partial t}} = \nu \frac{\partial^2 v}{\partial y^2} + \underbrace{\nu \frac{\partial^2 u_s}{\partial y^2} + \beta}_{=0}$$

$$\Rightarrow \quad \frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial y^2}$$

Boundary and initial conditions:

$$v(1,t) = u(1,t) - u_s(1)$$
$$= U - U = \underline{0}$$
$$v(-1,t) = u(-1,t) - u_s(-1)$$
$$= -U - (-U) = \underline{0}$$
$$v(y,0) = u(y,0) - u_s(y)$$
$$= Uy - \left(-\frac{\beta}{2\nu}(y^2-1) + Uy\right)$$
$$= \underline{\frac{\beta}{2\nu}(y^2-1)}$$

2d) Solve for $v(y,t)$ using separation of variables

$$v(y,t) = T(t)V(y)$$

Insert into gov. eq:

$$\dot{T}V = \nu T V'' \text{, where } \dot{T} = \frac{\partial T}{\partial t}\text{, } V'' = \frac{\partial^2 V}{\partial y^2}$$

$$\frac{\dot{T}}{\nu T} = \frac{V''}{V} = \text{constant}$$

, since left hand side depends only on t, and rhs only on y.
constant must be negative to obtain physically realistic results
set constant $= -\lambda^2$

Solve for T:

$$\dot{T} + \nu\lambda^2 T = 0$$

$$\underline{T(t) = e^{-\nu\lambda^2 t}}$$

2d)

Solve for V:

$$V'' + \lambda^2 V = 0$$

General solution:

$$V(y) = A\cos(\lambda y) + B\sin(\lambda y)$$

Boundary conditions:

$$V(\pm 1) = 0$$

Choose $\lambda_k = \dfrac{(2k-1)\pi}{2}$, $k = 1, 2, 3, \dots$

$$\Rightarrow B = 0$$

$$V_k(y) = A_k \cos(\lambda_k y)$$

Superposition gives complete solution:

$$v(y,t) = \sum_{k=1}^{\infty} A_k \cos(\lambda_k y) e^{-\nu \lambda^2 t}$$

A_k is found from initial condition

$$v(y,0) = \sum_{k=1}^{\infty} A_k \cos(\lambda_k y)$$

2d) Multiply by $\cos(\lambda_m y)$ and integrate over domain

$$\int_{-1}^{1} \frac{3}{2\nu}(y^2-1)\cos(\lambda_m y)\,dy = \int_{-1}^{1} \cos(\lambda_m y)\sum_{k=1}^{\infty} A_k \cos(\lambda_k y)\,dy$$

Use hint on lhs : $\int_{-1}^{1} \cos(\lambda_m y)(1-y^2)\,dy = \frac{4(-1)^{m-1}}{\lambda_m^3}$

$$-\frac{3}{2\nu}\frac{4(-1)^{m-1}}{\lambda_m^3} = A_m \underbrace{\int_{-1}^{1} \cos^2(\lambda_m y)\,dy}_{=1}$$

$$\underline{A_m = \frac{2\cdot 3}{\nu}\frac{(-1)^m}{\lambda_m^3}}$$

Complete solution:

$$u(y,t) = v(y,t) + u_s(y)$$

$$\underline{\underline{u(y,t) = \sum_{m=1}^{\infty} \frac{2\cdot 3(-1)^m}{\nu\lambda_m^3}\cos(\lambda_m y)e^{-\nu\lambda_m^2 t} - \frac{3}{2\nu}(y^2-1) + U_y}}$$

$$\underline{\underline{\lambda_m = \frac{(2m-1)\pi}{2}}}$$