Lecture Notes
Mek 4320
Hydrodynamic Wave theory

by

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Chapter 1

INTRODUCTION

It is difficult to give a general, and at the same time, precise definition of what wave motion is, but the following loose formulation can be a useful starting point: A wave is a disturbance that can propagate from one part of a medium to another with a characteristic speed. This requires the disturbance to be such that its position can be determined at any given time. The disturbance can change form, size, and propagation speed. Accordingly, both periodic disturbances (wave trains) and isolated disturbances (pulses) that move through the medium can be designated as waves.

Wave motion occurs in fluids, gasses, elastic media, and vacuums, but the waves can have different physical characteristics. Examples include surface waves on water caused by the forces of gravity or surface tension, inertial waves in the ocean and atmosphere caused by Earth's rotation, seismic waves caused by elastic forces, and electromagnetic waves caused by electric and magnetic forces. Even though the causes of wave propagation can be different, the different types of waves share common characteristics and a lot of the mathematical methods which are used in examinations of the various wave phenomena are the same. One can therefore correctly speak of a generic wave theory for all wave forms.

The main purpose of this lecture series is to describe important forms of wave propagation in fluids. We shall not seek to give a complete description of all wave types. Instead, we shall make the selection so that we get to common traits in different types of waves, and we seek to select them such that we get to know the essential mathematical methods in wave theory. For a more complete treatment of the field, refer to the books by Lamb (1932), Stoker (1957), Tolstoy (1973), Whitham (1974), and Lighthill (1978).

1.1 Elementary models and concepts

The simplest mathematical model for wave propagation is given by the equation

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \tag{1.1}$$

where x is a spatial coordinate, t is time c is a constant. This equation has the solution

$$\phi = f(x - ct)$$

where f is an arbitrary function that describes a disturbance that moves along the x-axis with constant speed c.

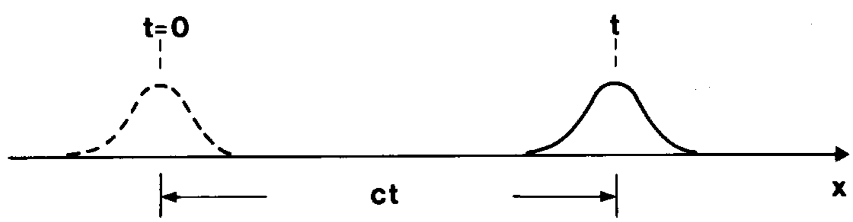

Figure 1.1: One-dimensional wave propagation.

The three-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0 \tag{1.2}$$

can describe propagation of sound waves, seismic waves, and electromagnetic waves in a medium with constant propagation speed c.

One solution of (1.2) is the spherically symmetric wave which can be written as

$$\phi = \frac{f(r - ct)}{r} + \frac{g(r + ct)}{r} \tag{1.3}$$

where r is the distance from a fixed reference point O in space. Here f and g are arbitrary functions describing disturbances that propagate out from or toward O, respectively. Notice that this wave attenuates with a factor $1/r$ because the energy is spread over continuously larger spherical shells (spherical attenuation) as r increases.

Another solution of (1.2) is the plane wave which can be written as

$$\phi = A \exp\{\mathrm{i}(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)\} \tag{1.4}$$

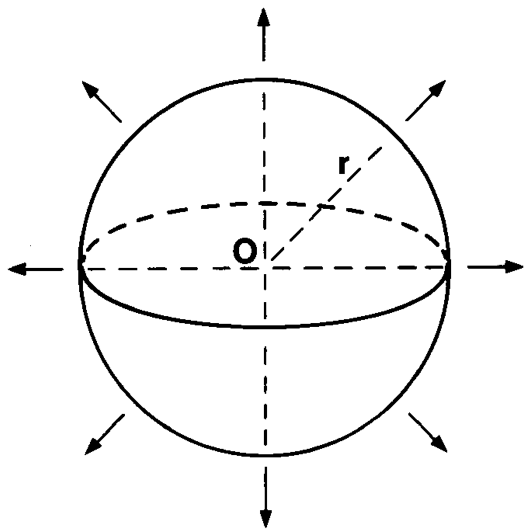

Figure 1.2: Spherically symmetric wave propagation.

where A can be called the complex wave amplitude. Sometimes we want the solution to be real, then we can write

$$\phi = \frac{A}{2}\exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t)\} + \text{c.c.} = |A|\cos(\boldsymbol{k}\cdot\boldsymbol{r} - \omega t + \arg A) \tag{1.5}$$

where A is complex and c.c. denotes "complex conjugate of the previous expression". Here $|A|$ can be called the real wave amplitude and $\arg A$ is a phase lag.

The other two constants ω and $\boldsymbol{k}$ above are related by the so-called dispersion relation, which for the wave equation (1.2) is

$$\omega^2 = (\boldsymbol{k}c)^2. \tag{1.6}$$

The vector $\boldsymbol{r}$ is a position vector relative to the reference point O, in a Cartesian coordinate system we have $\boldsymbol{r} = (x, y, z)$. The scalar

$$\chi = \boldsymbol{k}\cdot\boldsymbol{r} - \omega t \tag{1.7}$$

is called the *phase function*. The equation

$$\chi = \chi_0 = \text{constant}$$

describes how locations of constant phase move through space and time. If the phase function is held constant at a given time it defines a plane that can be called a *phase plane*. A trough or a crest corresponds to fixed values of the phase

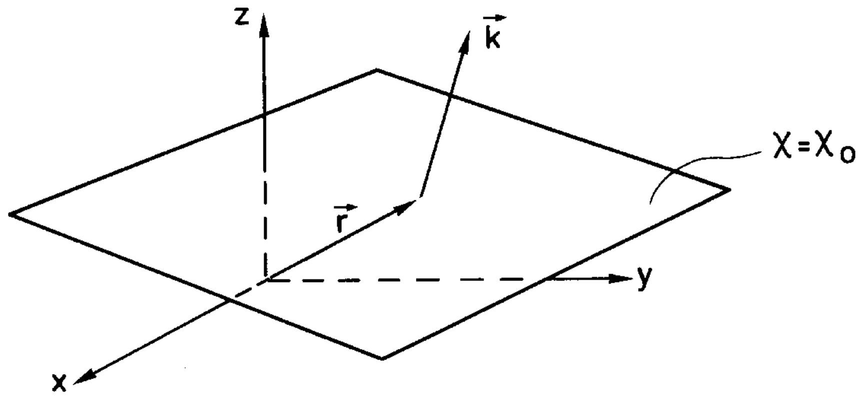

Figure 1.3: Phase plane.

function. The vector $\boldsymbol{k}$ is the *wavenumber vector*, or simply the *wave vector*, and the components along the axes x, y and z are, respectively, designated k_i $(i = 1, 2, 3)$ or k_x, k_y and k_z. It follows from (1.7) that

$$\boldsymbol{k} = \nabla\chi \tag{1.8}$$

which shows that the wave vector is orthogonal to the phase plane.

The *wavelength* λ is the distance between two phase planes for which the phase function changes by 2π. We have

$$\lambda = \frac{2\pi}{k} \tag{1.9}$$

where $k = |\boldsymbol{k}|$ is the *wavenumber*. The *angular frequency* is ω. The *period* T and *frequency* f are given by

$$\begin{aligned} T &= \frac{2\pi}{\omega} \\ f &= \frac{1}{T} \end{aligned} \tag{1.10}$$

The ratio between angular frequency and wavenumber

$$c = \frac{\omega}{k} \tag{1.11}$$

is known as the *phase speed*. For the wave equation (1.2) it is seen that the phase speed coincides with the speed c appearing in the equation (1.2). Since c is in this case independent of $\boldsymbol{k}$ and ω, we say that these waves are *non-dispersive* or *dispersionless*.

The phase speed is the speed that the phase plane moves in the normal direction, *i.e.* in the direction of the wave vector $\boldsymbol{k}$. However, in a direction different from $\boldsymbol{k}$ the phase plane moves faster than the phase speed (see relevant exercise below). Thus the phase speed does not satisfy the principles of vector projection, it is not a vector[1] and should not be called a velocity.[2]

For the linear equations (1.1) and (1.2) the principle of superposition is valid: A sum of the solutions of the form (1.4) or (1.5) is also a solution of the wave equation. By means of Fourier series or Fourier transform we can derive waves of arbitrary form.

The wave equation (1.2) appears quite frequently upon making simplifying assumptions on the nature of wave propagation and the medium, typical assumptions are:

1. Small amplitude such that the governing equations can be linearized.

2. Wave energy is conserved.

3. The medium is homogeneous and stationary.

In many applications these assumptions will not be fulfilled. Examples are nonlinear waves and waves in inhomogeneous media — topics of great current research interest. Through examples we shall show important properties of such wave phenomena.

In elementary expositions of wave theory the impression is often given that wave propagation is limited to hyperbolic equations of the type (1.1) or (1.2). This is not correct. One of the most conspicuous forms of waves, namely surface waves on water, are not of this type. Water waves belong to the class of *dispersive waves*, and for these waves, the phase speed depends on the wavelength. Dispersive waves can be plane waves like equation (1.4), but the dispersion relation is different from equation (1.6). In the general case ω is both dependent on the magnitude and the direction of the wave vector, and we can write the *dispersion relation* as

$$\omega = \omega(\boldsymbol{k}). \tag{1.12}$$

In many cases the dispersion relation is *isotropic*. This entails that ω is independent of the direction of the wave vector, and depends only on the wavenumber such that

$$\omega = \omega(k). \tag{1.13}$$

With isotropic dispersion, the phase speed is independent of the orientation of the phase plane.

With *anisotropic* dispersion the phase speed will be different depending on the orientation of the phase plane. Examples of this are elastic waves in sedimentary layers and ocean surface waves that propagate in areas of non-uniform currents.

[1] By *velocity* (Norwegian: *hastighet*) we mean a vector. By *speed* (Norwegian: *fart*) we mean a scalar. For a quantity to be a vector, it must satisfy the principles of vector projection.

[2] Despite these concerns, some authors *define* the phase velocity to be a vector with the same direction as $\boldsymbol{k}$, e.g. Whitham (1974) page 365.

Exercises

1. **The dispersion relation.**

   Which of these equations have solutions of the form $\eta = A\exp\{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)\}$? (In the last subproblem such an expression must be substituted for all of η, u and v.) Find and describe the dispersion relation, *i.e.* the relation between ω and $\boldsymbol{k}$. Which of these equations support waves?

   (a) $\dfrac{\partial^2\eta}{\partial t^2} = c_0^2\dfrac{\partial^2\eta}{\partial x^2}$

   (b) $\dfrac{\partial\eta}{\partial t} = c_0^2\dfrac{\partial^2\eta}{\partial x^2}$

   (c) $\dfrac{\partial\eta}{\partial t} = c_0\dfrac{\partial\eta}{\partial x} + \epsilon\dfrac{\partial^3\eta}{\partial x^3}$

   (d) $\dfrac{\partial^2\eta}{\partial t^2} + a\dfrac{\partial\eta}{\partial t} = c_0^2\dfrac{\partial^2\eta}{\partial x^2}$

   (e) $\dfrac{\partial^4\eta}{\partial t^4} - \dfrac{\partial^4\eta}{\partial t^2\partial x^2} + \epsilon\dfrac{\partial^4\eta}{\partial x^4} = 0$

   (f) $\dfrac{\partial^2\eta}{\partial t^2} = \nabla^2\eta + \dfrac{\epsilon}{3}\nabla^2\dfrac{\partial^2\eta}{\partial t^2}$

   (g) $\dfrac{\partial^2\eta}{\partial t^2} = \dfrac{\partial}{\partial x}(x\dfrac{\partial\eta}{\partial x})$

   (h) $\dfrac{\partial}{\partial x}(\dfrac{\partial\eta}{\partial t} + \dfrac{\partial\eta}{\partial x} + \dfrac{\mu^2}{6}\dfrac{\partial^3\eta}{\partial x^3}) + \dfrac{1}{2}\dfrac{\partial^2\eta}{\partial y^2} = 0$

   (i) $\dfrac{\partial u}{\partial t} = -\dfrac{\partial\eta}{\partial x},\ \dfrac{\partial v}{\partial t} = -\dfrac{\partial\eta}{\partial y},\ \dfrac{\partial\eta}{\partial t} = -\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}$

2. **Standard wave equations and initial conditions.**

   A function η is defined for all x and satisfies the equation given in exercise 1a. How many initial conditions are needed to ensure a unique solution? Give an example of such a set of initial conditions. Find the solution at an arbitrary later time. Repeat for 1c with $\epsilon = 0$. Discuss the case $\epsilon \neq 0$.

3. **Standard wave equation in spherical coordinates.**

   Express the Laplace operator ∇^2 in spherical coordinates $\{r, \theta, \varphi\}$, and show that equation (1.3) is a solution of equation (1.2).

   Hint: Avoid confusing the unknown ϕ and the azimuthal angle φ.

4. **Practical exercise for phase speed.**

   Next time you go to the coast, find a long pier that does not obstruct the water surface waves below. Suppose the waves are long-crested and have wave vector pointing in the x-direction, i.e., crests aligned in the y-direction. Suppose the pier is oriented at an angle θ with the x-axis. You want to run along the pier such that you follow one particular crest. Show that you need to run at a speed $c/\cos\theta$ where c is the phase speed of the waves. How fast would you have to run if the crest is parallel to the pier?

5. **Standard wave equation in cylindrical coordinates.**

   Consider the wave equation (1.2) in cylindrical coordinates $\{r, \theta, z\}$ and assume radial symmetry such that ϕ is only a function of r and t. Set $\phi(r,t) = u(r)\mathrm{e}^{-\mathrm{i}\omega t}$ and show that u satisfies the Bessel equation

$$\xi^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + \xi \frac{\mathrm{d}u}{\mathrm{d}\xi} + (\xi^2 - a^2)u = 0$$

   where $\xi = \omega r/c$ and where $a = 0$. The number a is called the order of the Bessel equation.

   Two independent solutions are the Bessel functions of the first and second kind, $\mathrm{J}_a(\xi)$ and $\mathrm{Y}_a(\xi)$, or alternatively the two Hankel functions $\mathrm{H}_a^{(1)}(\xi) = \mathrm{J}_a(\xi) + \mathrm{i}\mathrm{Y}_a(\xi)$ and $\mathrm{H}_a^{(2)}(\xi) = \mathrm{J}_a(\xi) - \mathrm{i}\mathrm{Y}_a(\xi)$. For large real and positive ξ we have the approximations

$$\mathrm{J}_0(\xi) \approx \sqrt{\frac{2}{\pi\xi}} \cos(\xi - \frac{\pi}{4})$$

$$\mathrm{Y}_0(\xi) \approx \sqrt{\frac{2}{\pi\xi}} \sin(\xi - \frac{\pi}{4})$$

$$\mathrm{H}_0^{(1)}(\xi) \approx \sqrt{\frac{2}{\pi\xi}} \mathrm{e}^{\mathrm{i}(\xi - \frac{\pi}{4})}$$

$$\mathrm{H}_0^{(2)}(\xi) \approx \sqrt{\frac{2}{\pi\xi}} \mathrm{e}^{-\mathrm{i}(\xi - \frac{\pi}{4})}$$

   Show that $\mathrm{J}_0(\xi)$ and $\mathrm{Y}_0(\xi)$ correspond to standing waves, $\mathrm{H}_0^{(1)}(\xi)$ corresponds to an outgoing wave, and $\mathrm{H}_0^{(2)}(\xi)$ corresponds to an incoming wave.

6. **Standard wave equation in spherical coordinates.**

   Consider the wave equation (1.2) in spherical coordinates $\{r, \theta, \varphi\}$ and assume radial symmetry such that ϕ is only a function of r and t. Set $\phi(r,t) = u(r)\mathrm{e}^{-\mathrm{i}\omega t}$ and show that u satisfies the equation

$$\xi^2 \frac{\mathrm{d}^2 u}{\mathrm{d}\xi^2} + 2\xi \frac{\mathrm{d}u}{\mathrm{d}\xi} + (\xi^2 - n(n+1))u = 0$$

   where $\xi = \omega r/c$ and where $n = 0$.

Two independent solutions are the spherical Bessel functions of the first and second kind, $\mathrm{j}_n(\xi)$ and $\mathrm{y}_n(\xi)$, or alternatively the two spherical Hankel functions $\mathrm{h}_n^{(1)}(\xi) = \mathrm{j}_n(\xi) + \mathrm{i}\mathrm{y}_n(\xi)$ and $\mathrm{h}_n^{(2)}(\xi) = \mathrm{j}_n(\xi) - \mathrm{i}\mathrm{y}_n(\xi)$. In particular we have

$$\mathrm{j}_0(\xi) = \frac{\sin(\xi)}{\xi}$$

$$\mathrm{y}_0(\xi) = -\frac{\cos(\xi)}{\xi}$$

Show that $\mathrm{j}_0(\xi)$ and $\mathrm{y}_0(\xi)$ correspond to standing waves, $\mathrm{h}_0^{(1)}(\xi)$ corresponds to an outgoing wave, and $\mathrm{h}_0^{(2)}(\xi)$ corresponds to an incoming wave.

Chapter 2

SURFACE WAVES IN FLUIDS

Surface waves caused by gravity or surface tension are well recognized day to day phenomena. This form of wave motion has been studied in numerous scientific works, and is known to share a multitude of properties with other forms of waves. It is therefore justified to let surface waves take a central place in the study of wave theory.

Depending on whether surface tension or gravity dominates the wave motion, we use the terms *capillary waves* or *gravity waves*. In some cases, wave motion will by influenced just as much by surface tension as by gravity, we then use the terms *capillary–gravity waves* or *gravity–capillary waves*. Both capillary and gravity waves are influenced by viscosity in the fluid in such a way that the shortest waves are dampened the most by the effect of friction. For wave motion in deep water, the viscosity leads to substantial dampening only for waves with a wavelength of less then about 1 cm. Compressibility in the fluid will generally be insignificant for surface waves.

Surface waves are closely related to waves on the interface between fluids of different densities. Some examples of this will be given in the section on internal waves.

2.1 Boundary conditions at the surface of the fluid

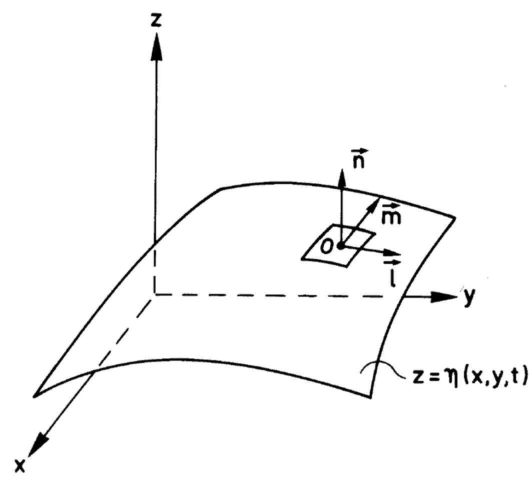

Figure 2.1: Orthogonal unit vectors on the surface

In the following, we often find use for the mathematical expression of boundary conditions at the surface of the fluid, and initially, we will derive these. Here we shall handle the case where a fluid borders an overlaying gas with much smaller density than the fluid such that one can neglect the motion in the overlaying gas. Because of that, we can expect the pressure over the fluid to be a constant isotropic pressure, which we designate as p_a.

We let $z = \eta(x, y, t)$ define the surface. The z-axis is in the vertical direction pointing up and the x- and y-axes are horizontal. We view the fluid as filling the space under the surface. When the fluid is stationary, and in equilibrium, the surface lies on the xy-plane. The unit normal vector $\boldsymbol{n}$ defines the surface normal at point O. This is chosen such that the unit vector points up from the surface. The unit vectors $\boldsymbol{l}$ and $\boldsymbol{m}$ lay in the tangent plane at point O, and the vectors $\boldsymbol{l}$, $\boldsymbol{m}$ and $\boldsymbol{n}$ constitute a set of orthogonal unit vectors. Such a set is given for example by

$$\begin{aligned} \boldsymbol{l} &= (0, 1, \eta_y)/\sqrt{1+\eta_y^2} \\ \boldsymbol{m} &= (-\eta_y^2 - 1, \eta_x\eta_y, -\eta_x)/\sqrt{(1+\eta_x^2+\eta_y^2)(1+\eta_y^2)} \\ \boldsymbol{n} &= (-\eta_x, -\eta_y, 1)/\sqrt{1+\eta_x^2+\eta_y^2} \end{aligned} \tag{2.1}$$

Here $\eta_x = \partial\eta/\partial x$ and $\eta_y = \partial\eta/\partial y$.

Below a surface element with area dA which lies in the surface with point O at its center (see fig. 2.1), forces act due to pressure and viscous stress in the fluid. With the assumptions we have made here there is also a pressure force acting on top of the surface element. The sum of these forces can be written as

$$[(p_s - p_a)\boldsymbol{n} - \boldsymbol{n} \cdot \boldsymbol{\mathcal{P}}]\mathrm{d}A \tag{2.2}$$

where $\boldsymbol{\mathcal{P}}$ is the viscous stress tensor and p_s is the pressure exerted by the fluid on the underside of the surface.

Along the edge of the surface element, additional forces act due to surface tension σ (force per unit of length), oriented along the tangent to the surface at the edge of the surface element. Let the point O be contained in a simply connected area Ω on the surface, limited by the edge Γ. The capillary force which acts on a linear line segment $\mathrm{d}\boldsymbol{s}$ can be written as $-\sigma\boldsymbol{n} \times \mathrm{d}\boldsymbol{s}$, where $\boldsymbol{n}$ is the surface normal for the line segment $\mathrm{d}\boldsymbol{s}$. The total capillary force acting on the edge Γ of the surface element is

$$\boldsymbol{F} = -\oint_\Gamma \sigma\boldsymbol{n} \times \mathrm{d}\boldsymbol{s}.$$

In the case that σ is constant on the surface, we can with the help of Stokes theorem, and with a view that the area of Ω, $\mathrm{d}A$, shrinks towards the point O, show that the capillary force on the surface element is in the direction of the normal vector $\boldsymbol{n}$ at O, and is given by

$$\mathrm{d}\boldsymbol{F} = \boldsymbol{n}\,\mathrm{d}F_n = -\sigma\boldsymbol{n}\,\nabla \cdot \boldsymbol{n}\,\mathrm{d}A.$$

The magnitude of the resultant capillary force can alternatively be expressed by the principal radii of curvature for the surface. In figure 2.2, the surface element is chosen such that the edges $\mathrm{d}s_1$ and $\mathrm{d}s_2$ fall along the principal directions of curvature for the surface. In the figure, we designate S_1 and S_2 the centers of curvature and R_1 and R_2 the principal radii of curvature for the surface. The angles $\mathrm{d}\beta_1$ and $\mathrm{d}\beta_2$ are given by the relations

$$\mathrm{d}\beta_1 = \frac{\mathrm{d}s_1}{R_1} \quad \text{and} \quad \mathrm{d}\beta_2 = \frac{\mathrm{d}s_2}{R_2}.$$

We find therefore that

$$F_n = -2\sigma\mathrm{d}s_2\frac{\mathrm{d}\beta_1}{2} - 2\sigma\mathrm{d}s_1\frac{\mathrm{d}\beta_2}{2} = -\sigma(\frac{1}{R_1} + \frac{1}{R_2})\mathrm{d}A \tag{2.3}$$

where R_1 and R_2 are chosen positive when the corresponding curvature centers are positioned inside the fluid. The quantity $\frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_2})$ is commonly called the mean curvature.

To avoid that the particles at the surface obtain infinitely or unrealistically large accelerations, the sum of forces due to pressure, gravity, capillarity and

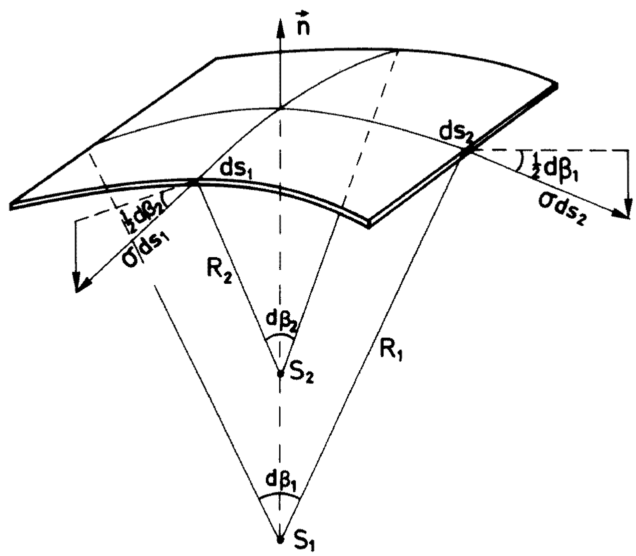

Figure 2.2: Principal radii of curvature for the surface

viscosity must be zero. This leads to the *dynamic boundary conditions* at $z = \eta(x, y, t)$

$$\begin{aligned} p_s - \boldsymbol{n} \cdot \boldsymbol{\mathcal{P}} \cdot \boldsymbol{n} - \sigma \nabla \cdot \boldsymbol{n} &= p_a \\ \boldsymbol{n} \cdot \boldsymbol{\mathcal{P}} \cdot \boldsymbol{l} &= 0 \\ \boldsymbol{n} \cdot \boldsymbol{\mathcal{P}} \cdot \boldsymbol{m} &= 0 \end{aligned} \tag{2.4}$$

In addition to the requirements (2.4) we impose the kinematic boundary condition at the surface. Normally, mass transport through the surface due to evaporation or diffusive processes is insignificant, and one can with good approximation assume that the particles at the surface should stay at the surface. Suppose a particle is at position $\boldsymbol{r} = (x, y, z)$ at time t, we have $z = \eta(x, y, t)$. Given the fluid velocity $\boldsymbol{v} = (u, v, w)$, after a small time interval dt the fluid particle is at the updated surface position

$$z + w\,\mathrm{d}t = \eta(x + u\,\mathrm{d}t, y + v\,\mathrm{d}t, t + \mathrm{d}t)$$

With Taylor expansion in powers of dt, we find the *kinematic boundary condition* at the surface.

$$w = \frac{\partial \eta}{\partial t} + \boldsymbol{v} \cdot \nabla \eta \tag{2.5}$$

The equations (2.4) and (2.5) are nonlinear in the variables u, v, w and η. In the following we will show how the boundary conditions simplify by linearization.

Given that one can assume the fluid motion is irrotational, the flow velocity can be deduced from a potential ϕ and

$$\boldsymbol{v} = \nabla\phi \tag{2.6}$$

If the fluid in addition is frictionless and homogeneous, the pressure in the fluid is given by Euler's pressure equation

$$p = -\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\boldsymbol{v}^2 + \Phi(z)\right) + f(t)$$

where ρ is the density of the fluid and Φ is the specific potential energy due to gravity. In this expression there is some ambiguity as $f(t)$ and the zero level for Φ may be chosen freely. Different choices then lead to velocity potentials, ϕ, which differ by a function of time only, and hence all correspond to the same velocity field $\boldsymbol{v}$ through (2.6). For the following derivations it is convenient to choose $f(t) = p_a$ and the zero level for Φ at $z = 0$. Then the gravity potential becomes $\Phi = gz$, where g is the acceleration of gravity, and the pressure under the surface can be written

$$p_s = -\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\boldsymbol{v}^2\right)_{z=\eta} - \rho g\eta + p_a \tag{2.7}$$

Combined with the normal component of (2.4) this relation yields the dynamic surface condition

$$-\rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\boldsymbol{v}^2\right)_{z=\eta} - \rho g\eta - \sigma\nabla\cdot\boldsymbol{n} = 0$$

For small wave heights this relation may be simplified through linearization. By Taylor expansion of the expression in parenthesis in powers of η, we get

$$p = -\rho\left[\left(\frac{\partial\phi}{\partial t}\right)_{z=0} + g\eta\right] - \rho\left[\frac{\partial^2\phi}{\partial t\partial z}\eta + \frac{1}{2}\boldsymbol{v}^2\right]_{z=0} + \cdots + p_a \tag{2.8}$$

where $\cdots$ designates higher order terms.

Given that the wave amplitude is small enough, quadratic or higher order terms of the flow variables $(\eta, \phi, \boldsymbol{v})$ may be dropped in relation to the terms which are linear in the flow variables. After we have found the wave motion that satisfies the linearized equations, we can set these linear solutions inside the nonlinear terms. From this we can find an estimate for how small the amplitude must be for the higher order terms to be dropped. We shall come back to this later.

The linearized expression for pressure in the fluid at $z = \eta$ can be written

$$p_s = -\rho\left[\left(\frac{\partial\phi}{\partial t}\right)_{z=0} + g\eta\right] + p_a \tag{2.9}$$

One can proceed similarly to linearize the terms in the boundary conditions (2.4). The expression where surface tension is included can be written

$$-\sigma\nabla\cdot\boldsymbol{n} = \sigma\frac{\eta_{xx}(1+\eta_y^2)+\eta_{yy}(1+\eta_x^2)-2\eta_x\eta_y\eta_{xy}}{(1+\eta_x^2+\eta_y^2)^{3/2}}$$

such that one by linearization can find

$$-\sigma\nabla\cdot\boldsymbol{n} = \sigma\left(\frac{\partial^2\eta}{\partial x^2}+\frac{\partial^2\eta}{\partial y^2}\right), \qquad (2.10)$$

which, combined with linearized expressions for the terms in p_s, provides the linear dynamic surface condition in (2.13).

Exercises

1. Consult any book on continuum mechanics, e.g. the compendium in MEK2200, to review the full expression for the viscous stress tensor $\boldsymbol{\mathcal{P}}$.

2. Find the linear expression for the viscous stress components in (2.4). Assume Newtonian fluids.

3. Find, by expansion, the linear and quadratic terms in the kinematic boundary condition (2.5).

2.2 Dispersion relation for linear capillary and gravity waves

We shall consider two-dimensional surface waves in a frictionless, homogeneous, and incompressible fluid restricted by a flat and horizontal bottom. We place the

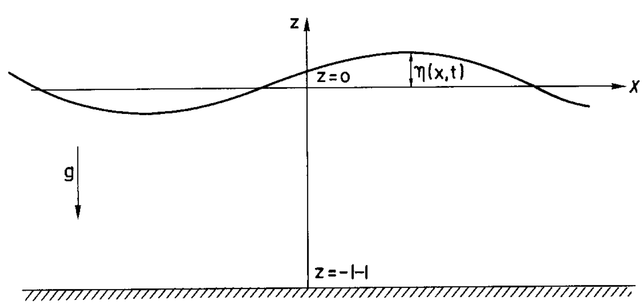

Figure 2.3: Two-dimensional wave

axes as shown in figure 2.3 where the x-axis forms the surface when the fluid is at rest and equilibrium. The depth of the fluid layer is then H.

As the wave motion is supposed to be irrotational and the fluid is incompressible the velocity potential satisfies the Laplace equation

$$\nabla^2 \phi = 0. \tag{2.11}$$

For linear waves the surface boundary conditions can, in accordance with (2.4), (2.9), (2.10) and results from the exercise under point 3 above, be written

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial z} \tag{2.12}$$

$$\frac{\partial \phi}{\partial t} + g\eta - \frac{\sigma}{\rho}\frac{\partial^2 \eta}{\partial x^2} = 0 \tag{2.13}$$

for $z = 0$. The boundary condition on the bottom, $z = -H$, is

$$\frac{\partial \phi}{\partial z} = 0. \tag{2.14}$$

Equations (2.11) through (2.14) may for instance be combined with initial conditions to describe temporal evolution of waves. We will return to this problem subsequently, but will start with the analysis of a simple wave component (monochromatic solution) of the form

$$\eta = a\sin(kx - \omega t), \quad \phi = A(z)\cos(kx - \omega t),$$

where a and k may be chosen freely, while A and ω must be determined accordingly. Such a simple solution is possible since the equation set is linear with constant coefficients. Using a sine function for η, equations (2.12) and (2.13) require a cosine for ϕ. Substitution of the expression for ϕ into the Laplace equation, (2.11), yields an ordinary differential equation for A, with a solution containing two constants of integration. These and ω are then determined by the three conditions (2.12), (2.13) and (2.14). The final result for η and ϕ can be written

$$\eta = a\,\sin(kx - \omega t) \tag{2.15}$$

$$\phi = -\frac{a\omega}{k\,\sinh kH}\cosh k(z+H)\cos(kx - \omega t) \tag{2.16}$$

where a is the amplitude of the wave. The wavenumber and angular frequency are connected by the dispersion relation

$$\omega^2 = (gk + \frac{\sigma k^3}{\rho})\tanh kH. \tag{2.17}$$

The phase speed for capillary–gravity waves is therefore given by

$$c = c_0\left(1 + \frac{\sigma k^2}{\rho g}\right)^{\frac{1}{2}}\left(\frac{\tanh kH}{kH}\right)^{\frac{1}{2}} \tag{2.18}$$

where $c_0 = \sqrt{gH}$. Equation (2.18) is little affected by surface tension given $\frac{\sigma k^2}{\rho g} \ll 1$. This entails that the surface tension can be neglected for waves with wavelength much larger then λ_m where

$$\lambda_m = 2\pi(\frac{\sigma}{\rho g})^{\frac{1}{2}}. \tag{2.19}$$

On the other hand, surface tension dominates wave motion when the wave length is much smaller then λ_m. For clean water at 20°C we have $\sigma = 7.4 \cdot 10^{-2}$ N/m, $\rho = 10^3$ kg/m^3, and with $g = 9.81$ m/s^2 we have $\lambda_m = 1.73$ cm.

In the case that the wavelength is much less than H and thereby $kH \gg 1$ we can with good approximation set $\tanh kH = 1$ and we find from (2.18) that

$$c = \left(\frac{g}{k} + \frac{\sigma k}{\rho}\right)^{\frac{1}{2}}. \tag{2.20}$$

In this case the phase speed has a minimum value

$$c_m = 2^{\frac{1}{2}}(\frac{\sigma g}{\rho})^{\frac{1}{4}}$$

for wavelength λ_m. For clean water at 20°C is $c_m = 23$ cm/s. Upon introducing the magnitudes λ_m and c_m we can write (2.20) as

$$\frac{c}{c_m} = \left[\frac{1}{2}(\frac{\lambda}{\lambda_m} + \frac{\lambda_m}{\lambda})\right]^{\frac{1}{2}}.$$

We notice that when the wavelength is larger than λ_m, the phase velocity is larger for long waves than it is for short, but the opposite is true for wavelength less than λ_m. Figure 2.4 shows c as a function of wavelength . Given that the wavelength is much larger than H such that $kH \ll 1$, we can set

$$\tanh kH = kH - \frac{(kH)^3}{3}$$

and we find from (2.18) that

$$c = c_0\left[1 + \left(\frac{\sigma}{\rho g H^2} - \frac{1}{3}\right)\frac{k^2H^2}{2}\right]. \tag{2.21}$$

If $\sigma = \frac{1}{3}\rho g H^2$, which for clean water happens for $H = 0.48$ cm, then (2.21) shows that the waves are non-dispersive, in analogy with acoustic waves. In this case water waves can be used in experimental studies as an analogy of non-dispersive sound waves.

With the solutions of the linear equations, which we have found, we are capable of giving complete conditions for the linearization to be valid. Looking back at

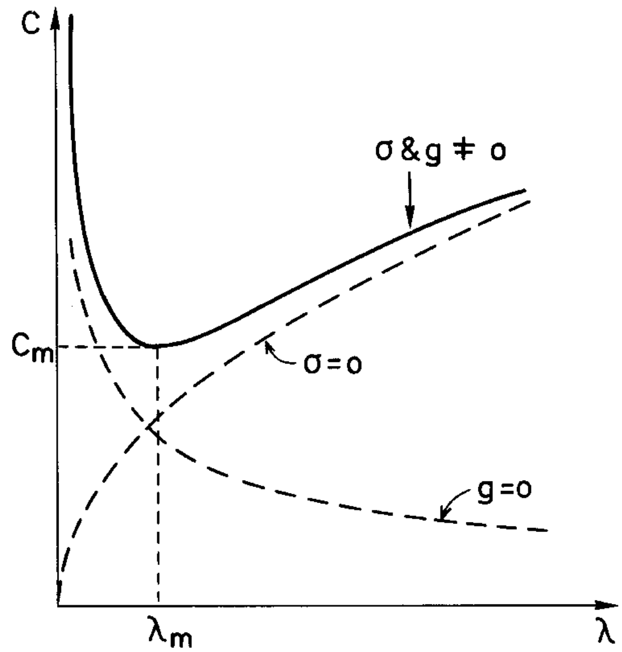

Figure 2.4: *Phase speed c as a function of the wavelength λ for waves in deep water.*

the expression (2.8), we see that $(\frac{\partial\phi}{\partial t})_{z=0}$ is a typical linear term and $u^2 = (\frac{\partial\phi}{\partial x})^2_{z=0}$ is a typical second order nonlinear term. If we use the solution (2.15)–(2.16), we find that the relation between linear and nonlinear terms is at most

$$\frac{ak}{\tanh kH}.$$

A necessary condition for justifying removal of nonlinear terms and keeping only linear terms is that this ratio be much less than 1. This implies that linearization is valid given that $a/\lambda \ll 1$ when $kH \gg 1$ and $a/H \ll 1$ when $kH \ll 1$. In both cases, linearization would also be valid when the steepness of the wave is adequately small.

Exercises

1. A plane wave has wavenumber vector $\boldsymbol{k} = (k_x, k_y)$ and angular frequency ω. Give a geometrical/physical interpretation of ω/k_x and ω/k_y. Find the wavenumber, wavelength, and phase speed for a gravity wave in deep water with period 10 s. Find the wavenumber vector when the wave propagates in a direction which is at a 30° angle with the x-axis.

2. Find the pressure in the fluid for gravity–capillary waves, and determine the pressure at the bottom. How does the pressure vary with z when $kH \ll 1$?

Which wave periods can be registered by a pressure sensor that lies on the bottom at 100 m depth with a sensitivity limited to 1/10000 part of the hydrostatic pressure? We let the wave steepness $2a/\lambda$ be 0.1.

3. Determine the kinetic energy density $\frac{\varrho}{2}\boldsymbol{v}^2$ for gravity waves in deep water. Which conditions must we set for the wave amplitude a such that the maximum value of the nonlinear term $\frac{\varrho}{2}\boldsymbol{v}^2$ is less then 10% of a typical linear term that occurs in Euler's pressure equation? Express this as a condition on the wave amplitude when it is required that the wavelength λ lies between 1 m and 300 m.

2.3 Particle motion in surface waves

Consider a fluid particle that is at position $\boldsymbol{r}_0$ at time t_0. We want to follow its motion. The particle has by a later time t a new position $\boldsymbol{r}(t)$. The particle velocity follows the velocity field of the fluid $\boldsymbol{v}$, and so $\boldsymbol{r}(t)$ is determined by the following integral equation:

$$\boldsymbol{r}(t) - \boldsymbol{r}_0 = \int_{t_0}^{t} \boldsymbol{v}(\boldsymbol{r}(\tau), \tau)\,\mathrm{d}\tau \tag{2.22}$$

We introduce the particles' displacement $\boldsymbol{R}(t)$ such that

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{R}(t)$$

and set this expression into the integrand in (2.22). Series expansion in terms of powers of $\boldsymbol{R}(t)$ gives

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \int_{t_0}^{t} \boldsymbol{v}(\boldsymbol{r}_0, \tau)\,\mathrm{d}\tau + \int_{t_0}^{t} (\boldsymbol{R}(\tau)\cdot\nabla)\boldsymbol{v}(\boldsymbol{r}_0, \tau)\,\mathrm{d}\tau + \cdots$$

Given that the displacement is small and the velocity gradient is small, higher order terms can be neglected. By linearization we find therefore that

$$\boldsymbol{R}(t) = \boldsymbol{r}(t) - \boldsymbol{r}_0 = \int_{t_0}^{t} \boldsymbol{v}(\boldsymbol{r}_0, \tau)\,\mathrm{d}\tau \tag{2.23}$$

We shall use this equation to find the particle motion for two-dimensional capillary and gravity waves. The velocity components in wave motion are according to (2.6) and (2.16)

$$\begin{aligned} u &= \frac{\partial\phi}{\partial x} = \frac{a\omega}{\sinh kH}\cosh k(z+H)\,\sin(kx-\omega t) \\ w &= \frac{\partial\phi}{\partial z} = -\frac{a\omega}{\sinh kH}\sinh k(z+H)\,\cos(kx-\omega t) \end{aligned}$$

We set $\boldsymbol{r}_0 = (x_0, z_0)$ and $\boldsymbol{r} = (x, z)$, then it follows from (2.23) that

$$\begin{aligned} x - x_0 &= \frac{a}{\sinh kH} \cosh k(z_0 + H) \cos(kx_0 - \omega t) + K_1 \\ z - z_0 &= \frac{a}{\sinh kH} \sinh k(z_0 + H) \sin(kx_0 - \omega t) + K_2 \end{aligned} \tag{2.24}$$

where K_1 and K_2 are integration constants.

Both here and later we will need the time-average of a function $F(t)$, and we define this by

$$\overline{F}(t_0) = \frac{1}{T} \int_{t_0 - T/2}^{t_0 + T/2} F(t)\,\mathrm{d}t, \tag{2.25}$$

where t_0 is a constant. If the function $F(t)$ is periodic with period T, the time average $\overline{F}$ is independent of t_0.

By averaging (2.24) over a period we find, since cos and sin are periodic functions,

$$\begin{aligned} \bar{x} - x_0 &= K_1 \\ \bar{z} - z_0 &= K_2 \end{aligned}$$

where $(\bar{x}, \bar{z})$ is the average position of the particles. With the help of these relations (2.24) can be written

$$\begin{aligned} x - \bar{x} &= A \cos(kx_0 - \omega t) \\ z - \bar{z} &= B \sin(kx_0 - \omega t) \end{aligned} \tag{2.26}$$

where

$$A = a\frac{\cosh k(z_0 + H)}{\sinh kH} \quad \text{and} \quad B = a\frac{\sinh k(z_0 + H)}{\sinh kH}.$$

From (2.26) it follows that

$$(\frac{x - \bar{x}}{A})^2 + (\frac{z - \bar{z}}{B})^2 = 1 \tag{2.27}$$

which shows us that the water particles follow elliptical paths, and that the orbital period is the wave period $T = 2\pi/\omega$. The semi-axes of the ellipses are A and B. For particles that are near the bottom ($z_0 \to -H$) it is seen that $A \to a/\sinh kH$ and $B \to 0$, and the ellipses degenerate to straight lines. If H is much larger than the wavelength such that $kH \gg 1$, then $A = B = a\mathrm{e}^{kz_0}$. In this case, for waves in deep water, the particle paths are circular as shown in figure 2.5. The radius of the circular paths decreases deeper in the water, and at depth $z_0 = -\lambda$ the radius is a factor of $e^{-2\pi} \simeq 2 \cdot 10^{-3}$ smaller than at the surface. The orbital direction of the particle paths follows from (2.26), and for waves that propagate to the right, the orbital direction is indicated with arrows in figure 2.5. For comparison the particle paths in long waves on shallow water $H/\lambda = 0.15$ are sketched in figure 2.6.

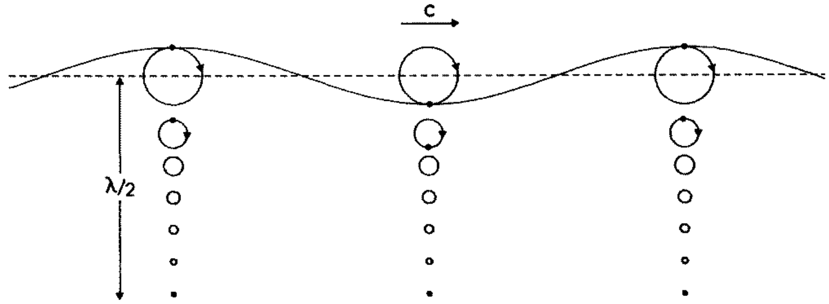

Figure 2.5: *Particle paths for waves in deep water $a/\lambda = 0.06$.*

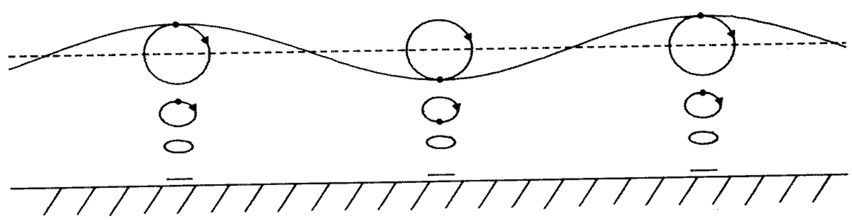

Figure 2.6: *Particle paths for waves in shallow water $a/H = 0.25$.*

2.4 The mechanical energy in surface waves

For wave motion periodic in time with period T, one can find a simple expression for the average mechanical energy per surface area in the horizontal plane. For surface waves, we shall use the term average *energy density* for this quantity. (Normally the term is used for energy per unit of volume.) The energy consists of kinetic and potential energy, and the potential energy in surface waves is caused by surface tension and the force of gravity. When the surface deforms, the surface tension performs work which is σ multiplied by the change in the surface area. The average potential energy per surface area caused by surface tension can therefore be written ("c" = capillary, overline = average)

$$\overline{E_p^c} = \frac{1}{T}\int_{t_0}^{t_0+T} \sigma \left[\sqrt{1+(\frac{\partial \eta}{\partial x})^2+(\frac{\partial \eta}{\partial y})^2} - 1\right] \mathrm{d}t$$

Taylor expansion of the integrand gives for small values of $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$

$$\overline{E_p^c} = \frac{1}{2T}\int_{t_0}^{t_0+T} \sigma \left[(\frac{\partial \eta}{\partial x})^2+(\frac{\partial \eta}{\partial y})^2\right] \mathrm{d}t \qquad (2.28)$$

For the *potential energy* due to the force of gravity we can chose the zero-level anywhere, it is most convenient to chose the quiescent water surface $z = 0$.

Energy per volume unit is

$$e_p = \rho g z$$

and the average potential energy per surface area due to the force of gravity can therefore be written as ("g" = gravity, overline = average)

$$\overline{E_p^g} = \frac{1}{T}\int_{t_0}^{t_0+T}\int_{-H}^{\eta} e_p\,\mathrm{d}z\,\mathrm{d}t = \frac{\rho g}{2T}\int_{t_0}^{t_0+T}\eta^2\,\mathrm{d}t \tag{2.29}$$

The total average potential energy per surface unit for surface waves is

$$\overline{E_p} = \overline{E_p^g} + \overline{E_p^c} \tag{2.30}$$

The *kinetic energy* per volume is

$$e_k = \frac{\rho}{2}\boldsymbol{v}^2$$

and the average kinetic energy per surface area is

$$\overline{E_k} = \frac{1}{T}\int_{t_0}^{t_0+T}\int_{-H}^{\eta} e_k\,\mathrm{d}z\,\mathrm{d}t$$

By Taylor expansion in powers of η we find

$$\int_{-H}^{\eta} e_k\,\mathrm{d}z = \int_{-H}^{0} e_k\,\mathrm{d}z + \left.(e_k)\right|_{z=0}\eta + \ldots$$

For waves with small amplitude it is enough to take the first term in this series expansion, and by introduction of the velocity potential, we can write

$$\overline{E_k} = \frac{\rho}{2T}\int_{t_0}^{t_0+T}\int_{-H}^{0}(\nabla\phi)^2\,\mathrm{d}z\,\mathrm{d}t$$

With the help of Green's theorem and the boundary conditions at the bottom (2.14) the expression can be changed, and we find

$$\overline{E_k} = \frac{\rho}{2T}\int_{t_0}^{t_0+T}\left.\left(\phi\frac{\partial\phi}{\partial z}\right)\right|_{z=0}\mathrm{d}t \tag{2.31}$$

Now we set in the expressions (2.16) for velocity potential and surface displacement in (2.28)–(2.31) we find after some simple manipulations

$$\begin{aligned}\overline{E_p^c} &= \frac{1}{4}\sigma k^2 a^2 \\ \overline{E_p^g} &= \frac{1}{4}\rho g a^2 \\ \overline{E_p} &= \overline{E_k} = \frac{1}{4}\rho g a^2 (1 + \frac{\sigma k^2}{\rho g})\end{aligned} \tag{2.32}$$

The last equation shows that the average energy densities of potential and kinetic energy are equal (*equipartition of energy*). Even if this seems to be valid for most types of linear wave motion, it is possible to find examples of wave types where one does not have equipartition of energy: Severdrup-waves, which are a type of gravity–inertia waves, is such an example.

The sum of $\overline{E_p}$ and $\overline{E_k}$ is the average mechanical energy density and

$$\overline{E} = \overline{E_p} + \overline{E_k} = 2\overline{E_p} = 2\overline{E_k} \tag{2.33}$$

From the last equation in (2.32) it appears that for wavelengths $\lambda \gg \lambda_m$, where λ_m is defined in (2.19), the energy density due to surface tension is insignificant in proportion to the energy density due to gravity. We therefore have that for $\lambda \gg \lambda_m$

$$\overline{E} = \frac{1}{2}\rho g a^2 \tag{2.34}$$

which is the average mechanical energy density for gravity waves. For wavelengths $\lambda \ll \lambda_m$ the energy density due to gravity is insignificant, and we find that the average mechanical energy density for capillary waves is

$$\overline{E} = \frac{1}{2}\sigma k^2 a^2 \tag{2.35}$$

We also want to calculate the energy flow or *energy flux* in surface waves. Let us imagine that we limit an area at the water surface WS within a closed curve Γ. Vertically under the surface WS lies a volume of water Ω, limited by the vertical wall Π. The total energy in Ω and in WS is given by the expression

$$\int_\Omega e_p \, \mathrm{d}V + \int_\Omega e_k \, \mathrm{d}V + \int_{WS} \sigma \, \mathrm{d}S$$

where e_p is the potential gravitational energy per volume, e_k is the kinetic energy per volume, and the surface tension σ is the potential energy per surface area. Under the assumption that energy is not created or destroyed in the domain we shall set up an integrated energy equation of the form

$$\frac{d}{dt}\left\{\int_\Omega e_p \, \mathrm{d}V + \int_\Omega e_k \, \mathrm{d}V + \int_{WS} \sigma \, \mathrm{d}S\right\} = -\int_\Gamma \boldsymbol{q}_\Gamma \cdot \hat{\boldsymbol{n}} \, \mathrm{d}s - \int_\Pi \boldsymbol{q}_\Pi \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S \tag{2.36}$$

where $\boldsymbol{q}_\Gamma$ is the surface energy flux density along the free surface WS, $\boldsymbol{q}_\Pi$ is the energy flux density in the volume Ω and $\hat{\boldsymbol{n}}$ is the unit normal vector to the vertical side Π.

If we had included viscous effects a dissipative term would have appeared. A nonconservative volume force would also have given rise to a source term in the form of an integral over Ω.

We need a special case of Leibniz's rule

$$\frac{d}{dt}\int_\Omega G\,\mathrm{d}V = \int_\Omega \frac{\partial G}{\partial t}\,\mathrm{d}V + \int_{WS} G|_{z=\eta}\frac{\partial \eta}{\partial t}\,\mathrm{d}A \tag{2.37}$$

where $\mathrm{d}A$ is an infinitesimal element in the horizontal xy-plane, and the last term is due to time variation of the free surface. One can note that this term alone gives the contribution to the change of e_p because the gravity potential is time independent.

We now set up the equations which control the motion and which we will deduce the energy equation (2.36) from. Some of these are given earlier, but we repeat them here for the sake of overview. First we have the equation of motion for an ideal fluid

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla(\frac{\boldsymbol{v}^2}{2}) - \frac{1}{\rho}\nabla p - \nabla\Phi \tag{2.38}$$

where we have assumed irrotationality to restate the convective term. Instead of introducing the acceleration of gravity explicitly, we use a general force potential Φ. The fluid is incompressible and the continuity equation is then

$$\nabla\cdot\boldsymbol{v} = 0. \tag{2.39}$$

At the free surface we have the dynamic condition

$$p = \sigma\nabla\cdot\boldsymbol{n} \tag{2.40}$$

where $\boldsymbol{n}$ is the unit normal vector to the free surface WS. In addition, we have the kinematic surface condition (2.5) which can be written in the form of

$$\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = \frac{\partial\eta}{\partial t}\,\mathrm{d}A \tag{2.41}$$

where $\mathrm{d}A$ is the projection of $\mathrm{d}S$ down in the xy-plane.

Now following some manipulation we obtain the energy equation in the desired form. We start by taking the dot product of (2.38) with the velocity vector and integrate the result over Ω. Then we use Gauss' theorem to arrive at the equation

$$\int_\Omega \frac{\partial}{\partial t}(\frac{1}{2}\rho\boldsymbol{v}^2)\,\mathrm{d}V = -\int_{\partial\Omega}(p + \frac{1}{2}\rho\boldsymbol{v}^2 + \rho\Phi)\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S \tag{2.42}$$

where $\partial\Omega$ designates the edge of Ω, i.e. the union of the free surface WS, the vertical side Π and the bottom. There is however no contribution from the bottom due to the kinematic bottom condition (2.14). The unit normal vector $\boldsymbol{n}$ points outwards. We now use (2.37) to write the equation in a form which provides the time derivative of the integrated kinetic and potential energy in a gravity field. The kinematic surface condition (2.41) gives us that most of the terms in the surface integral cancel and we end up with

$$\frac{d}{dt}\left\{\int_\Omega e_p\,\mathrm{d}V + \int_\Omega e_k\,\mathrm{d}V\right\} = -\int_{WS} p\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S - \int_\Pi \left(p + \frac{\rho}{2}\boldsymbol{v}^2 + \rho\Phi\right)\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}V \tag{2.43}$$

Now using (2.40) and (2.41) the first term in the right side in (2.43) we write

$$-\int_{WS} p\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = -\int_{WS}\sigma\nabla\cdot\boldsymbol{n}\frac{\partial\eta}{\partial t}\,\mathrm{d}A. \tag{2.44}$$

By further using $\boldsymbol{n} = (\boldsymbol{e}_z - \nabla\eta)/\sqrt{1+(\nabla\eta)^2}$ we have the identity

$$\nabla\cdot(\boldsymbol{n}\eta_t) = \nabla\cdot\boldsymbol{n}\eta_t + \boldsymbol{n}\cdot\nabla\eta_t = \nabla\cdot\boldsymbol{n}\eta_t - \frac{\nabla\eta\cdot\nabla\eta_t}{\sqrt{1+(\nabla\eta)^2}}$$

and therefore we have, with use of both Leibnitz' rule and Gauss' theorem,

$$-\int_{WS}\sigma\nabla\cdot\boldsymbol{n}\frac{\partial\eta}{\partial t}\,\mathrm{d}A = -\frac{\mathrm{d}}{\mathrm{d}t}\int_{WS}\sigma\,\mathrm{d}S - \int_\Gamma \sigma\boldsymbol{n}\cdot\hat{\boldsymbol{n}}\frac{\partial\eta}{\partial t}\,\mathrm{d}s \tag{2.45}$$

where $\hat{\boldsymbol{n}}$ is the unit normal vector to Π and $\boldsymbol{n}$ is the unit normal vector to the free surface WS, and $\mathrm{d}s$ is an infinitesimal curve element along the projection of Γ into the xy-plane. The first term on the right side is the change in total surface energy, while the second term is the total energy flux of the surface energy (both advection of surface energy through Γ and the surface tension acting on the curve Γ).

Let us sum up everything

$$\begin{aligned}&\frac{d}{dt}\left\{\int_\Omega \frac{\rho}{2}\boldsymbol{v}^2\,\mathrm{d}V + \int_\Omega \rho\Phi\,\mathrm{d}V + \int_{WS}\sigma\,\mathrm{d}S\right\}\\ &= -\int_\Gamma \sigma\hat{\boldsymbol{n}}\cdot\boldsymbol{n}\frac{\partial\eta}{\partial t}\,\mathrm{d}s - \int_\Pi\left(p+\frac{\rho}{2}\boldsymbol{v}^2+\rho\Phi\right)\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S\end{aligned} \tag{2.46}$$

where the contributions on the left side are the change in kinetic energy, the change in potential gravitational energy, and the change in surface energy. The contributions on the right side are the flux of surface energy through the curve Γ, the action of pressure on the vertical wall Π, transport of kinetic energy through Π and transport of potential gravity energy through Π. With the help of Euler's pressure formula, the three terms in the brackets can be simplified

$$\frac{d}{dt}\left\{\int_\Omega \frac{\rho}{2}\boldsymbol{v}^2\,\mathrm{d}V + \int_\Omega \rho\Phi\,\mathrm{d}V + \int_{WS}\sigma\,\mathrm{d}S\right\} = -\int_\Gamma \sigma\hat{\boldsymbol{n}}\cdot\boldsymbol{n}\frac{\partial\eta}{\partial t}\,\mathrm{d}s + \int_\Pi \rho\frac{\partial\phi}{\partial t}\boldsymbol{v}\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S \tag{2.47}$$

A simple illustration of the right side in (2.47) for the two-dimensional geometry is given in figure 2.7. For two-dimensional geometry $\hat{\boldsymbol{n}} \cdot \boldsymbol{n} = \sin\alpha$ where α is the sloping angle to the surface that is defined in the figure. Work per unit of time which the fluid to the left of the section A–A performs on the fluid to the right is

$$W = -\sigma\frac{\partial\eta}{\partial t}\sin\alpha - \int_{-h}^{\eta}\rho\frac{\partial\phi}{\partial t}u\,\mathrm{d}z.$$

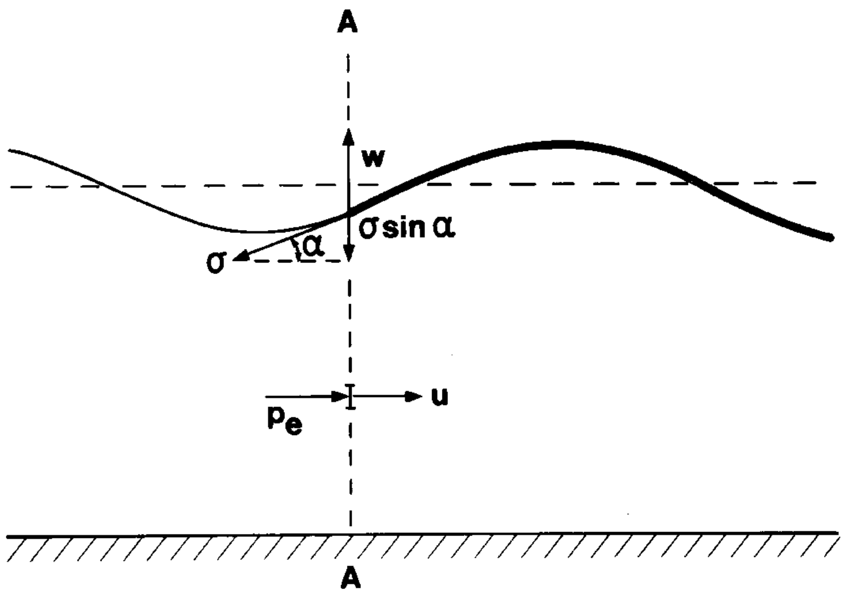

Figure 2.7: Illustration of energy flux.

We shall now consider the energy flux density integrated in the vertical direction, $\boldsymbol{F}$, defined such that the right side of the equation (2.47) can be written in the form

$$-\int_{\Gamma}\hat{\boldsymbol{n}} \cdot \boldsymbol{F}\,\mathrm{d}s$$

where the infinitesimal curve element ds is along the projection of Γ into the xy-plane. Taylor expansion of the horizontal energy flux density for small values of the derivative of η and ϕ now gives

$$\hat{\boldsymbol{n}} \cdot \boldsymbol{F} = \hat{\boldsymbol{n}} \cdot \left[\sigma\nabla\eta\frac{\partial\eta}{\partial t} + \int_{-h}^{0}\rho\frac{\partial\phi}{\partial t}\nabla\phi\,\mathrm{d}z\right].$$

In agreement with previous assumptions, only those terms for which the wave amplitude occurs to quadratic order have been included. Introducing the velocity potential and surface displacement (2.15)–(2.16), we therefore find, with the help of the relations (2.17)–(2.18) and with the time averaging (2.25) that the average energy flux density $\overline{\boldsymbol{F}}$ (power per length along the wave) is

$$\overline{\boldsymbol{F}} = \overline{E}\boldsymbol{c}_g \tag{2.48}$$

where the magnitude

$$\boldsymbol{c}_g = \frac{c}{2}\left[\frac{1+\frac{3\sigma k^2}{\rho g}}{1+\frac{\sigma k^2}{\rho g}} + \frac{2kH}{\sinh 2kH}\right]\frac{\boldsymbol{k}}{k} \tag{2.49}$$

is a vector in the same direction as the wave vector $\boldsymbol{k}$, and which has the same dimensions as velocity, and E is the energy density for gravity–capillary waves. By differentiation of (2.17) we find that the group velocity is

$$\boldsymbol{c}_g = \frac{\partial \omega}{\partial \boldsymbol{k}}.$$

The vector $\boldsymbol{c}_g$ defines the *group velocity* and we shall discuss this quantity in the next section. So far we have noticed that the energy flux density is equal to energy density multiplied by the group velocity. This is valid for linear wave motion.

For gravity waves in deep water ($kH \gg 1$) the energy flux density (power per length along the wave crest) is

$$\overline{F} = \frac{1}{4}\rho g a^2 c. \tag{2.50}$$

The table below shows the phase speed, period, and energy flux density for gravity waves with wavelengths from 1–150 m and amplitude 0.1 m. The water depth is assumed to be much larger than the wavelength such that we can employ the theory of infinitely deep water.

λ(m)	c(m/s)	T(s)	$\overline{F}$(W/m)
5	2.79	1.79	69
10	3.95	2.53	97
25	6.25	4.00	153
50	8.84	5.66	217
75	10.82	6.93	265
100	12.50	8.00	307
150	15.30	9.80	375

Exercises

1. We look at a sea basin with depth 10 m and with a straight coastline. A buoy registers waves with period 7 s and amplitude 0.5 m. Assume that the coast absorbs all incoming wave energy. Estimate the importance of finite depth and/or surface tension. Compute the wavelength, phase speed, and group velocity. How much power is absorbed by a 500 m long coastline? How long a coastline would be sufficient to cover the energy needs for a household?

2. A tsunami has amplitude 2 m in 5000 m deep water. Determine the phase speed and group velocity (km/h), energy density (kWh/m^2) and energy flux density (kW/m). If you want to you may assume a wavelength of, say, 200 km. However, you should also explain why this is not important for the quantities you have just calculated.
   Assuming the tsunami to be periodic (it is not, but ...), find the maximum horizontal velocities and the maximum particle motion when the period is half an hour.
   Next we assume periodic wind-generated waves, of length 200m, on the same depth and with the same amplitude as the tsunami. Compare energy density, phase speed and group velocity for the wind wave and the tsunami.

2.5 A simple kinematic interpretation of group velocity. Group velocity for surface waves.

In the previous section we have seen that the velocity $c_g = \frac{d\omega}{dk}$ is related to the energy flux in plane surface waves. In this section we shall give a simple kinematic interpretation of the magnitude c_g which also justifies the definition group velocity. We add up two wave components η_1 and η_2 with the same amplitude but with slightly different values for wave number and angular frequency.

$$\eta_1 = \frac{1}{2}a\sin[(k+\Delta k)x - (\omega+\Delta\omega)t]$$

and

$$\eta_2 = \frac{1}{2}a\sin[(k-\Delta k)x - (\omega-\Delta\omega)t]$$

where $2\Delta k$ and $2\Delta\omega$ are, respectively, the differences in wavenumber and angular frequency. By the formula for the summation of two sine functions we find

$$\eta = \eta_1 + \eta_2 = a\cos(\Delta kx - \Delta\omega t)\sin(kx - \omega t). \tag{2.51}$$

Equation (2.51) depicts a series of wave groups or wave packets where the amplitude of the individual waves within a group varies from 0 to a. The length of the wave group is $\lambda_g = 2\pi/\Delta k$, and the wavelength for the individual waves in the group is $\lambda = 2\pi/k$. The individual crests propagate with phase speed $c = \omega/k$, while the wave group propagates with velocity

$$\frac{\Delta\omega}{\Delta k} \approx c_g = \frac{d\omega}{dk}. \tag{2.52}$$

For wave motion in more spatial dimensions we find that the group velocity is a vector defined as the gradient of the angular frequency ω with respect to the wavenumber vector $\boldsymbol{k}$, $\boldsymbol{c}_g = \partial\omega/\partial\boldsymbol{k}$. For isotropic waves, i.e., when the dispersion relation only depends on wavenumber and not the direction of the wavenumber vector, we see that the group velocity vector has the same direction as the wavenumber vector.

With the help of the relation $\omega = ck$ we can write

$$c_g = c + k\frac{dc}{dk} = c - \lambda\frac{dc}{d\lambda} \tag{2.53}$$

which shows that for dispersive waves, the magnitude of the group velocity is different from the phase speed. For nondispersive waves, the magnitude of the group velocity is equal to the phase speed. The group velocity is larger or smaller than the phase speed depending on the phase speed decreasing or increasing with wavelength. For gravity waves $c_g < c$, while for capillary waves $c_g > c$. Since individual crests propagate with a speed different from the group velocity, an observer following an individual wave in the group will see the wave propagate through the group with varying amplitude. Waves will therefore be lost from sight at the leading or trailing edge of the group depending on c being larger or smaller than c_g. Because the wave amplitude is zero at the leading and trailing edge of a group, this suggests that energy within the group spreads with the group velocity. We have already seen that this interpretation is correct for surface waves.

With the help of (2.53) it is possible to find c_g graphically by drawing the tangent to the graph $c = c(\lambda)$. This method is shown in figure 2.9. Group velocity for surface waves is given by (2.49). By these expressions we see that for surface waves in deep water, $kH \gg 1$, we have

$$c_g = \frac{1}{2}\,c\,\frac{1 + \frac{3\sigma k^2}{\rho g}}{1 + \frac{\sigma k^2}{\rho g}} \tag{2.54}$$

where the phase speed c is known by (2.20). Depending on the wave length being much larger or much smaller than λ_m (see equation 2.19) we find

$$\begin{aligned} c_g &= \frac{1}{2}c \qquad \text{when} \qquad \lambda \gg \lambda_m \\ c_g &= \frac{3}{2}c \qquad \text{when} \qquad \lambda \ll \lambda_m \end{aligned} \tag{2.55}$$

These are expressions for group velocity for gravity and capillary waves, respectively, in deep water.

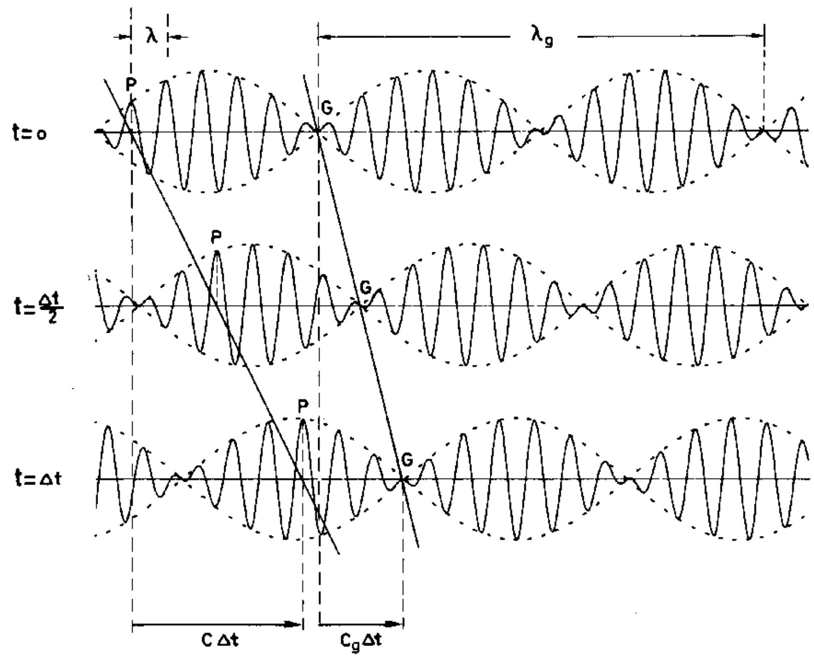

Figure 2.8: Graph of the function $\eta = a\cos(\Delta kx - \Delta\omega t)\sin(kx - \omega t)$ with envelopes $a\cos(\Delta kx - \Delta\omega t)$ at three different times for a wave train with $c_g = c/2$. Forward movement of the wave group (G) and an individual wave (P) are marked.

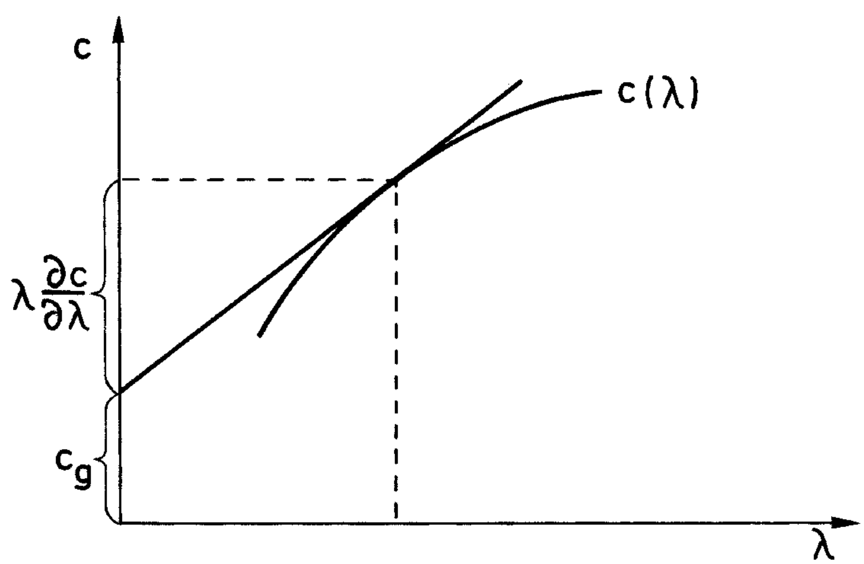

Figure 2.9: Phase speed and group velocity.

For surface waves in shallow water , $kH \ll 1$, we find from earlier methods

$$\begin{aligned} c_g &= c = c_0 \qquad \text{when} \qquad \lambda \gg \lambda_m \\ c_g &= 2c \qquad \text{when} \qquad \lambda \ll \lambda_m \end{aligned}$$

See figure 2.10 and 2.11.

In this section we have shown that a series of wave groups appear by the interference of two harmonic wave components. In the exercises with section 2.7 it will become evident that a single wave group or wave packet can be composed by a number of harmonic wave components.

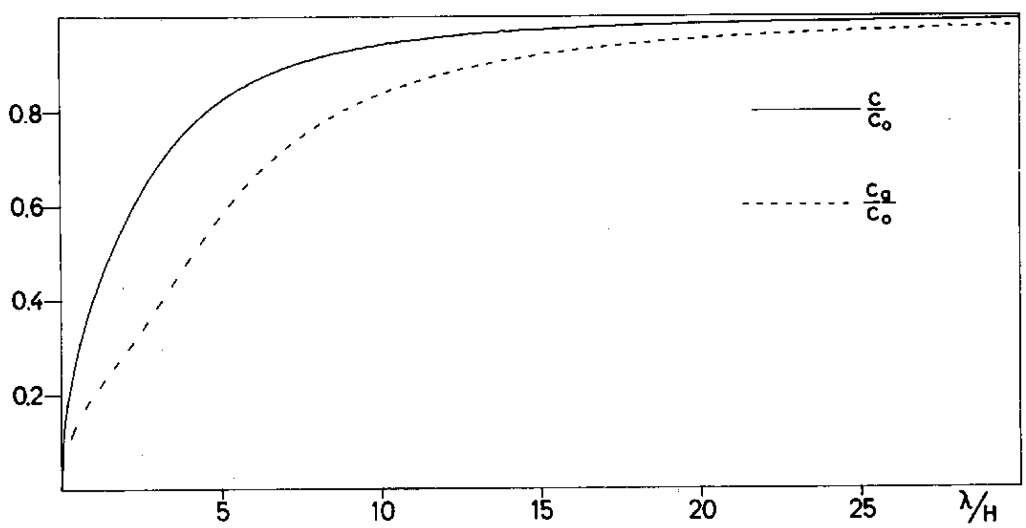

Figure 2.10: Phase speed and group velocity for gravity waves.

Wave groups often appear in nature, and a typical example of this is swells. Figure 2.12 shows wave groups registered in swells at Trænabanken. The seismograph in figure 3.1 in section 2.13 also shows wave groups.

2.6 The Klein-Gordon equation

As an example to illustrate the two previous sections we shall consider the Klein-Gordon equation

$$\frac{\partial^2 \eta}{\partial t^2} - c_0^2 \frac{\partial^2 \eta}{\partial x^2} + c_0^2 q^2 \eta = 0 \tag{2.56}$$

where c_0 and q are constant and η is a functions of x and t.

This equation describes, for example, the motion of an oscillating string where in addition to tension in the string, spring forces pull the string in towards the

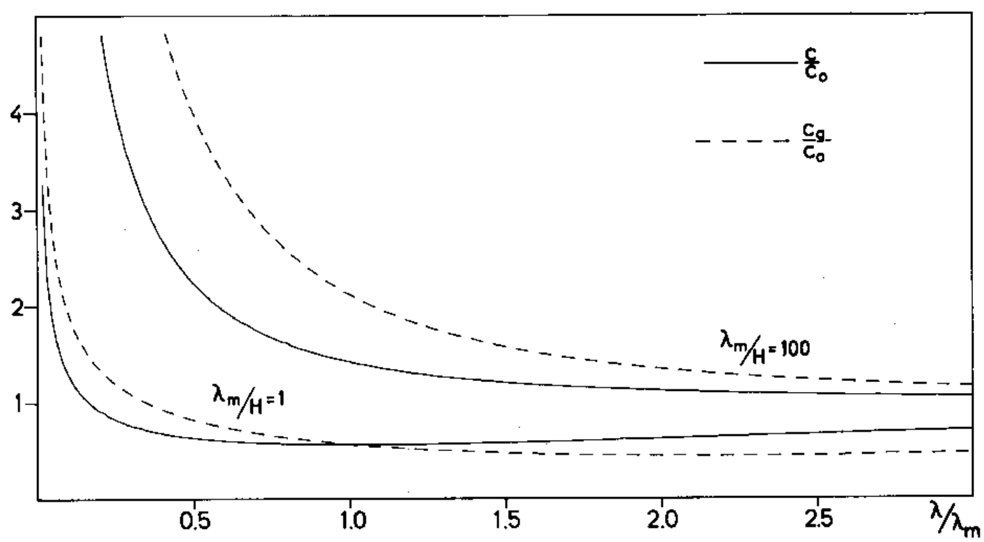

Figure 2.11: Phase speed and group velocity for gravity and capillary waves.

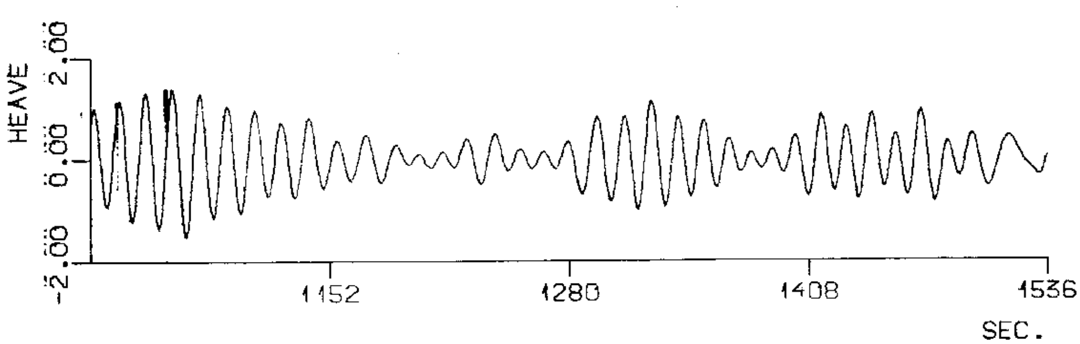

Figure 2.12: Long periodic rising and sinking of the water surface (in meters) registered on 24 January 1982 at 9 GMT by a buoy anchored at Trænabanken off the coast of Helgeland. The swells came from a storm-center in the North-Atlantic east of New-Foundland (Gjevik, Lygre and Krogstad, 1984).

equilibrium position. The Klein–Gordon equation furthermore describes phenomena within relativistic quantum mechanics, and it occurs also for long gravity–inertia waves in rotating fluids.

The equation allows for waves of the form

$$\eta = a \sin k(x - ct)$$

where the phase speed is

$$c = c_0(1 + q^2/k^2)^{\frac{1}{2}}$$

The group velocity is

$$c_g = \frac{d(ck)}{dk} = \frac{c_0^2}{c}$$

The energy equation comes out by multiplying the Klein–Gordon equation by $\frac{\partial \eta}{\partial t}$, and we find after reordering

$$\frac{\partial E}{\partial t} + \frac{\partial F}{\partial x} = 0$$

where the energy density (per mass unit) is

$$E = \frac{1}{2}\left[(\frac{\partial \eta}{\partial t})^2 + c_0^2(\frac{\partial \eta}{\partial x})^2 + c_0^2 q^2 \eta^2\right]$$

and the energy flux is

$$F = -c_0^2 \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial t}$$

With use of the wave solution we can estimate the average energy density and energy flux, and we find

$$\begin{aligned} \overline{E} &= \frac{1}{2}a^2k^2c^2 \\ \overline{F} &= \frac{1}{2}a^2k^2c_0^2c \end{aligned}$$

The average propagation velocity for the energy is therefore

$$\frac{\overline{F}}{\overline{E}} = \frac{c_0^2}{c} = c_g$$

2.7 Surface waves generated by a local disturbance in the fluid

When a stone is thrown into still and relatively deep water, a train of regular waves which propagate radially out from the point where the stone hit the surface is formed. If the stone is comparatively large, we will see long waves in the front of the wave train and gradually shorter waves in towards the center of the motion. The cause of this is that the mechanical energy which is contributed to the water by the stone will distribute itself into different wave components, which propagate out with a velocity which is dependent on wavelength. After a time, the long waves will have outran the shorter, and the original disturbance is canceled out into a regular wave train with gradually varying wavelengths. If the stone is tossed into shallow water, then all of the wave components with wavelength greater then the water depth will move with nearly the same velocity. In this case the disturbance can extend over a long distance without noticeably altering form. If an adequately small stone is released into the water, the long waves generated will have a very small amplitude such that only the wave components with wavelength $\lambda \leq \lambda_m$ are visible. In this case, a wave train will develop with shorter waves in front and longer waves behind.

We shall now handle some of these phenomena mathematically. For simplicity we shall limit to two-dimensional wave motion which can be thought of as arising from an elongated disturbance. This case is mathematically easier to treat than the case where the waves propagate in all directions and the wave amplitude decreases because the energy constantly spreads out over a larger area. Here we shall also handle the case where the motion starts from rest with an elevation or depression of the surface. The initial conditions can therefore be written

$$\begin{aligned} \phi &= 0 \qquad \text{for} \qquad t = 0 \\ \eta &= \eta_0(x) \qquad \text{for} \qquad t = 0 \end{aligned} \tag{2.57}$$

To solve the equations (2.11) with the boundary conditions (2.12)–(2.14) and the initial conditions (2.57) we introduce the Fourier transform with respect to that x which we define as

$$\hat{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(x) e^{-ikx}\, dx \tag{2.58}$$

where $f(x)$ is a function of x such that the integral (2.58) exists. The corresponding inverse transform is

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(k) e^{ikx}\, dk \tag{2.59}$$

We assume that the Fourier transforms of ϕ and η and their derivatives exist, and that they attenuate to zero as $x \to \pm\infty$. By integration by parts we find that

$$\widehat{\frac{\partial^n \phi}{\partial x^n}} = (ik)^n \hat{\phi} \qquad \text{for} \qquad (n = 1, 2, ...)$$

It follows that the Fourier transform of the Laplace equation (2.11) can be written

$$\frac{\partial^2 \hat{\phi}}{\partial z^2} - k^2 \hat{\phi} = 0 \tag{2.60}$$

The transform of the first of the two boundary conditions (2.12) gives

$$\hat{\eta} = -\frac{1}{g(1+\frac{\sigma k^2}{\rho g})}(\frac{\partial \hat{\phi}}{\partial t})_{z=0} \tag{2.61}$$

and by elimination of $\hat{\eta}$ with the help of the other conditions in (2.12) we find

$$\frac{\partial^2 \hat{\phi}}{\partial t^2} + g(1+\frac{\sigma k^2}{\rho g})\frac{\partial \hat{\phi}}{\partial z} = 0 \quad \text{for} \;\; z = 0 \tag{2.62}$$

The transform of the boundary conditions (2.14) is

$$\frac{\partial \hat{\phi}}{\partial z} = 0 \quad \text{for} \;\; z = -H \tag{2.63}$$

Finally the Fourier transforms of the initial conditions (2.57) are

$$\hat{\phi} = 0 \qquad \text{and} \qquad \hat{\eta} = \hat{\eta}_0 \qquad \text{for} \qquad t = 0 \tag{2.64}$$

A solution of (2.60) that satisfies the boundary conditions (2.63) can be written

$$\hat{\phi} = A(t) \cosh\, k(z+H) \tag{2.65}$$

where $A(t)$ is an undefined function of t.

Substituting into (2.62) gives

$$\frac{d^2 A}{dt^2} = -\omega^2 A$$

where ω is given by the dispersion relation (2.17). The initial conditions (2.64) result in that $A = 0$ for $t = 0$, and the solution for the last equation can therefore be written

$$A(t) = A_0 \sin \omega t \tag{2.66}$$

where the constant A_0 is determined with the help of (2.61), (2.65) and the initial conditions (2.64) for $\hat{\eta}$. We find that

$$A_0 = -\frac{g(1+\frac{\sigma k^2}{\rho g})}{\omega \cosh kH}\hat{\eta}_0 \tag{2.67}$$

The inverse of the transformed expression (2.65) with $A(t)$ given by (2.66) and (2.67) gives the velocity potential

$$\phi = -g\int_{-\infty}^{+\infty} \frac{\sin \omega t}{\omega}\frac{\cosh k(z+H)}{\cosh kH}(1+\frac{\sigma k^2}{\rho g})\hat{\eta}_0 e^{ikx}\, dk \tag{2.68}$$

Setting the velocity potential (2.68) into the other condition in (2.12) we find after integration that the surface displacement can be written

$$\eta = \frac{1}{2}\left(\int_{-\infty}^{+\infty} \hat{\eta}_0 e^{i(kx+\omega t)}\, dk + \int_{-\infty}^{+\infty} \hat{\eta}_0 e^{i(kx-\omega t)}\, dk\right) \tag{2.69}$$

To arrive at (2.69) we also have to use the dispersion relation (2.17). Since $\hat{\eta}_0(-k)$ is the complex conjugate of $\hat{\eta}_0(k)$ and $\omega(-k) = \omega(k)$ the integrals can be converted such that one only integrates over positive values of k. Thereby we find

$$\eta = \int_0^{\infty} |\hat{\eta}_0(k)|(\cos(kx - \omega t + \gamma) + \cos(kx + \omega t + \gamma))\, dk$$

where γ is the argument of $\hat{\eta}_0(k)$. This expression shows that the motion can be seen as consisting of a spectrum of harmonic wave components. We see that the original disturbance splits up into wave components which move in the positive or negative x-directions. The complex Fourier transform $\hat{\eta}_0(k)$ determines the amplitude and phase distribution for the various wave components. The modulus squared $|\hat{\eta}_0(k)|^2$ is proportional to the so-called *wave spectrum*, which in this case is the wave number spectrum.

If the spectrum is such that only the wave components where $\lambda \gg H$ are substantial, we can, according to (2.21) with good approximation, set $\omega = c_0|k|$ and integrate over the long-wave portion of the spectrum. Then the Fourier integral in (2.69) can be evaluated as

$$\eta = \frac{1}{2}\eta_0(x + c_0 t) + \frac{1}{2}\eta_0(x - c_0 t)$$

This shows that the original disturbance will split into two identical pulses which move with constant velocity and without changing form in the positive and negative x-directions respectively. The amplitude for each of the pulses is half of the amplitude for the original disturbance (see figure 2.13).

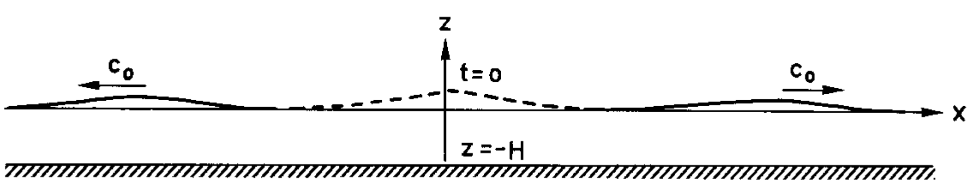

Figure 2.13: Wave motion excited from the original disturbance.

To be able to discuss the wave motion given by (2.69) in detail we shall use a value of the function $\eta_0(x)$ which leads to a relatively simple integral by setting in (2.68) or (2.69). We set

$$\eta_0(x) = \frac{Q}{2L\sqrt{\pi}} e^{-(x/2L)^2} \tag{2.70}$$

where Q and L are constants. The functions are sketched for different values of L in figure 2.14. For the function (2.70) we have that

$$\int_{-\infty}^{+\infty} \eta_0(x)\, dx = Q$$

and

$$\hat{\eta}_0(x) = \frac{Q}{2\pi} e^{-(kL)^2} \tag{2.71}$$

The modulus $|\hat{\eta}_0|$ is also symmetric around $k = 0$, and since $\hat{\eta}_0$ is real, the phase is the same for all values of the wave components.

If $L \to 0$, then $\hat{\eta}_0(x)$ degenerates to $Q\delta(x)$ where $\delta(x)$ is the Dirac delta generalized function with the properties

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \qquad \delta(x) = 0 \qquad \text{for} \qquad x \neq 0$$

and

$$\hat{\delta}(k) = \frac{1}{2\pi}$$

The Dirac delta function has a spectrum with equal amplitude for all the wave components (this is known as a "white" spectrum).

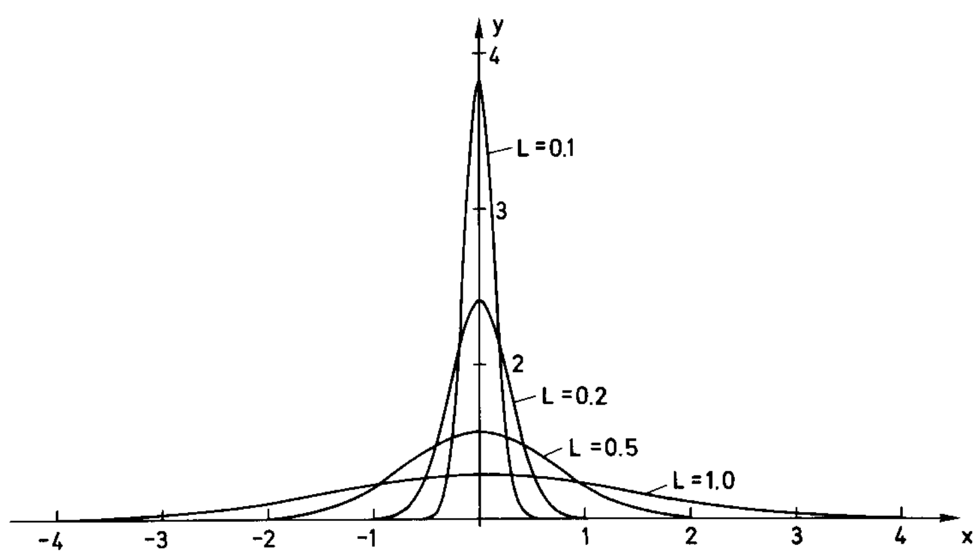

Figure 2.14: Gaussian start profile

For finite L (2.71) shows that the wave components in the spectrum with a wavenumber larger then $2\pi/L$ have vanishingly small amplitudes. This entails that the substantial contribution to the integral (2.69) comes from the portion of the spectrum where the wavelength is larger than L. We see, without further ado, that a disturbance of the form (2.70) will not generate capillary waves of any significance unless $L < \lambda_m$ where λ_m is defined earlier.

When $\hat{\eta}_0$ is given by (2.71), surface displacement is symmetric around the origin, and the surface displacement can be written

$$\eta = \frac{Q}{2\pi}\int_0^\infty e^{-(kL)^2}[\cos(kx+\omega t)+\cos(kx-\omega t)]\,dk \tag{2.72}$$

There are difficulties in discussing the full content of (2.72), and we shall limit the discussion here to gravity waves in deep water. In this case $\omega^2 = gk$. We introduce u which is a new variable of integration and set it in the first and second terms in the integral (2.72) respectively

$$\omega = (\frac{g}{x})^{\frac{1}{2}}(u \mp r)$$

$$k = \frac{1}{x}(u^2 \mp 2ur + r^2)$$

$$kx \pm \omega t = u^2 - r^2$$

where

$$r = (\frac{gt^2}{4x})^{\frac{1}{2}}$$

This now gives $L \to 0$

$$\eta = \frac{2Q}{\pi}\frac{r}{x}\int_0^r \cos(u^2 - r^2)\,du \tag{2.73}$$

The derivation of (2.73) implies some difficult boundary transitions and a few will perhaps prefer to go the route of the potential. One considers the potential around $z = 0$ and undertakes further substitutions in this expression. After one has found the converted expression for the potential, the surface displacement can be found with the help of the boundary conditions. Details are given by Lamb (1932). The equation (2.73) is an exact expression for surface displacement when the motion starts from rest, and the surface displacement at $t = 0$ has the form of a Dirac delta function at the origin. This shows that surface displacement can be expressed by the Fresnel integrals

$$C(r) = \sqrt{\frac{2}{\pi}}\int_0^r \cos u^2\,du$$

and

$$S(r) = \sqrt{\frac{2}{\pi}}\int_0^r \sin u^2\,du$$

which are tabulated. Since (2.73) leads to large values for η near the origin, it is clear that the solution in this case clashes with the stipulation for linear wave theory. Far from the origin will, nevertheless, (2.73) give values of η which fulfill the requirements and which describe wave motion with a good approximation. With finite values for L, one will, on the other hand, find that the solution (2.72) gives reasonable values of η near the origin.

The surface displacement, equation (2.73), which is a function of t at a point at the distance $x = x_0$ from the origin is shown in figure 2.15. It is evident from

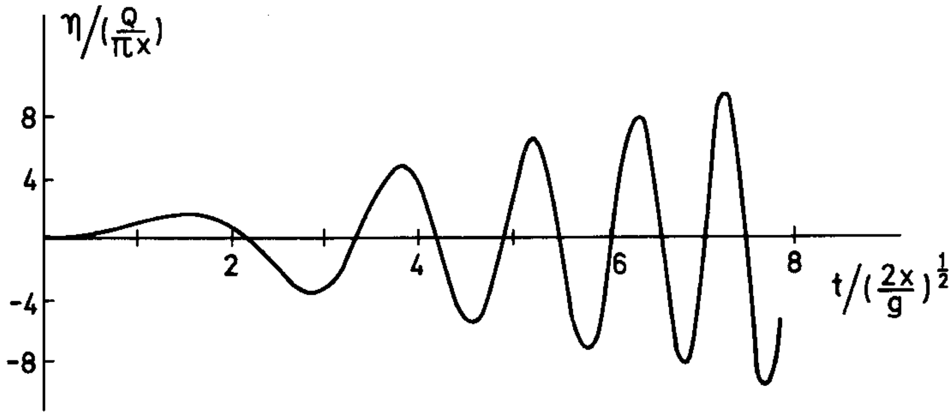

Figure 2.15: Surface displacement at a point.

the figure that the motion at every position starts with long periodic oscillations followed by oscillations with gradually shorter periods. We notice that the motion always starts at $t = 0$ even at points that are far away from the origin. This is related to the fact that in deep water, the longest gravity waves are spread with an infinitely large velocity, and the disturbance will be detected immediately in all points in the fluid. For $r \to \infty$

$$C(r) = S(r) = \frac{1}{2}$$

and it follows from (2.73) that the surface displacement can be written

$$\eta = \frac{Q}{2\sqrt{\pi}} \frac{g^{\frac{1}{2}} t}{x^{\frac{3}{2}}} \cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right) \tag{2.74}$$

The graph for the function (2.74) at an indeterminate point in time $t = t_0$ is sketched in figure 2.16.

Let us now study the surface from a point in time t_0, and again at point $x = x_0$. Taylor expansion of the phase function gives

$$\frac{gt_0^2}{4x} = \frac{gt_0^2}{4x_0} - \frac{gt_0^2}{4x_0}\left(\frac{x - x_0}{x_0}\right) + \ldots$$

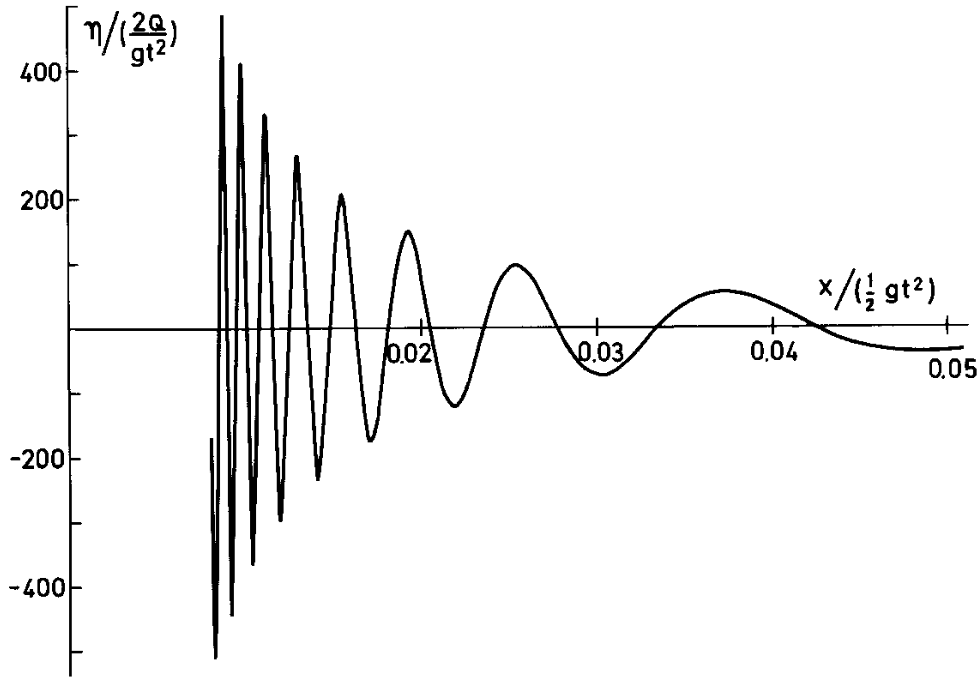

Figure 2.16: Surface displacement at an indeterminate point in time.

The wavelength λ corresponds to the change in the phase function of 2π

$$\lambda = |x - x_0| = \frac{8\pi x_0^2}{g t_0^2} \tag{2.75}$$

In the vicinity of the point $x = x_0$ the surface displacement will therefore be approximately periodic with the wavelength given above.

A fixed phase of the wave train, which may be a the zero-point $\eta = 0$, will be characterized by a fixed value for the phase function. This entails that the null points I_1, I_2 , and I_3 in figure 2.17 propagate in the x-direction such that

$$\frac{gt^2}{4x} = \text{constant}$$

This means that the points propagate with velocity

$$\dot{x} = \frac{2x}{t}$$

At a definite point in time, the velocity for I_1 is larger then for I_2, and the velocity for I_2 is again larger then for I_3. The waves are as such continuously extending in length.

Exercises

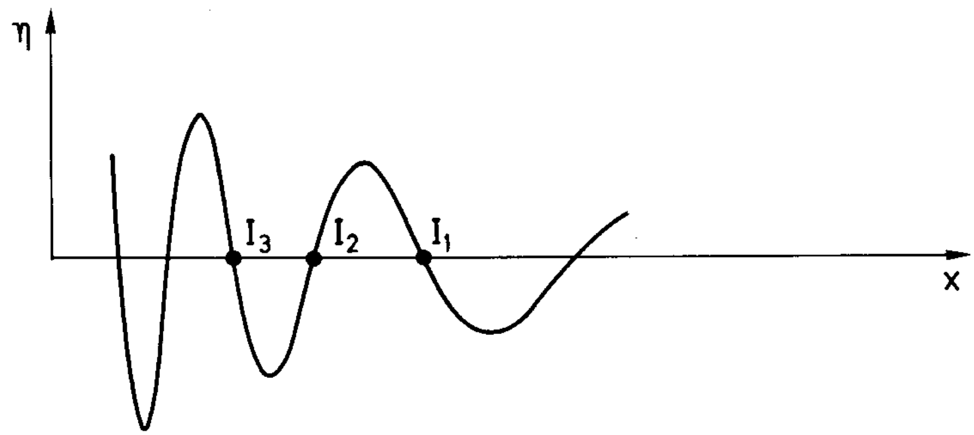

Figure 2.17: Sketch of the surface displacement.

1. **The narrow band spectrum.**

   We write the surface displacement, which is the sum of an infinite number of harmonic wave components.

$$\eta(x,t) = \mathrm{Re}\int_{-\infty}^{+\infty} a(k)e^{i(kx-\omega t)}\,dk$$

   with

$$a(k) = \frac{a_0}{\kappa\sqrt{\pi}}e^{-(\frac{k-k_0}{\kappa})^2}$$

   where a_0, κ and κ_0 are constants. Note that the wavenumber spectrum is proportional to $|a(k)|^2$. Show that at $t=0$ the surface has the form of a Gaussian wave packet given by

$$\eta(x,t=0) = a_0 e^{-(\frac{\kappa x}{2})^2}\cos k_0 x$$

   We assume that the wavenumber spectrum is narrowband (κ is small) such that in the neighborhood of $k=k_0$, where we find the contributions to the integral, we can set

$$\omega(k) \simeq \omega(k_0) + (\frac{d\omega}{dk})_{k_0}(k-k_0)$$

   Show that

$$\eta(x,t) = a_0 e^{-\frac{\kappa^2(x-c_g t)^2}{4}}\cos(k_0 x - \omega(k_0)t)$$

   where $c_g = (\frac{d\omega}{dk})_{k_0}$. Sketch the surface displacement for different values of the parameters. Explain why the next term in the series expansion for ω will lead to the wave packet changing form by stretching out.

   Hint: In an integral of the form $\int_{-\infty}^{+\infty} e^{-au^2+bu}\,du$ substitute $u = v + b/2a$. The integral $\int_{-\infty}^{+\infty} e^{-av^2}\,dv = \sqrt{\frac{\pi}{a}}$.

2. Find the expression for the velocity potential ϕ and surface displacement η in the case that the fluid at $t = 0$ contributes an impulse such that $\phi(x, z = 0, t = 0) = \phi_0(x)$ while the surface $\eta(x, t = 0) = 0$.

2.8 Stationary phase approximation. Asymptotic expressions of the Fourier integral.

In the previous section we have shown that if the disturbance in the fluid is local, the velocity potential and surface displacement of the ensuing motion can be expressed with a Fourier integral of the type

$$\int_{-\infty}^{+\infty} F(k) e^{i(kx \pm \omega(k)t)}\, dk$$

where $F(k)$ is a function of k. It is, as we have seen, difficult to discuss the integral in general, and we shall find an asymptotic expression for the Fourier integral valid for large values of t and x. This derivation was originally done by Kelvin (1887). We shall treat the case where $t \to \infty$ subject to x/t having a fixed value. We write the integral in the form

$$I(x,t) = \int_{-\infty}^{\infty} F(k) e^{it\chi(k)}\, dk \tag{2.76}$$

where

$$\chi(k) = \frac{x}{t} k \pm \omega(k)$$

For large values of t, the principal contributions to the integral (2.76) will come from values of k in a region around the values $k = k_0$ where the derivative of χ with respect to k is zero. Thus

$$\chi'(k_0) = \frac{x}{t} \pm \omega'(k_0) = 0$$

For other values of k the integrand will oscillate rapidly. If $F(k)$ is a comparatively slowly varying function of k, the positive and negative contributions will abolish each other. The net contribution to the integral from these values of k will be negligible, and we can therefore limit the integral into a small neighborhood of point $k = k_0$ only

$$I(x,t) = \int_{k_0-\epsilon}^{k_0+\epsilon} F(k) e^{it\chi(k)}\, dk \tag{2.77}$$

As the integral is restricted to a narrow region around $k = k_0$ we can develop the functions $F(k)$ and $\chi(k)$ in Taylor series around that point,

$$\begin{aligned} F(k) &\simeq F(k_0) \\ \chi(k) &\simeq \chi(k_0) + \frac{1}{2}\chi''(k_0)(k-k_0)^2 \end{aligned}$$

where we assume that $\chi''(k_0) \neq 0$. These series expansions are set into (2.77) and we find

$$I(x,t) \simeq F(k_0)e^{it\chi(k_0)} \int_{k_0-\epsilon}^{k_0+\epsilon} e^{\frac{i}{2}t\chi''(k_0)(k-k_0)^2}\, dk.$$

Since the integrand in this integral also oscillates rapidly for large values of $|k-k_0|$, we can without significant error return to the original limits $\pm\infty$. From known formulas we have that the integral

$$\int_{-\infty}^{+\infty} e^{\pm im^2u^2}\, du = \frac{\sqrt{\pi}}{m} e^{\pm i\frac{\pi}{4}}$$

where m is a positive constant. This gives

$$I(x,t) \simeq \frac{\sqrt{2\pi}F(k_0)}{\sqrt{t|\chi''(k_0)|}} e^{i(\chi(k_0)t \pm \frac{\pi}{4})} \tag{2.78}$$

where the upper or lower sign in the exponent is valid when $\chi''(k_0)$ is positive or negative, respectively. This is the stationary phase approximation for the Fourier integral.

If $\chi'(k) = 0$ at several points k_0 sufficiently spread from each other, we would have to sum the contributions from each point, every term having the form of equation (2.78). In the case that $\chi''(k_0) = 0$ we would have to continue the Taylor expansion to higher order until the first nonzero term.

Restricting to the case that $\chi''(k_0) \neq 0$, we can include one additional term in the series expansion for χ,

$$\frac{1}{6}\chi'''(k_0)(k-k_0)^3.$$

It is then seen that the formula (2.78) is valid when

$$|(k-k_0)\chi'''(k_0)/\chi''(k_0)| \ll 1$$

in the region for k which contribution to the integral, that is when the integrand oscillates slowly, see figure 2.18. One therefore finds the contribution to the integrand for k-values where

$$t(k-k_0)^2\chi''(k_0) \leq 2\pi n$$

where n is a small integer. Therefor we must have

$$q = t^{-\frac{1}{2}}|\chi'''(k_0)|/|\chi''(k_0)|^{3/2} \ll 1$$

for the approximation (2.78) to be valid.

Let us now use the stationary phase approximation to find an asymptotic expression for the surface displacement for gravity waves in deep water generated

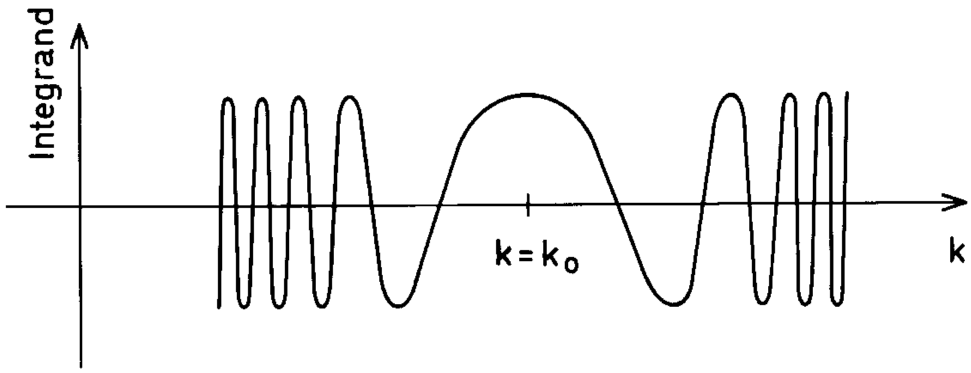

Figure 2.18: Sketch of the principle for stationary phase.

by a surface displacement in the form of a Dirac delta function at the origin ($x = 0$). The motion of the surface for $x > 0$ due to waves which propagate in the x-direction, and the surface displacement for $x > 0$ can, according to the results in section 2.7 be written

$$\eta(x,t) = \frac{Q}{2\pi}\frac{1}{2}\int_{-\infty}^{\infty} e^{i(kx-\omega t)}\, dk.$$

We set the following

$$\begin{aligned} F(k) &= \frac{Q}{2\pi}, \\ \chi(k) &= k\frac{x}{t} - \sqrt{gk}. \end{aligned}$$

We have

$$\chi'(k) = \frac{x}{t} - \frac{1}{2}\sqrt{\frac{g}{k}}$$

and therefore the criterion $\chi'(k_0) = 0$ is satisfied for

$$k_0 = \frac{gt^2}{4x^2} \qquad \text{and} \qquad \chi(k_0) = -\frac{gt}{4x}$$

and we also have

$$\chi''(k_0) = \frac{2x^3}{gt^3} \qquad \text{and} \qquad \chi'''(k_0) = -12\frac{x^5}{g^2t^5}.$$

By equation (2.78) we find an approximation for surface displacement identical to the earlier expressions we found in (2.74). The magnitude q is proportional to $(2x/gt^2)^{\frac{1}{2}}$ which shows that this approximation is valid given that $\frac{1}{2}gt^2 \gg x$. This is also in agreement with what we have found earlier.

Exercises

1. Use the stationary phase approximation to find the surface displacement for gravity waves in deep water when $L \neq 0$, see (2.70). Discuss the solution.

2. Use the stationary phase approximation to find the surface displacement for capillary waves in deep water. Let the initial disturbance have the form of a Dirac delta function. Compare with the solution for gravity waves.

3. We have given the Klein–Gordon equation in the form:

$$\frac{\partial^2\eta}{\partial t^2} - c_0^2\frac{\partial^2\eta}{\partial x^2} + q\eta = 0$$

which is valid for $t > 0$ and $-\infty < x < \infty$. Waves are generated from the initial disturbance

$$\eta(x,0) = A_0 e^{-\left(\frac{x}{L}\right)^2}, \qquad \frac{\partial\eta(x,0)}{\partial t} = 0.$$

Find an approximate solution for large x and t and discuss where this is valid.

2.9 Asymptotic generation of the wave front

If we assume infinite depth, we will not have any clear limitation of the front of the wave train generated from an initial disturbance except that the amplitude will decrease gradually in accordance with the distribution of energy in the spectrum. If we assume finite depth, we have a limitation in the velocity of gravity waves and we should expect that there is a clear limitation of the front of wave train. This will be studied below.

The Fourier transform used for the initial value problem gives the inversion integral

$$\eta(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} \hat{\eta}_0 \mathrm{e}^{i\chi}\,\mathrm{d}k; \qquad \chi \equiv kx - \omega(k)t \tag{2.79}$$

for the portion of the wave train which propagates in the positive x-direction. For large values of x and t, the dominant contribution will come from the stationary point $k = k_s$ where

$$\frac{d\chi(k_s)}{dk} = 0$$

which corresponds to

$$c_g(k_s) = \frac{x}{t}. \tag{2.80}$$

If we limit to gravity waves, the group velocity is confined by $c_g \leq c_0 = \sqrt{gH}$. We therefore only find stationary points for $\frac{x}{t} \leq c_0$. For larger values of x/t we can anticipate that waves have not yet had time to arrive. For the condition $\frac{x}{t} = c_0$, we find a stationary point for $k_s = 0$. This is a second order stationary point in

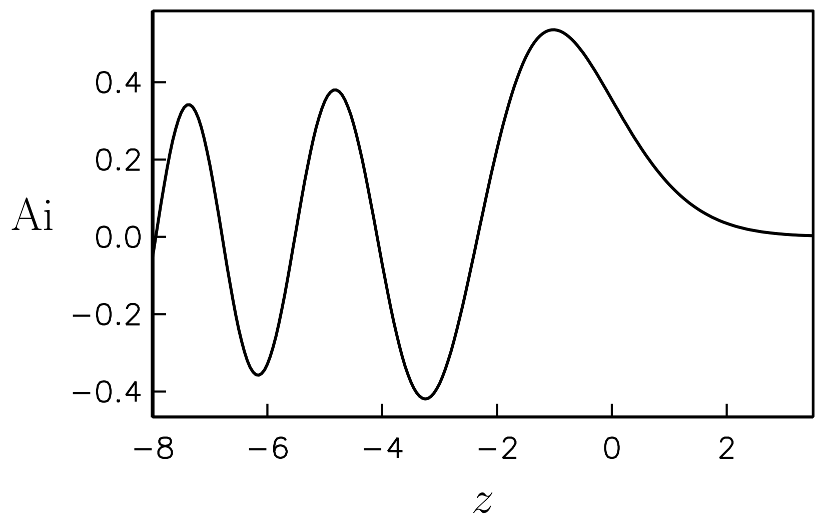

Figure 2.19: The Airy function.

the sense that also $\chi''(k_s) = 0$. We therefore continue the Taylor expansion to the third term and find

$$\chi \approx kx - c_0 \left(k - \frac{H^2}{6} k^3 \right) t \tag{2.81}$$

and the integral becomes

$$\eta(x,t) \approx \frac{1}{2} \int_{-\infty}^{\infty} \hat{\eta}_0(0) \mathrm{e}^{i\left(kx - (c_0 k - \frac{H^2}{6} c_0 k^3)t\right)} \, \mathrm{d}k. \tag{2.82}$$

The above integral can be expressed in closed form by the Airy function

$$\mathrm{Ai}(z) = \frac{1}{\pi} \int_0^{\infty} \cos \left(\frac{1}{3} s^3 + zs \right) \, \mathrm{d}s \tag{2.83}$$

which is a solution of the Airy equation

$$\frac{d^2 F}{dz^2} - zF = 0. \tag{2.84}$$

The Airy function plays an important role in applied mathematics, having an exponential behavior for $z > 0$ and an oscillatory behavior for $z < 0$. The Airy function often shows up as an approximation to the outer edge of a wave pattern or wave front. The Airy function Ai is a solution of (2.84) that is exponentially decreasing when $z \to \infty$ and that fulfills the normalization condition $\int_{-\infty}^{\infty} F \, \mathrm{d}z = 1$. The Airy function is rendered in figure 2.19.

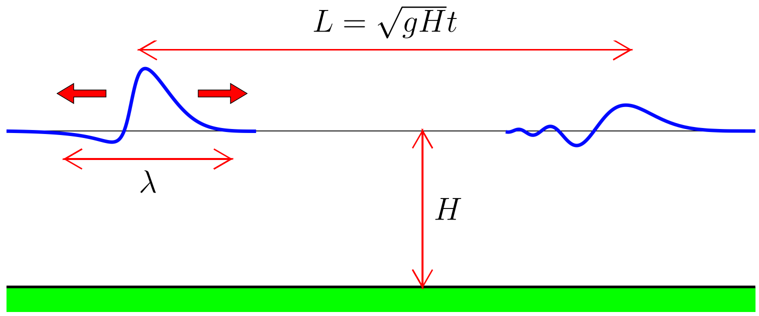

Figure 2.20: Definition sketch of initial conditions and wave spreading.

Near $x = c_0 t$ we find the asymptotic approximation

$$\eta \sim \frac{\pi\hat{\eta}(0)}{(\frac{1}{2}c_0 H^2 t)^{\frac{1}{3}}} \mathrm{Ai}\left(\frac{x - c_0 t}{(\frac{1}{2}c_0 H^2 t)^{\frac{1}{3}}}\right) \tag{2.85}$$

where $\hat{\eta}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \eta(x,0)\,dx$.

When $\frac{x-c_0t}{(\frac{1}{2}c_0H^2t)^{\frac{1}{3}}} \to -\infty$ we can use the asymptotic expression for Ai. This can be used to match the asymptotic approximation with the normal stationary phase approximation for $x \to c_0 t^-$. We skip the details, but we see instead an example of this sort of spreading in tsunamis (figure 2.20). The mathematical solution is presented in figures 2.21 and 2.22. After the theory of long waves has been worked through, it is a good idea to look closer at the relationship between the results.

Exercises

1. In the range between gravity and capillary waves, there is one particular wavenumber k_0 at which the dispersion relation $\omega(k)$ has a turning point, i.e. where $\omega''(k_0) = 0$. Waves with this wavenumber k_0 have the smallest group velocity achieved in the wave train, therefore we can anticipate that this wavenumber could represent the rear end of a wave train. Find an approximate solution for such a rear end of a wave train.

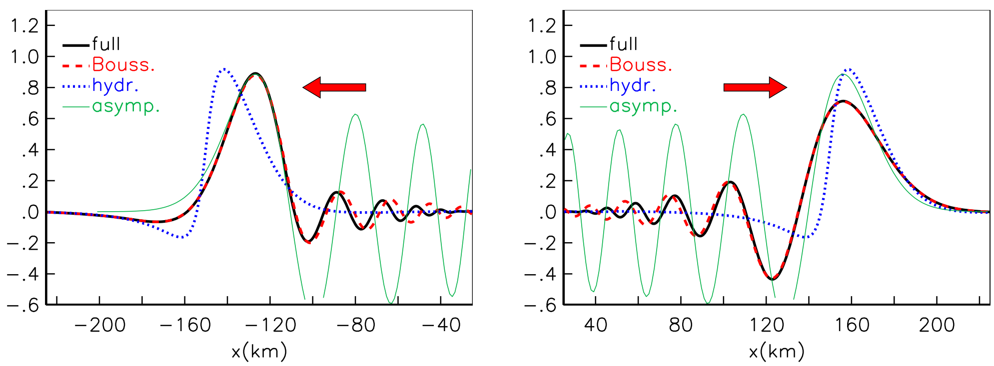

Figure 2.21: Spreading of surface waves generated by an earthquake (tsunami). Surface displacement (meter) is drawn after $t = 11.3$ minutes which corresponds to a distance $L = c_0 t = 150$ km when $H = 5$ km. The curve marked "full" is an exact numeric solution of the Laplace equation with boundary conditions, "Bouss" is a solution of the Boussinesq equation (for explanation of *Boussinesq equation* see section 2.11.1), "hydr." is a solution of the shallow water equations, while "asymp" is the solution given by (2.85). We note that "hydr." has results of the same form as the initial conditions, but has half the amplitude.

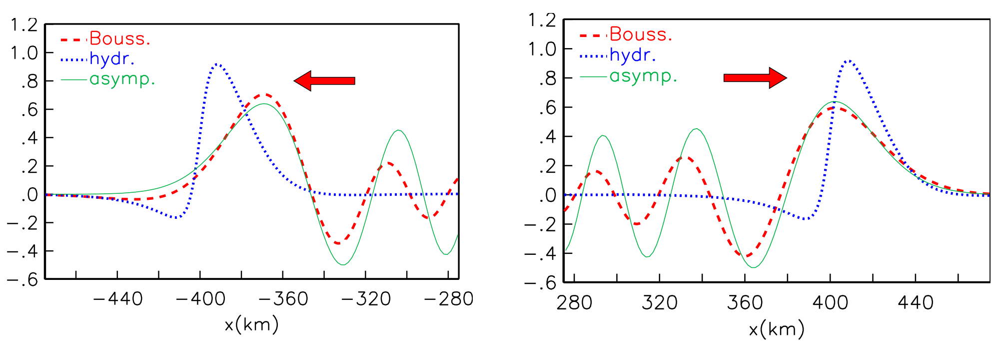

Figure 2.22: Surface displacement (m) after $t = 30$ minutes, $L = 400$ km. We note that the difference between the numeric results and the asymptotic results are reduced in the second figure 2.21.

2.10 Long waves in shallow water

If the wavelength is large in comparison to the depth and the amplitude is sufficiently small, the water particles move in long funnels of elliptical paths and the horizontal velocity is much larger than the vertical velocity, then the waves are considered to be long waves in shallow water. Then the pressure in the fluid is approximately hydrostatic and determined by the pressure of the overlaying fluid. These relationships can be exploited to derive the equation that describes the propagation of long waves in shallow water.

It is convenient to introduce a notation that distinguishes between vertical and horizontal components of the gradient and the velocity vector

$$\mathbf{v} = \mathbf{v}_h + w\boldsymbol{i}_z, \quad \nabla = \nabla_h + \boldsymbol{i}_z \frac{\partial}{\partial z},$$

where

$$\mathbf{v}_h = u\boldsymbol{i}_x + v\boldsymbol{i}_y, \quad \nabla_h = \boldsymbol{i}_x \frac{\partial}{\partial x} + \boldsymbol{i}_y \frac{\partial}{\partial y},$$

The actual depth is $h(x, y, t)$, whereas H will denote a typical value of h. In long wave theory we will employ a depth integrated version of the continuity equation. To this end we define a general vertical cylindrical volume, S, with a footprint in the xy-plane denoted by Ω. The volume is confined by the bottom, $z = -h$, and reach beyond the free surface η. The volume of fluid within S is given by the integral

$$\mathcal{V} = \iint_{\Omega} (\eta + h) dx dy,$$

While the total flux of volume out through the boundary of S is

$$q = \int_{\Gamma} \int_{-h}^{\eta} \mathbf{v}_h \cdot \mathbf{n} dz ds,$$

where Γ is the circumference of Ω, ds is arc length along Γ, and $\mathbf{n}$ is a unit normal vector which is horizontal. Conservation of mass in S implies

$$\frac{d\mathcal{V}}{dt} = -q.$$

We now define the depth integrated velocity

$$\mathbf{U} = U\boldsymbol{i}_x + V\boldsymbol{i}_y = \int_{-h}^{\eta} \mathbf{v}_h dz,$$

Since $\mathbf{n}$ is independent of z, mass conservation may be expressed as

$$\iint\limits_{\Omega} \frac{\partial}{\partial t}(\eta + h)dxdy = -\int\limits_{\Gamma} \mathbf{U}\cdot\mathbf{n}ds.$$

Use of Gauss theorem then yields

$$\iint\limits_{\Omega} \left\{\frac{\partial}{\partial t}(\eta + h) + \nabla_h\cdot\mathbf{U}\right\} dxdy = 0.$$

This is valid for any Ω. Hence

$$\frac{\partial\eta}{\partial t} + \frac{\partial h}{\partial t} = -\nabla_h\cdot\mathbf{U}. \tag{2.86}$$

This is an exact depth integrated continuity equation, but it is applicable only if we have some kind of expression for $\mathbf{U}$.

Alternatively we may derive (2.86) by direct integration of the continuity equation on the form

$$\frac{\partial w}{\partial z} = -\nabla_h\cdot\mathbf{v}_h.$$

Integration from bottom to surface yields

$$w(x,y,\eta,t) - w(x,y,-h,t) = \int\limits_{-h}^{\eta} \frac{\partial w}{\partial z}dz = -\int\limits_{-h}^{\eta} \nabla_h\cdot\mathbf{v}_h dz. \tag{2.87}$$

The kinematic condition at the free surface

$$w(x,y,\eta,t) = \frac{\partial\eta}{\partial t} + \mathbf{v}_h(x,y,\eta,t)\cdot\nabla_h\eta,$$

and the kinematic condition at the bottom

$$w(x,y,-h,t) = -\left(\frac{\partial h}{\partial t} + \mathbf{v}_h(x,y,-h,t)\cdot\nabla_h h\right),$$

are invoked for the left hand side of (2.87). Then the rightmost integral in (2.87) is rewritten

$$\int\limits_{-h}^{\eta} \nabla_h\cdot\mathbf{v}_h dz = \nabla_h\cdot\int\limits_{-h}^{\eta} \mathbf{v}_h dz - \nabla_h\eta\cdot\mathbf{v}_h(x,y,\eta,t) - \nabla_h h\cdot\mathbf{v}_h(x,y,-h,t).$$

Recognizing the integral for $\mathbf{U}$ and inserting in (2.87) we observe that some terms cancel out and

$$\frac{\partial\eta}{\partial t} + \frac{\partial h}{\partial t} = -\nabla_h\cdot\mathbf{U}$$

is again obtained.

The above derivation is valid for a moving depth. However, from this point we will assume that the depth is independent of time, $\frac{\partial h}{\partial t} = 0$.

Euler's equations of motion (horizontal and vertical) read

$$\begin{aligned}\frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} &= -\frac{1}{\rho}\nabla_h p, \\ \frac{\mathrm{D}w}{\mathrm{D}t} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,\end{aligned}$$

where $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{v}_h \cdot \nabla_h + w\frac{\partial}{\partial z}$. Integration of the z component from the surface (where we let $p_a = 0$) yields

$$p = \rho g(\eta - z) - \rho \int_\eta^z \frac{\mathrm{D}w}{\mathrm{D}t} dz. \tag{2.88}$$

The term $g\eta$ balances the extra weight from surface elevation, while the last term comes from vertical accelerations. In the linear case we may assess the vertical acceleration term. First

$$\rho \int_\eta^z \frac{\mathrm{D}w}{\mathrm{D}t} dz \approx \rho h \frac{\partial w(x, y, 0, t)}{\partial t} \approx \rho h \frac{\partial^2 \eta}{\partial t^2},$$

where we have invoked the kinematic condition at the surface in the last step. For a harmonic wave mode we then have

$$h\frac{\partial^2 \eta}{\partial t^2} = -h\omega^2 \eta.$$

The acceleration term in (2.88) is much less than $\rho g \eta$ if

$$h\omega^2 \ll g \Rightarrow (kh)^2 \ll \frac{gh}{c^2}$$

For long waves $c \approx \sqrt{gh}$ and this then corresponds to $(kh)^2 \ll 1$.

We now assume

$$\rho g \eta \gg \rho \int_\eta^z \frac{\mathrm{D}w}{\mathrm{D}t} dz,$$

also for variable depth and nonlinear waves. Then

$$p = \rho g(\eta - z),$$

and the horizontal part of the momentum equation becomes

$$\frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} = -g\nabla_h \eta.$$

Hence, the horizontal particle acceleration is independent of z which implies that $\mathbf{v}_h$ remains independent of z if initially so; for instance for waves propagating into quiescent water. Consequences are

- $\mathbf{v}_h = \mathbf{v}_h(x, y, t) \to \frac{\mathrm{D}\mathbf{v}_h}{\mathrm{D}t} = \frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h$
- $\mathbf{U} = \int\limits_{-h}^{\eta} \mathbf{v}_h dz = (h + \eta)\mathbf{v}_h$

Substitution of these into the horizontal component of Euler's equation of motion and the depth integrated continuity equation give

$$\begin{aligned} &\frac{\partial \mathbf{v}_h}{\partial t} + \mathbf{v}_h \cdot \nabla_h \mathbf{v}_h = -g \nabla_h \eta, \\ &\frac{\partial \eta}{\partial t} = -\nabla_h \cdot ((h + \eta)\mathbf{v}_h) \,. \end{aligned} \tag{2.89}$$

These are the nonlinear shallow water equations, generally named by the acronym *NLSW*. In these equations the number of spatial free variables is reduced by one in comparison with the Euler's equation and the standard continuity equation. Moreover, the time dependence of the fluid domain, which is a challenge for nonlinear solutions of the general equations, is now represented simply by the field $\eta(x, y, t)$. The NLSW equations are hyperbolic and efficient numerical solution procedures are available. These procedures may be extended to include bores, Coriolis effects and bottom drag, but frequency dispersion is lost. Long wave models are the tools of the trade in Ocean modeling, including tides, tsunamis and storm surges.

For plane waves the NLSW equations reduce to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial \eta}{\partial x} \tag{2.90}$$

and

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}[u(h + \eta)]. \tag{2.91}$$

These equations will be discussed again in section 7.1.

If we delete the nonlinear terms in (2.89) we obtain

$$\begin{aligned} &\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}, \quad \frac{\partial v}{\partial t} = -g \frac{\partial \eta}{\partial y}, \\ &\frac{\partial \eta}{\partial t} = -\frac{\partial (hu)}{\partial x} - \frac{\partial (hv)}{\partial y}. \end{aligned} \tag{2.92}$$

Velocities are eliminated by applying temporal derivation to the continuity equation. If h is independent of time the time derivatives of the velocities are replaced by means of the components of the momentum equation. Defining $c_0 = \sqrt{gh}$ the result is written

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial}{\partial x}(c_0^2 \frac{\partial \eta}{\partial x}) + \frac{\partial}{\partial y}(c_0^2 \frac{\partial \eta}{\partial y}). \tag{2.93}$$

If the bottom is a plane (c_0 independent of x and y), (2.93) reduces to the normal two-dimensional wave equation for non-dispersive waves. A wave solution of (2.92) can where c_0 is constant, be written

$$\begin{aligned} \eta &= a \sin k(x - c_0 t), \\ u &= \frac{a}{H} c_0 \sin k(x - c_0 t), \\ v &= 0. \end{aligned}$$

With the help of this solution, we can easily find the conditions for the linearized solutions to be valid. Linearization implies, for example, that the term $u\frac{\partial u}{\partial x}$ is removed in comparison with $\frac{\partial u}{\partial t}$. The relationship between the first and last of these terms is maximum at a/H. The nonlinear term can therefore be easily cut away from the relation when $a/H \ll 1$. This is in good agreement with what we have found earlier in section 2.2.

We shall now see how the linearized shallow water equation can be used to study the propagation of waves over a sloping bottom. When plane waves and and a plane bathymetry, meaning $h = h(x)$, are assumed (2.93) simplifies to

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(gh(x)\frac{\partial \eta}{\partial x}\right) = 0,$$

Since the coefficients depend only on x we may find standing waves on the form (separation of variables)

$$\eta(x,t) = \hat{\eta}(x)\sin(\omega t + \Delta),$$

where ω is given and Δ may chosen freely. Substitution into the wave equation given above then gives

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(h(x)\frac{\mathrm{d}\hat{\eta}}{\mathrm{d}x}\right) + \frac{\omega^2}{g}\hat{\eta} = 0.$$

In general this equation cannot be solved in closed form, but analytic solutions do exist for special $h(x)$. Most studied is the plane slope

$$h = \alpha x$$

where $\alpha = \tan\theta$, and θ is the angle of inclination of the bottom with respect to the horizontal plane (x-axis). In fact, some other bottom profiles yield simpler solutions, but the linear one is much used in the literature.

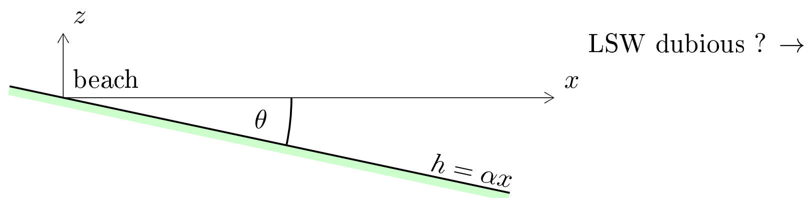

Figure 2.23: Linear profile $h = \alpha x$

With the linear profile we obtain the following equation for $\hat{\eta}$

$$\frac{\partial}{\partial x}(x\frac{\partial \hat{\eta}}{\partial x}) + \kappa\hat{\eta} = 0 \tag{2.94}$$

where $\kappa = \omega^2/\alpha g$. Ordinary second order equations with low order polynomials as coefficients may often be transformed to equations of standard form with known solutions. For (2.94) the substitution $s = 2\sqrt{\kappa x}$ yields

$$s\frac{\partial^2 \hat{\eta}}{\partial s^2} + \frac{\partial \hat{\eta}}{\partial s} + s\hat{\eta} = 0$$

which is known as the Bessel equation of zeroth order and the general solution is

$$\hat{\eta} = A\mathrm{J}_0(s) + B\mathrm{Y}_0(s). \tag{2.95}$$

Here J_0 and Y_0 are the Bessel and Neumann functions, respectively. These functions are described in mathematical handbooks and are available in software libraries for computation. For small s we have the Fröbenius series (see, for instance, Bender & Orzag: Advanced mathematical methods for scientists and engineers.)

$$\mathrm{J}_0(s) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{4}s^2\right)^n}{(n!)^2}, \quad \mathrm{Y}_0(s) = \frac{2}{\pi}(\log(\frac{1}{2}s) + \gamma)\mathrm{J}_0(s) + \frac{1}{\pi}\sum_{n=0}^{\infty} a_n s^{2n},$$

where γ and a_n are constants. We observe that J_0 is analytic at $s = 0$, while Y_0 is singular there. For large s we have the asymptotic approximations

$$\mathrm{J}_0(s) \sim \sqrt{\frac{2}{\pi s}}\cos(s - \frac{\pi}{4}), \quad \mathrm{Y}_0(s) \sim \sqrt{\frac{2}{\pi s}}\sin(s - \frac{\pi}{4}).$$

The Bessel and Neumann functions are depicted in figure 2.24.

If we stay away from the shoreline, meaning that $x = 0$ is excluded from the domain, both A and B may be non-zero in (2.95). Using the the asymptotic approximations for large $s = 2\sqrt{\kappa x}$ we may rewrite the corresponding approximation for $\hat{\eta}$ as

$$\hat{\eta} \sim a_0\,(\kappa x)^{-\frac{1}{4}}\cos(2\sqrt{\kappa x} + \delta) \equiv a(x)\cos\chi,$$

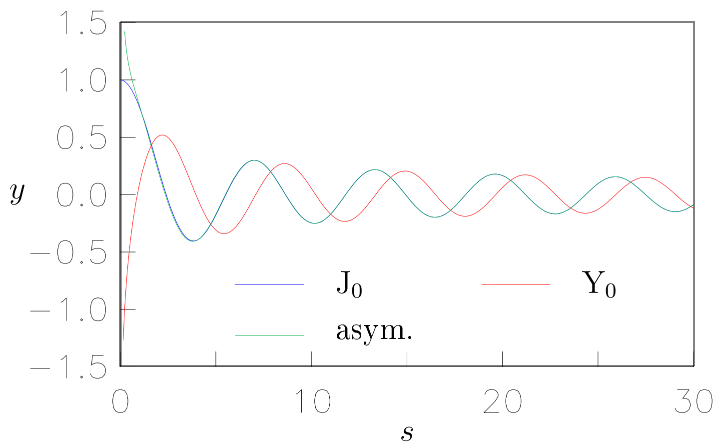

Figure 2.24: The Bessel and Neumann functions of zeroth order. Also the asymptotic approximation to the Bessel function is included.

where a_0 and δ are related to A and B. The amplitude part of $\hat{\eta}$, a, is proportional to $h^{-\frac{1}{4}}$. This is a special case of Green's law. The rate of change of the phase part ($\frac{\mathrm{d}\chi}{\mathrm{d}x}$) decreases with x, while $\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2}$ decreases even faster. Hence, we may define a local wave length for large x through $\lambda\frac{\mathrm{d}\chi}{\mathrm{d}x} = 2$ which gives

$$\lambda = 2\pi\sqrt{\frac{x}{\kappa}}.$$

Hence, the wave-length increases with x (and thus h).
The solutions for large x (large s) are nearly periodic in x with slow variations of amplitude and wave length. In a later chapter we will return to such solutions in the more general context of ray theory.

In the preceding expressions a_0, δ and Δ may be chosen freely. Each choice then gives a standing wave. Restoring the time dependence and using trigonometric formulas we may combine such standing waves to a general solution for propagating waves

$$\eta = a_1(\kappa x)^{-\frac{1}{4}}\cos(2\sqrt{\kappa x} - \omega t + \delta_1) + a_2(\kappa x)^{-\frac{1}{4}}\cos(2\sqrt{\kappa x} + \omega t + \delta_2),$$

where a_1, a_2, δ_1 and δ_2 are arbitrary. It is stressed that this is still an approximation for large $\sqrt{\kappa x}$.

When $x = 0$ (the beach) is *included* we must expect that the solution is a standing wave consisting of an incident and a reflected wave. Now, since $\mathrm{Y}_0(s)$ is singular at $s = 0$ ($x = 0$) we must require $B = 0$ in (2.95) to avoid singularity in $\hat{\eta}$. Hence the solution becomes

$$\hat{\eta} = A\mathrm{J}_0(2\sqrt{\kappa x}), \quad \eta = A\mathrm{J}_0(2\sqrt{\kappa x})\cos(\omega t + \Delta),$$

which is a simple solution for runup of period waves on a sloping beach. Away from the shoreline, we may again invoke the asymptotic expression for J_0. The solution is depicted in figures 2.25 and 2.26.

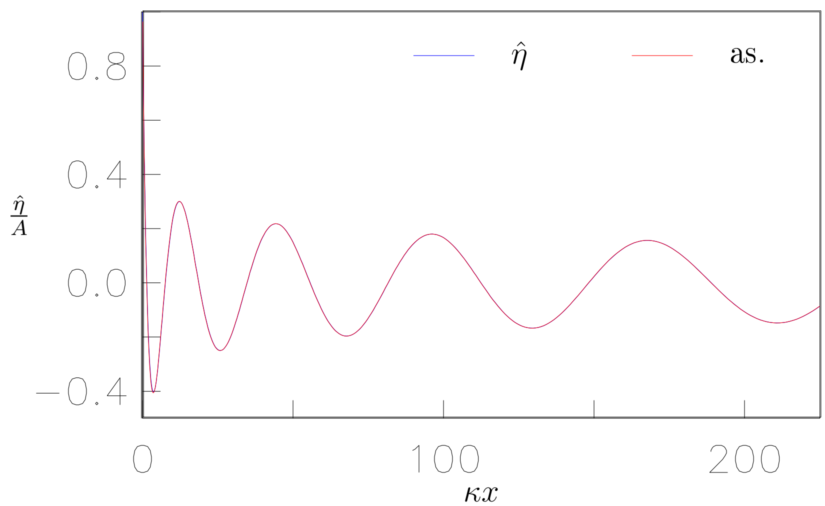

Figure 2.25: The incident and reflected waves from the shore combined to a standing wave at a time for maximum runup. The use of the asymptotic expression for J_0 produces the curve marked *as.*

Exercises

1. Derive (2.93) from (2.92).

2. In an infinitely long canal with a plane bottom and side walls which are parallel to each other, the water depth is H when the water is at rest, and the width of the canal is B. We set the x-axis along the middle of the canal and the y-axis perpendicular to this at the sidewalls. Assume that $kH \ll 1$, and show that waves with surface displacement

$$\eta = \hat{\eta}(y)e^{i(kx-\omega t)}$$

are valid under the given conditions. Determine $\hat{\eta}(y)$, and show that the wave motion can be thought of as waves that are reflect from the side walls.

3. We assume two-dimensional wave motion in a fluid layer where the bottom is a horizontal plane and the depth is H. On the bottom there lies a submerged block with height h and length L. Towards the block a sinusoidal wave is approaching with amplitude a and wavelength λ. Find the amplitude of the wave that is reflected by the block, and the wave that

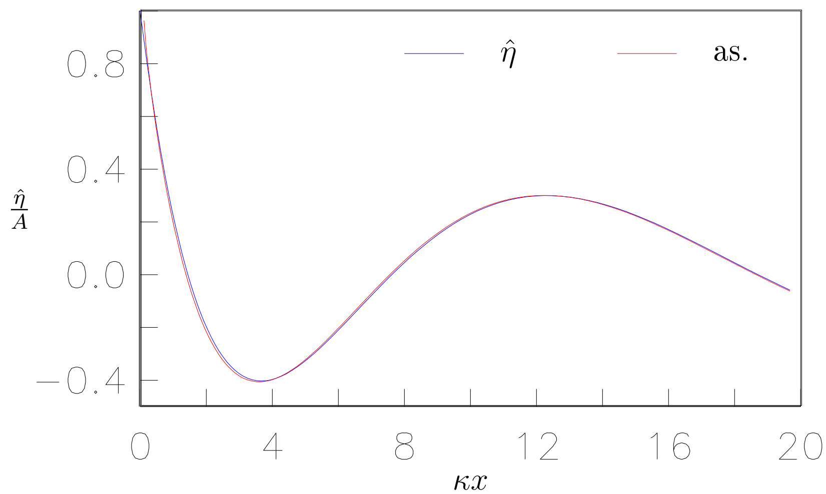

Figure 2.26: Near-shore blow-up of the solution in figure 2.25. The deviation between the full and the asymptotic solution is slightly visible near $x = 0$.

propagates into the area behind the block. We assume that the linearized hydrostatic shallow water theory is valid for the entire area. Find the maximum force that acts on the block in the horizontal direction. Insert values characteristic for the foundation for the oil platform Statfjord C: $h = 60$m, $L = 150$m and long period swell $\lambda = 800$m and $a = 1$m. The water depth is $H = 150$m. Determine the force on the foundation. What are the primary reasons why this estimate must be presumed to be laden with large errors?

4. **Reflection from a shelf.**

   A bottom profile is given as

$$h = \begin{cases} h_1 & x < 0 \\ h_2 & x > 0 \end{cases}$$

   We assume that linear hydrostatic shallow water theory is applicable both to the right and to the left of the shelf ($x = 0$). At the shelf we assume that the surface displacement and the mass flux are continuous.

   A sinusoidal wave with amplitude A and frequency ω arrives towards the shelf from the left. Find the reflected and the transmitted waves.

5. **Reflection from a slope.**

We have given a depth function

$$h = \begin{cases} h_1 & x < 0 \\ h_1 - \frac{x}{L}(h_1 - h_2) & 0 < x < L \\ h_2 & x > L \end{cases}$$

We assume that a linear hydrostatic shallow water theory is applicable for all x. A sinusoidal wave with amplitude A and frequency ω is approaching from the left, $x < 0$. Find the resulting wave system.

You will have to differentiate the two linearly independent solutions of Bessel's differential equations:

$$\frac{\partial J_0}{\partial x} = -J_1, \quad \frac{\partial Y_0}{\partial x} = -Y_1.$$

To approximate different asymptotic behaviors you will need the equations

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{4} - n\frac{\pi}{2}), \quad Y_n(x) \sim \sqrt{\frac{2}{\pi x}} \sin(x - \frac{\pi}{4} - n\frac{\pi}{2}),$$

for large x and

$$J_0(x) \sim 1, \quad J_1(x) \sim \frac{1}{2}x, \quad Y_0(x) \sim \frac{2}{\pi} \ln x, \quad Y_1(x) \sim -\frac{2}{\pi x},$$

for small x.

6. **Reflection of a pulse from a shelf.**

A bottom profile is given as

$$h = \begin{cases} h_1 & x < 0 \\ h_2 & x > 0 \end{cases}$$

We assume that linear, hydrostatic shallow water theory is applicable on both sides of the jump ($x = 0$). A wave pulse approaches from the left and is given as

$$\eta(s) = \begin{cases} 0 & s < -L \\ A\cos^2(\pi\frac{s}{2L}) & -L < s < L \\ 0 & L < s \end{cases}$$

where $s = x - c_1 t$. We assume that the pulse will be split into a transmitted and a reflected pulse. We further assume that both of these have a form similar to the original pulse, but with other values for L and A. Which lengths should these pulses have? Find the amplitudes by insisting on conservation of mass and energy.

7. **Energy transfer in long waves in shallow water.**

   In this we shall exercise assume non-linear long wave theory (NLSW).

   (a) Find the energy density E per horizontal surface unit by direct calculation. (Energy in vertical columns of water.)
   (b) Find, correspondingly, the horizontal components of the energy flux $\boldsymbol{F}$ integrated over the entire depth.
   (c) What mathematical relation must exist between E and $\boldsymbol{F}$? Show that this is correct by use of the NLSW equations.
   (d) We now assume a flat bottom, linear equations and a harmonic, progressive wave. Show how the averaged E and $\boldsymbol{F}$ now can be related by the group velocity.

2.11 Derivation of dispersive long wave equations

2.11.1 Derivation of Boussinesq equations

We assume two-dimensional motion with the x-axis horizontal and the z-axis vertical. The bottom is variable and described by $z^\star = -h^\star(x^\star)$, where $\star$ denotes magnitudes with dimension. We can make the equation dimensionless with the following considerations:

- A typical depth, H, is used to scale vertical magnitudes.
- A typical wavelength, ℓ, is used to scale horizontal magnitudes.
- A typical amplitude, a, characterizes the vertical displacement of the surface elevation. The small dimensionless parameter $\alpha = a/H$ is used to characterize the smallness of all field quantities.

Linearization requires that α is small. The long wave approximation requires

$$\epsilon \equiv \frac{H^2}{\ell^2} \tag{2.96}$$

is small. These parameters will appear in the derivation below. In the following we will not introduce a formal perturbation expansion for the unknowns, rather we will iterate the governing equations.

We employ the following scalings:

$$\begin{array}{lll} z^\star = Hz & x^\star = \ell x & t^\star = \ell (gH)^{-\frac{1}{2}} t \\ h^\star = Hh(x) & \eta^\star = \alpha H \eta & u^\star = \alpha (gH)^{\frac{1}{2}} u \\ w^\star = \epsilon^{\frac{1}{2}} \alpha (gH)^{\frac{1}{2}} w & p^\star = \rho g H p & \end{array}$$

At the surface $z = \alpha\eta$ we have the dynamic and kinematic boundary conditions

$$p = 0, \qquad \eta_t + \alpha u \eta_x = w. \tag{2.97}$$

At the bottom we have the kinematic boundary condition

$$w = -h_x u. \tag{2.98}$$

Within the fluid we have Euler's equations of motion

$$u_t + \alpha u u_x + \alpha w u_z = -\alpha^{-1} p_x \tag{2.99}$$
$$\epsilon(w_t + \alpha u w_x + \alpha w w_z) = -\alpha^{-1}(p_z + 1) \tag{2.100}$$

and the continuity equation

$$u_x + w_z = 0. \tag{2.101}$$

Eliminating terms of order ϵ we obtain the hydrostatic description. Equation (2.100) then gives the pressure

$$p = \alpha\eta - z + O(\alpha\epsilon) \tag{2.102}$$

which implies that $p_x = \alpha\eta_x + O(\alpha\epsilon)$, and therefore the horizontal pressure gradient is independent of z to the leading order of ϵ. It follows that all of the particles in a vertical cross-section, x=constant, have the same horizontal acceleration. If they start out with the same horizontal velocity, e.g. starting from rest, they necessarily have the same horizontal velocity at all later times, to the leading order in ϵ. We can now write

$$u_z = O(\epsilon) \tag{2.103}$$

or $u \approx u(x,t)$. Equation (2.99) gives

$$u_t + \alpha u u_x = -\eta_x + O(\epsilon). \tag{2.104}$$

We will now derive a higher order equation by assuming that both ϵ and α are small. This means that we retain the terms of order α and ϵ, but the terms which contain products of these or ϵ^2 can be dropped. We start by defining a vertical average

$$\overline{u} = (h + \alpha\eta)^{-1} \int_{-h}^{\alpha\eta} u \,\mathrm{d}z. \tag{2.105}$$

A consequence of (2.103) is that $\overline{u}(x,t) - u(x,z,t) = O(\epsilon)$. An average depth continuity equation can now be written

$$\eta_t = -\{(h + \alpha\eta)\overline{u}\}_x \tag{2.106}$$

which is exact.

A correction to the hydrostatic pressure can be found from (2.100). We can make a correction of the order ϵ and α into the parenthesis in the left side, without that, the pressure finds the relative corrections of the lower orders in $\alpha\epsilon$ and ϵ^2. First we must find an expression for w. From (2.97), (2.103) og (2.101) follows

$$w = \eta_t - z\overline{u}_x + O(\alpha, \epsilon). \tag{2.107}$$

Setting these into (2.100) we find

$$p = \alpha\eta - z - \epsilon\alpha(z\eta_{tt} - \frac{1}{2}z^2\overline{u}_{xt}) + O(\alpha\epsilon^2, \alpha^2\epsilon). \tag{2.108}$$

From (2.106) we have $\eta_{tt} = -(h\overline{u})_{xt} + O(\alpha)$ which can be used to write the expressions for p. The next step is to average the horizontal components of the equation of motion (2.99) over the fluid depth. Averaging the first term in the left side gives

$$\overline{(u_t)} = (h + \alpha\eta)^{-1} \int_{-h}^{\alpha\eta} u_t \,\mathrm{d}z = \overline{u}_t + \alpha(h + \alpha\eta)^{-1}(\overline{u} - u|_{z=\alpha\eta})\eta_t. \tag{2.109}$$

Equation (2.103) now gives the last terms of order $\alpha\epsilon$ which can be dropped. For the next term in (2.99) we find equivalently

$$\overline{(\frac{1}{2}\alpha(u^2)_x)} = \alpha\frac{1}{2}\left(\overline{(u^2)}\right)_x + O(\alpha^2\epsilon). \tag{2.110}$$

A consequence of (2.103) is that we can write $u = \overline{u} + \epsilon u_1$ where u_1 is of order 1, but the average is zero. A simple calculation gives

$$\frac{1}{2}\left(\overline{(u^2)}\right)_x = \frac{1}{2}\overline{(\overline{u}^2 + 2\epsilon\overline{u}u_1)}_x + O(\epsilon^2) = \overline{u}\,\overline{u}_x + O(\epsilon)^2. \tag{2.111}$$

The last contribution to acceleration in (2.99) can be dropped immediately; this follows from (2.103). We set in p from (2.108) in the left side, and by averaging these we find

$$\overline{u}_t + \alpha\overline{u}\,\overline{u}_x = -\eta_x + \epsilon\{\frac{1}{2}h(h\overline{u}_t)_{xx} - \frac{1}{6}h^2\overline{u}_{xxt}\} + O(\epsilon^2, \alpha\epsilon). \tag{2.112}$$

These together with (2.106) constitute a set of two equations in x and t for the unknowns η and $\overline{u}$. The equations can be presented in many different forms by rewriting the higher order terms, or by introducing $U = (h + \alpha\eta)\overline{u}$ as unknown. Notice that (2.112) is valid for variable depth. In relation to the NLSW equations the important new feature of (2.112) is that the new term on the right-hand side gives dispersion. If we delete the nonlinear terms, assume constant depth, and insert a harmonic mode in (2.112) and (2.106) we find the dimensionless dispersion relation

$$\omega^2 = \frac{k^2}{1 + \frac{\epsilon}{3}k^2}.$$

Restoring the dimensions we may recast this into

$$\omega^2 = \frac{c_0^2 k^2}{1 + \frac{1}{3}(kh)^2},$$

where $c_0 = \sqrt{gh}$.

The relative importance of nonlinear relative to dispersion effects in (2.112) and (2.106) is measured by the Ursell parameter

$$U_r = \frac{\alpha}{\epsilon}.$$

If U_r is small dispersion effects dominate over nonlinear effects, while nonlinearity dominates for large U_r. If we insert the harmonic wave mode from the linear solution into the different terms of the equations it is seen that nonlinear and dispersive terms are of comparable magnitude when $U_r \approx 10$.

2.11.2 Derivation of KdV-equation

If waves are propagating in one direction on constant depth, the Boussinesq-equations can be replaced by the KdV (Korteweg–de Vries) equation. This equation can be derived in several ways:

1. Use a nonlinear term from hydrostatic nonlinear theory and the leading dispersive term from the linear dispersion relation. These can be combined directly.

2. Change the coordinate system such that it propagates with the linear shallow water speed c_0. For one-directional waves all scalars will change slowly in time in this system.

3. Find corrections for the Riemann-invariants within the framework of Boussinesq-equations.

Method 1 will be scrutinized later, but here method 2 will be derived in detail.

We will do two things: We modify the description such that η is the only unknown, and we will reduce the order of the equation. First, the change of coordinates

$$\xi = x - t, \qquad \tau = \epsilon t. \tag{2.113}$$

The wave propagates towards increasing x and the small factor ϵ in front of t denotes the slow time variation in the new coordinate system.

We could have chosen α instead of ϵ for rescaling of the time as these parameters are assumed to be of the same order. The change of coordinates in the continuity equation, (2.106), gives

$$\epsilon\eta_\tau - \eta_\xi = -(1+\alpha\eta)\overline{u}_\xi - \alpha\overline{u}\eta_\xi \tag{2.114}$$

where we have limited the depth to be constant, $h \equiv 1$. To the leading order: $\eta_\xi = \overline{u}_\xi + O(\alpha, \epsilon)$. If η and $\overline{u}$ are equal somewhere, e.g. outside a finite wave disturbance, they will be similar to this order everywhere. The equation above can therefore be rewritten within the accuracy used herein

$$\epsilon\eta_\tau - \eta_\xi = -\overline{u}_\xi - 2\alpha\eta\eta_\xi + O(\epsilon^2, \alpha\epsilon). \tag{2.115}$$

Using this, most of $\overline{u}$ can be eliminated in the transformed equation of motion. Equation (2.112) can be rewritten as

$$\overline{u}_\xi = \eta_\xi + \epsilon\eta_\tau + \alpha\eta\eta_\xi + \frac{1}{3}\epsilon\eta_{\xi\xi\xi} + O(\epsilon^2, \alpha\epsilon). \tag{2.116}$$

Equation (2.115) can now be rearranged as a variant of the KdV equation

$$\epsilon\eta_\tau + \frac{3}{2}\alpha\eta\eta_\xi + \frac{1}{6}\epsilon\eta_{\xi\xi\xi} = O(\epsilon^2, \alpha\epsilon). \tag{2.117}$$

This equation expresses the balance between slow time variation, nonlinearity and dispersion. If we introduce the coordinates x and t in (2.117) we find

$$\eta_t + (1 + \frac{3}{2}\alpha\eta)\eta_x + \frac{1}{6}\epsilon\eta_{xxx} = O(\epsilon^2, \alpha\epsilon), \tag{2.118}$$

which is the same as will be presented later. This can be rewritten in different ways. For example we can use the leading order balance in (2.118), $\eta_t + \eta_x = O(\epsilon, \alpha)$, to rewrite the dispersive term

$$\eta_t + (1 + \frac{3}{2}\alpha\eta)\eta_x - \frac{1}{6}\epsilon\eta_{xxt} = O(\epsilon^2, \alpha\epsilon), \tag{2.119}$$

which is better to solve numerically. The different variants of the KdV equation will not give the same answer, but they should give solutions that are consistent within the common order of accuracy.

It is possible to generalize the KdV-equation such that a slowly varying depth, $h_x = O(\epsilon, \alpha)$, can be included.

2.12 The effect of viscosity on surface waves

So far we have dealt with an ideal fluid, without friction, and thereby neglected the effect of viscosity on the wave motion. Even though we may do this with good approximation for many wave phenomena, the viscosity can in some cases have a dominant effect on the wave motion. We should therefore take a closer look at the effect of friction on surface waves.

We shall here suffice by considering long-crested surface waves in a homogeneous and incompressible Newtonian liquid of infinite depth. The linearized governing equations are

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{2.120}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\nabla^2 w - g \tag{2.121}$$

and the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0.$$

In these equations, and in the following equations in this section, the symbol ∇^2 denotes the two-dimensional Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}.$$

ν is the kinematic viskosity.

We introduce the potential ϕ and the stream function ψ and write

$$u = \frac{\partial \phi}{\partial x} - \frac{\partial \psi}{\partial z}, \qquad w = \frac{\partial \phi}{\partial z} + \frac{\partial \psi}{\partial x}. \tag{2.122}$$

Both the vorticity equation, which can be found by eliminating the pressure from (2.120) and (2.121), and the continuity equation are satisfied if

$$\frac{\partial \psi}{\partial t} - \nu\nabla^2\psi = 0 \tag{2.123}$$

and

$$\nabla^2\phi = 0. \tag{2.124}$$

Thereby it follows from (2.120) and (2.121) that the pressure p can be written

$$p = -\rho(\frac{\partial \phi}{\partial t} + gz) + f(t)$$

where $f(t)$ is an arbitrary function of t which we can merge into ϕ by the well-known integral technique. If we employ the results from section 2.1, the linearized surface conditions take the form

$$\begin{aligned} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} &= 0 \\ \frac{\partial \phi}{\partial t} + g\eta + 2\nu\frac{\partial w}{\partial z} - \frac{\sigma}{\rho}\frac{\partial^2 \eta}{\partial x^2} &= 0 \\ \frac{\partial \eta}{\partial t} &= w \end{aligned} \tag{2.125}$$

where all three equations are to be evaluated at $z = 0$. The first of these equations expresses that the shear stress vanishes at the surface, the next equation is the condition for the normal stress, and the last equation is the kinematic condition. The surface elevation can be eliminated from the last two equations, and we get

$$\frac{\partial^2 \phi}{\partial t^2} + gw + 2\nu \frac{\partial^2 w}{\partial z \partial t} - \frac{\sigma}{\rho} \frac{\partial^2 w}{\partial x^2} = 0 \tag{2.126}$$

for $z = 0$.

Equations (2.123) and (2.124) have solutions

$$\begin{aligned} \psi &= Ae^{mz} \mathrm{e}^{\mathrm{i}(kx - \omega t)} \\ \phi &= Be^{kz} \mathrm{e}^{\mathrm{i}(kx - \omega t)} \end{aligned} \tag{2.127}$$

where A, B, k and ω are constants, and

$$m = (k^2 - \frac{\mathrm{i}\omega}{\nu})^{\frac{1}{2}}$$

For the motion to die out when $z \to -\infty$ we insist that the real part of m is positive. The corresponding velocity components can be found by substituting the expressions (2.127) into (2.122). The first surface conditon in (2.125) together with equation (2.126) leads to a homogeneous set of equations for the constants A and B. If this set is to have non-trivial solutions for A and B, the determinant must be zero, and this requires the following condition to be satisfied

$$(\frac{c}{c_s} + \mathrm{i}\beta)^2 - 1 + \beta^{3/2}(\beta - 2\mathrm{i}\frac{c}{c_s})^{\frac{1}{2}} = 0 \tag{2.128}$$

where $c_s = (\frac{g}{k} + k\frac{\sigma}{\rho})^{\frac{1}{2}}$ is the phase speed for gravity–capillary waves in a frictionless liquid ($\nu = 0$) and c denotes as before the phase speed ω/k.

The parameter β that appears in equation (2.128) is given by

$$\beta = \frac{2\nu k}{c_s}.$$

For liquids with little viscosity β is very small except for very short wavelengths. Table 2.1 shows some values of β for clean water ($\sigma = 7.4 \cdot 10^{-2}$N/m, $\nu = 10^{-6}$m^2/s).

With the exception of very short wavelengths one, one can eliminate the term that has $\beta^{3/2}$ as factor in equation (2.128), and we get

$$c = c_s(1 - \mathrm{i}\beta).$$

For a real wavenumber we see that the phase speed is complex. The real part of c is the phase speed of the waves, while the imaginary part corresponds to an

λ cm	$\beta \times 10^3$	z_e cm
1	5.01	0.01
10	0.31	0.03
100	0.01	0.05
1000	0.0001	0.09

Table 2.1: Damping at different wavelengths.

exponential damping. By employing these results, the surface elevation can be written

$$\eta = a\mathrm{e}^{-2\nu k^2 t} \cos k(x - c_s t).$$

This expression shows that the propagation speed is not affected by friction, but the friction causes the wave amplitude to be damped and this damping is strongest for short waves. The e-folding time is

$$t_e = \frac{1}{2\nu k^2} = \frac{\lambda^2}{8\pi^2\nu}.$$

Since the period fo the waves is $T = 2\pi/kc_s$ we see that the e-folding time is a number of periods

$$\frac{t_e}{T} = \frac{1}{4\pi}(\frac{c_s}{\nu k}) = \frac{1}{2\pi\beta}.$$

Even for as short waves as $\lambda = 1$cm the e-folding time in water is about 30 periods. In other words, after the waves have propagated a distance of $30\,\lambda$ the amplitude has been reduced by a factor e^{-1}.

For sufficiently small values of β we have

$$m = \pm\frac{k}{\beta^{\frac{1}{2}}}(1 - \mathrm{i}).$$

The potential ψ which describes how the irrotational velocity field is deformed by the action of friction, will in this case contain a damping factor $\exp(\frac{z}{z_e})$ where

$$z_e = \frac{\beta^{\frac{1}{2}}\lambda}{2\pi}.$$

The velocity field is therefore only noticably affected by the friction in a thin zone of thickness z_e near the surface. Some numerical values for z_e for surface wave in water are given in table 2.1. The above computations show that the friction has little influence on long waves, and the e-folding time t_e grows without bound when $k \to 0$. If the depth of the liquid is limited, the longest waves will have appreciable velocities near the bottom. Due to friction there will also be a boundary layer near the bottom affecting the damping of the waves. It is possible to estimate the effect of this damping, but we will not do it here.

2.13 Oscillations in a basin

For linear wave motion the principle of superposition guarantees that a sum of wave compontents is also possble solution. Standing waves appear by adding wave components propagating in opposite directions. If we add the wave component given by (2.15)–(2.16) to another with wavenumber $-k$ and amplitude a, we get

$$\begin{aligned} \eta &= A\cos kx \sin \omega t \\ \phi &= \frac{A\omega}{k\sinh kH}\cosh k(z+H)\cos kx\cos\omega t \end{aligned}$$

where $A = -2a$. If we furthermore require that the horizontal velocity shall be zero at vertical planes at $x = 0$ and $x = L$, the motion is only possible when the wavenumber has certain discrete values k_n given by

$$k_n L = n\pi$$

where $n = (1, 2, 3, \ldots)$. The corresponding value for the angular frequency ω_n is found by setting $k = k_n$ in the dispersion relation (2.17). For gravity wave we get

$$\omega_n = [g\frac{n\pi}{L}\tanh(\frac{n\pi H}{L})]^{\frac{1}{2}}.$$

Thereby one can determine the frequency and the period for standing oscillations in a container with horizontal bottom and vertical walls. The surface elevation at the time $t = 0$ for the two lowest modes, the first harmonic $n = 1$ and the second harmonic $n = 2$, are sketched in figure 2.27. If $\frac{H}{L} \ll 1$, the lowest modes can be approximated by $\tanh(\frac{n\pi H}{L}) \simeq \frac{n\pi H}{L}$, and the period for the oscillations becomes

$$T_n = \frac{2L}{nc_0}.$$

This demonstrates that the longest oscillation period ($n = 1$) corresponds to the time it takes for a wave to move back and forth in the basin.

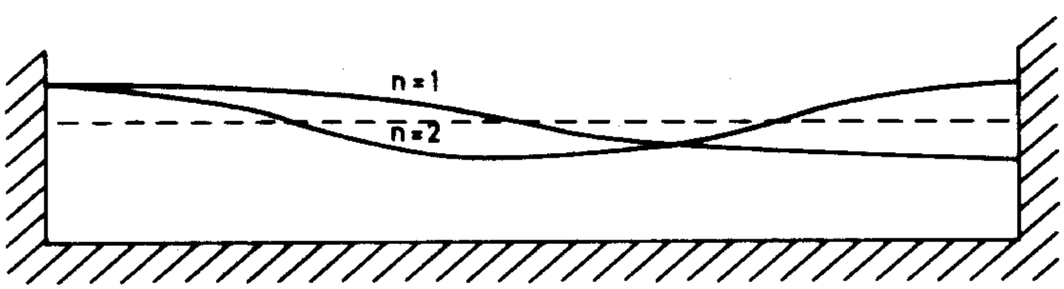

Figure 2.27: Two lowest eigenmods in basin.

Standing oscillations can also appear in a basin where the depth is not uniform. From the solutions of equation (2.94) one may for example find the period for standing oscillations in a basin with linearly sloping bottom. It is left as an exercise to do this. More complicated shapes of basins can present mathematical difficulties for the determination of the period, but there are a number of calculations for various geometries. Interested reader are referred to a review paper by Miles (1974).

Standing wave of the type described here can appear in lakes and harbors. In lakes it is usually the wind that provokes the oscillations, but they may also be caused by avalanches or earthquakes. For example, after an earthquake in India in 1950 there were observed oscillations in varous Norwegian lakes and fiords (Kvale 1955).

For oscillations in harbor basins the analysis is complicated by the fact that the basin is open to the ocean and some of the energy can leak out through the opening. As an example of how the oscillation period can be determined for open basins we refer to one of the following exercises. This type of oscillation has a clear analogy to acoustic oscillations in organ pipes.

The oscillations in harbors, (*Norwegian mariners call it "drag"*), usually arise under certian weather conditions, and the oscillations probably get their energy from periodic disturbances in the ocean outside the basin. The oscillations influence the harbor conditions and have therefore great practical interest. Investigations of the oscillations in two Norwegian harbors, Sørvær in Finnmark and Sirevåg in Rogaland, are described by Viggosson and Rye (1971) and by Gjertveit (1971). In figure 2.28 we see observations of oscillations in Sørvær harbor. The amplitude of the oscillations are up to 0.8 m, and the oscillation time is about 6 min.

Exercises

1. A two-dimensional model of a harbor consists of a shallow part of length

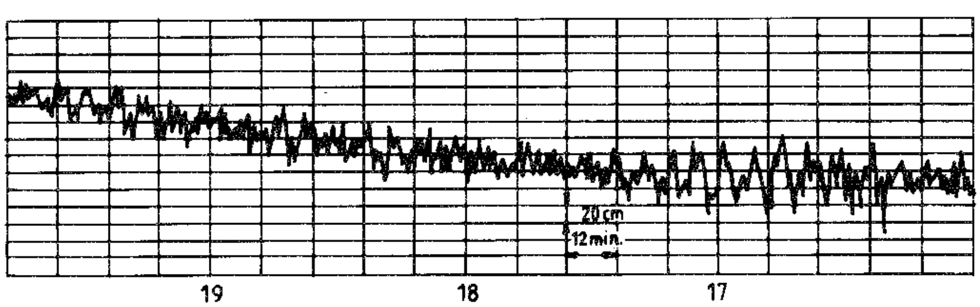

Figure 2.28: Measurement of oscillations in Sørvær harbor. Reproduced from Viggosson and Rye.

b with uniform depth h and an unlimied deep ocean with uniform depth H. We shall investigate the possibility for long periodic oscillations in the harbor at the same time as waves propagate out into the deep ocean. We assume that the oscillations are long periodic and use the linearized hydrostatic shallow water theory. Within the harbor the surface elevation is set to

$$\eta = \hat{\eta}(x)\mathrm{e}^{-\mathrm{i}\omega t}$$

where x is oriented outward from the coast and ω is the angular frequency and $\hat{\eta}(x)$ is a function of x. In the deep ocean we have

$$\eta = a\mathrm{e}^{\mathrm{i}(kx-\omega t)}$$

where the wavenumber $k = \omega/c_2$ and the amplitude a are determined by the oscillations in the harbor. Show that ω is determined by

$$\tan(\frac{\omega b}{c_1}) = \mathrm{i}\frac{c_2}{c_1}$$

where $c_1 = \sqrt{gh}$ and $c_2 = \sqrt{gH}$. Find the oscillation time when $h/H \ll 1$, and show that one in this case has a node for the oscillations at the entrance of the harbor.

2. Show that the velocity potential

$$\phi = a[\sinh ky \sin kz + \sin ky \sinh kz] \cos \omega t$$

describes transversal oscillations in a canal where the sidewalls are straight lines forming an angle of 45° with the vertical. The origin is placed at the deepest point of the canal with the y- and z-axes horizontal and vertical, respectively. Also show that for the lowest mode we have

$$\phi = ayz \cos \omega_1 t$$

and hte angular frequency is determined by

$$\omega_1 = \sqrt{\frac{g}{H}}$$

H denotes the height of the water surface above the origin at equilibrium.

3. **Oscillations in a basin**

A basin has length L and constant depth h. The wave motion in the basin is assumed to be well described by linear potential theory. At the endpoints of the basin η has zero derivative. (What does this mean physcially?)

The initial condition is given by

$$\eta(x,0) = \frac{A}{f^4}(x+f)^2(x-f)^2$$

for $-f < x < f$, $\eta(x,0) = 0$ in the rest of the basin. The motion starts from rest.

(a) Sketch the initial condition and find a suitable Fourier representation for $\eta(x,t)$. (The Fourier coefficients should be computed).

(b) Set $L = 200$ and $f = 10$. Compute and plot the shapes of the surface at $t = 0.25\frac{L}{\sqrt{gh}}$ for $h = 0.1$, $h = 1.0$, $h = 10.0$ and $h = 100.0$. Repeat for $t = 0.75\frac{L}{\sqrt{gh}}$.

(c) Set $L = 0.03$, $h = 0.01$ and $f = 0.001$. Study the solution for various times. How is the pulse broken up?

(d) Can this solution method be generalized to three dimensions? Explain how.

Chapter 3

GENERAL PROPERTIES OF PERIODIC AND NEARLY PERIODIC WAVE TRAINS

Previously we have seen examples of wave trains where the characteristics of the individual waves such as wavelength, frequency, and amplitude change very little from one wave to the next. The gradual changes arose from dispersion because of inhomogeneity in the medium. We have also seen that the viscosity can lead to gradual changes in the wave train.

Nearly periodic wave trains of this type often appear in nature, and these phenomena are, of course, not just tied to waves in fluids. To show this, we will look at an earthquake registered from five sensors at the seismic registration installation NORSAR (figure 3.1). The center of the installation is at Kjeller,

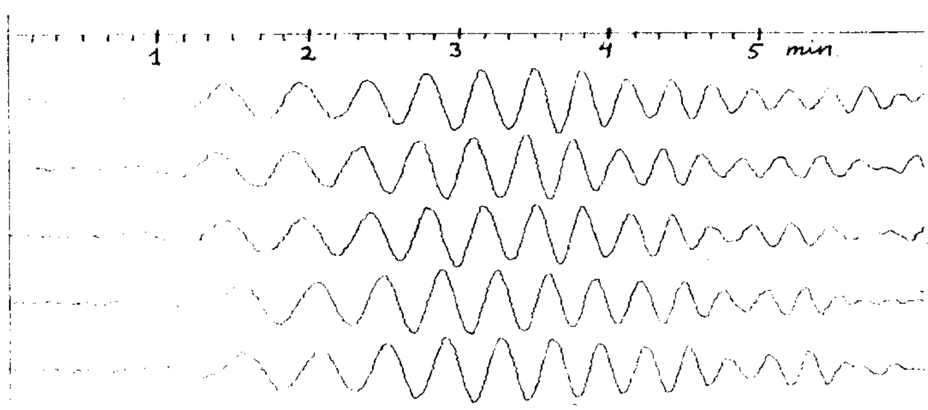

Figure 3.1: Seismograph

outside of Oslo, and the sensors were places over the south-eastern part of Norway.

These waves are seismic waves (Rayleigh waves) which propagate along the earth's surface, and the waves, in this case, are due to an earthquake at the Azores. The regular wave train mainly arises from the dispersion.

3.1 Wave kinematics for one-dimensional wave propagation

We assume that at least one parameter (velocity, pressure, displacement, etc) which describes wave motion can be written in the form

$$A(x,t)\mathrm{e}^{\mathrm{i}\chi(x,t)}. \tag{3.1}$$

This implies that either the real or the imaginary parts of the equation above represent the physical magnitude. From the phase function χ we can determine k and ω by

$$k = \frac{\partial \chi}{\partial x} \qquad \omega = -\frac{\partial \chi}{\partial t}. \tag{3.2}$$

k, ω and A are assumed to be slowy varying functions of x and t in the sense that they vary little over a distance comparable to a wavelength or over a time comparable to a period. k and ω thus represent local values for the wavenumber and angular frequency. The relation to the corresponding quantities for a single, uniform and harmonic modes may be demonstrated by a Taylor expansion

$$\chi(x,t) = \chi(x_0,t_0) + k(x-x_0) - \omega(t-t_0) + ...$$

From this, it is reasonable to assume that ω and k are locally connected by the dispersion relation. In an inhomogeneous medium the dispersion relation involves x and t, in addition to k. Hence, we can write

$$\omega = W(k,x,t).$$

While ω is to be considered as function of x and t, W is a function of k, x and t as defined by the dispersion relation. For example, for long waves in shallow water the frequncy is $k\sqrt{gH}$. In case of a slowly variable depth x and t may enter the dispersion relation explicitly through H. Hence, we write

$$W(k,x,t) = k\sqrt{gH(x,t)}.$$

Generally, a bottom will only have spatial, and not temporal, variations. Explicit time dependence in the dispersion relation is more likely to appear due to interaction with currents such as tides or large scale eddies.

It follows from (3.2) that

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{3.3}$$

Since $k/2\pi$ is the number of waves per unit of length and $\omega/2\pi$ is the number of waves per unit of time, k and ω can be interpreted respectively as wave density and wave flux. Equation (3.3) therefore expresses that the number of waves within a certain interval changes due to the net flux of waves through the interval's end points. Differentiating the dispersion relation, $\omega = W$, with respect to x while holding t constant, we find

$$\frac{\partial \omega}{\partial x} = \frac{\partial W}{\partial k}\frac{\partial k}{\partial x} + \frac{\partial W}{\partial x}.$$

Substitution into (3.3) then gives

$$\frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial x} = -\frac{\partial W}{\partial x} \tag{3.4}$$

where $c_g = \frac{\partial W}{\partial k}$ is the group velocity.

In a homogeneous medium, the dispersion relation will not depend explicitly on x and $\frac{\partial W}{\partial x} = 0$. In this case, (3.4) simplifies to

$$\frac{\partial k}{\partial t} + c_g \frac{\partial k}{\partial x} = 0. \tag{3.5}$$

This shows that in a homogeneous medium, an observer that moves with the group velocity will follow a wave with fixed wavenumber. If we consider a point with coordinate x which moves with the group velocity, then

$$\frac{dx}{dt} = \frac{\partial W}{\partial k} = c_g. \tag{3.6}$$

This defines a curve in the x, t plane that is a characteristic. Introducing the temporal differentiation along a characteristic according to

$$\frac{d()}{dt} = \frac{\partial()}{\partial t} + c_g \frac{\partial()}{\partial x},$$

equation (3.4) can be rewritten as

$$\frac{dk}{dt} = -\frac{\partial W}{\partial x} \tag{3.7}$$

which expresses the change in k with time at a point which moves with velocity c_g.

Differentiation of $\omega = W$ with respect to t yields

$$\frac{\partial \omega}{\partial t} = \frac{\partial W}{\partial k}\frac{\partial k}{\partial t} + \frac{\partial W}{\partial t}.$$

Removing the derivative of k by means of (3.3) and introducing the temporal differentiation along a characteristic we may write

$$\frac{d\omega}{dt} = \frac{\partial W}{\partial t}. \tag{3.8}$$

If the medium does not change in time, W is not explicitly dependent on t and

$$\frac{d\omega}{dt} = 0.$$

The last equation shows that ω is constant when one moves with the group velocity. In other words: A wave with a definite period moves with a velocity that corresponds to the group velocity for this wave. This makes it feasible to determine the group velocity by observation. For the earthquake waves shown in figure 3.1, one can identify waves with periods from 15 to 35 seconds. If the occurrence time for the quake is known, the transit time, t_g, (i.e. the time the waves have used to get to the observation point from the epicenter) for the various wave periods can be determined. The waves will follow a path which lies close to a great circle through the epicenter and the observation place. If one knows the length, L, of the path, then the group velocity for the waves is

$$c_g = L/t_g.$$

It must be emphasized that since the group velocity can vary along the path, this method will find an average group velocity for the path.

We notice that equations (3.4) and (3.5) only give information on frequency and wavenumber, and thereby the phase function, but the wave amplitude is undetermined. For this reason, the term wave kinematics is often used for this theory and the results obtained with it.

For a homogeneous medium where c_g is a function of k, equation (3.5) has a solution such that k is determined by

$$c_g = \frac{x}{t}. \tag{3.9}$$

This is correctly realized by differentiation of the last equations with respect to x and t such that the magnitudes $\frac{\partial k}{\partial t}$ and $\frac{\partial k}{\partial x}$ appear. The equation expresses that a wave component with a fixed wavenumber k which at time $t = 0$, starts from the origin $x = 0$, has propagated a distance x in the time t.

The expression (3.9) can be used to derive the results we have derived earlier after comprehensive analysis. This is valid for wave forms at a large offset from the disturbance that created the waves.

For gravity waves in deep water is

$$c_g = \frac{1}{2}\sqrt{\frac{g}{k}}$$

and with the help of (3.9), we find that

$$k = \frac{gt^2}{4x^2}.$$

The corresponding value for ω is

$$\omega = \frac{gt}{2x}.$$

This gives a phase function corresponding with what we have found for waves generated by an isolated disturbance (see equation 2.74).

With an equivalent method, we can find wavenumber, angular frequency, and phase function for capillary waves generated by an isolated disturbance. For capillary waves in deep water is

$$c_g = \frac{3}{2}\sqrt{\frac{\sigma k}{\rho}}$$

and (3.9) gives

$$k = \frac{4}{9}\frac{\rho}{\sigma}\frac{x^2}{t^2}.$$

It must be emphasized that this expression for k is valid when one is a long distance away from the initial disturbance.

As an example of the solution of equation (3.4) when $\frac{\partial \omega}{\partial x}$ is nonzero, we shall look at long gravity waves over a sloping bottom, see equation (2.94). In this case

$$W = \sqrt{\alpha g x}k.$$

We assume the stationary state is $\frac{\partial k}{\partial t} = 0$, and thereby

$$\frac{\partial k}{\partial x} = -\frac{1}{2}\frac{k}{x}.$$

This equation has the solution

$$k = k_0 x^{-\frac{1}{2}}$$

where k_0 is a constant. We find that the results are in agreement with what we have found earlier.

Exercises

1. **Wave generator; counting of crests**

   In one end of a wave tank, a wave is generated with length $\lambda = 1$ m. The wave generator has been on for 100 seconds. Estimate the number of wave crests in the channel in the three cases where the water depth is $h = 4$ m, $h = 0.5$ m and $h = 0.1$ m. Assume that the wave channel is long enough that reflection from the opposite end does not happen.

3.2 Hamilton's equations. Wave-particle analogy

Readers who have knowledge of classical mechanics will immediately recognize equations (3.6)–(3.8). Hamilton's equations of motion are

$$\begin{aligned} \frac{dq}{dt} &= \frac{\partial H}{\partial p} \\ \frac{dp}{dt} &= -\frac{\partial H}{\partial q} \\ \frac{dH}{dt} &= \frac{\partial H}{\partial t} \end{aligned} \tag{3.10}$$

where q is the generalized coordinate, p is the generalized momentum, and H is the Hamiltonian function.

In the case that the Hamiltonian function does not depend on time, the system is autonomous and is called a conservative system. For a conservative system, the value of H is conserved.

We see that the set of equations in (3.10) is of the same form as equations (3.6)–(3.8). Therefore there exists a formal analogy between particle motion and wave motion such that

$$\begin{array}{ccc} x & \text{corresponds to} & q \\ k & \text{corresponds to} & p \\ \omega & \text{corresponds to} & H \end{array}$$

It was Einstein (1905) who first gave physical content to the formal analogy between particle motion and wave motion by setting the energy and momentum for light particles (photons) to, respectively, $H = \hbar\omega$ and $p = \hbar k$, where $\hbar$ is Planck's constant. This is the foundation for the two complementary descriptions of light as particle motion and wave motion.

3.3 Wave kinematics for multi-dimensional wave propagation. Ray theory.

In the general case, the phase function will be dependent on the position vector $\boldsymbol{r}$ such that $\chi = \chi(\boldsymbol{r}, t)$. Corresponding to what we have used in section 2.13, we can determine the local values for the wavenumber and angular frequency

$$\boldsymbol{k} = \nabla\chi \qquad \text{and} \qquad \omega = -\frac{\partial \chi}{\partial t}.$$

We assume that $\boldsymbol{k}$ and ω are slowly varying functions of the space and time coordinates. For a given point in time the equation

$$\chi(\boldsymbol{r}, t) = constant$$

will represent a phase surface and the vector $\boldsymbol{k}$ is normal to the surface. Because of the assumption that the functions vary slowly, the phase surface will bend faintly, see figure 3.2. The spatial curve that arises by setting

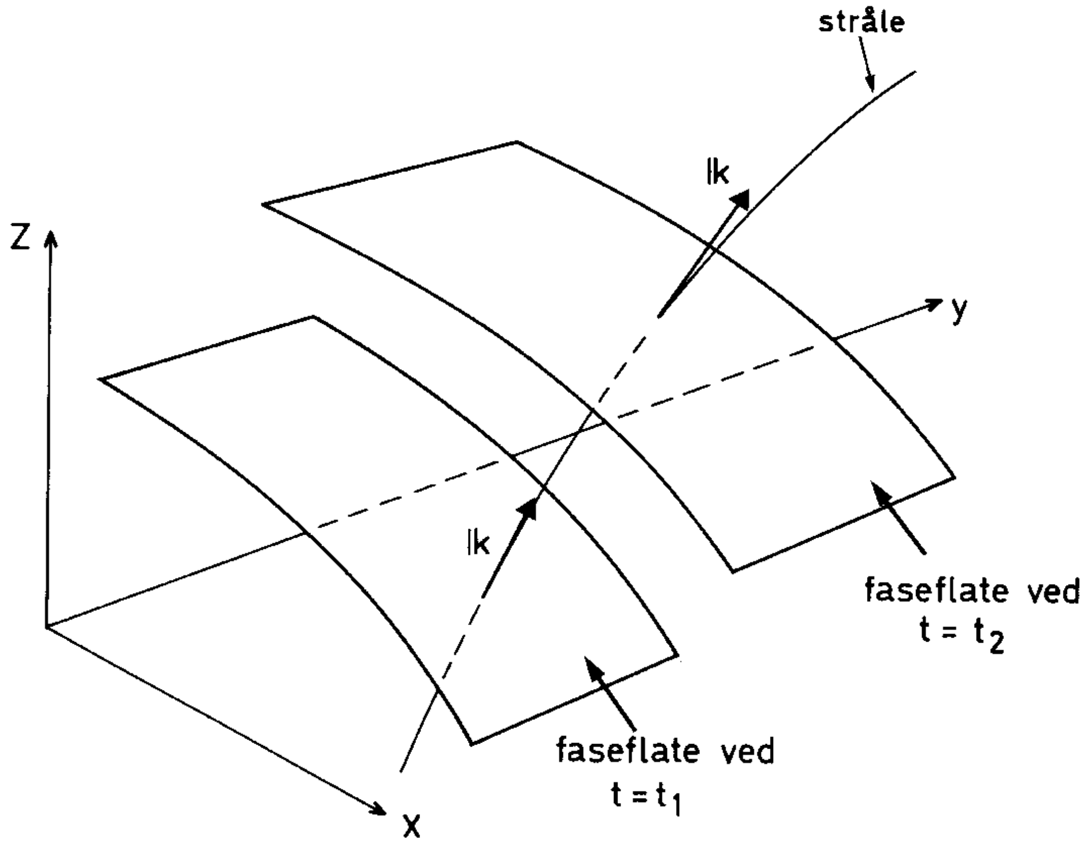

Figure 3.2: Rays and phase surface.

$$\boldsymbol{k} \times d\boldsymbol{r} = 0$$

where $d\boldsymbol{r}$ is a vectorial arch element for the curve, denotes a *ray*. The wavenumber vector is also tangent to the ray. As in the one-dimensional case, $\boldsymbol{k}$ and ω are related by the dispersion relation, which we in this case assume to be time independent

$$\omega = W(\boldsymbol{k}, \boldsymbol{r}). \tag{3.11}$$

The explicit dependence on $\boldsymbol{r}$ is an expression for spatial inhomogeneity in the medium. The change over time of $\boldsymbol{k}$ and ω at a point with coordinate $\boldsymbol{r}$, which moves with the group velocity, will be given by the equations for all of the coordinate directions

$$\begin{aligned} \frac{dx_i}{dt} &= \frac{\partial W}{\partial k_i} \\ \frac{dk_i}{dt} &= -\frac{\partial W}{\partial x_i} \\ \frac{d\omega}{dt} &= 0 \end{aligned} \tag{3.12}$$

where the indices $i = 1, 2, 3$ denote the vector components along the three axial directions. Equations (3.11) and (3.12) together with the initial conditions for x_i, k_i and ω designate a *wave path* (or particle path) $\boldsymbol{r}(t)$. The form of these equations is particularly well suited for numerical computations. In a medium with isotropic dispersion where W is a function of $k = |\boldsymbol{k}|$, but not of the direction of $\boldsymbol{k}$, we find that

$$\frac{\partial \omega}{\partial k_i} = \frac{\partial W}{\partial k}\frac{\partial k}{\partial k_i} = \frac{\partial W}{\partial k}\frac{k_i}{k}.$$

This shows that for media with isotropic dispersion, the group velocity has the same direction as the wavenumber vector, and the wave paths coincide with the rays.

We shall look at two examples in order to show explicit solutions of (3.12). The first deals with refraction of long waves in shallow water. We assume that the bathymetry is such that the water depth, H, is a function of x given by

$$c(x) = \sqrt{gH(x)}.$$

The dispersion relation can be written

$$W = c(x)k.$$

Since the propagation velocity depends on depth, the waves will refract. If the water depth increases with x, the rays will be as sketched in figure 3.3. We designate the wavenumber vector components along the x- and y-axes with k_x and k_y respectively. From equation (3.12) we find that along the ray is

$$\begin{aligned} \omega &= constant \\ k_y &= constant \\ k_x &= \pm\left(\frac{\omega^2}{c^2(x)} - k_y^2\right)^{\frac{1}{2}} \end{aligned}$$

For this case, which is sketched in the figure, one must select the minus sign in front of the square root in the last expression.

If the angle between the x-axis and the wavenumber vector is designated by θ, we find

$$\frac{\sin\theta}{c} = \frac{k_y}{kc} = \frac{k_y}{\omega} = constant$$

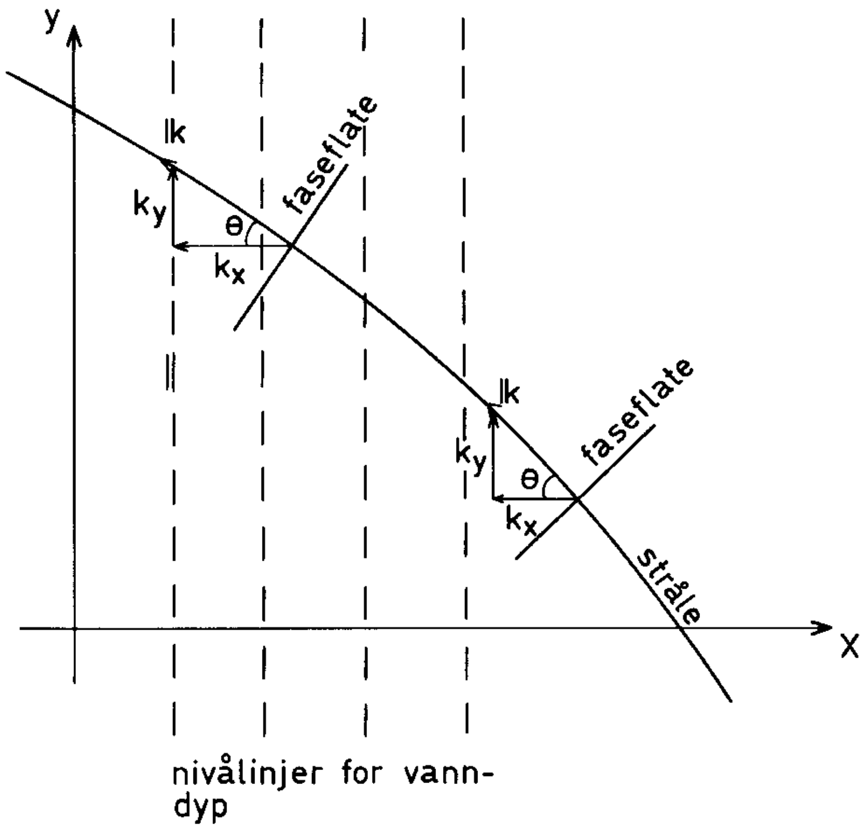

Figure 3.3: Refraction in shallow water.

which shows that the refraction obeys *Snell's law*. The equation for rays is

$$\frac{dy}{dx} = \frac{k_y}{k_x}.$$

For special values of the function $H(x)$, one can find simple analytic expressions for rays. With refraction of this type, one can notice when the waves propagate in towards a straight coast in shoaling depth. One will see that waves which come in on the slope relative to the bottom are deflected such that the crests are eventually parallel to the coast.

The next example of refraction is of seismic waves in a symmetric sphere. We assume that the velocity for the waves is a function of distance $r = |\boldsymbol{r}|$ from the sphere's center and that waves grow without dispersion such that

$$W = c(r)k.$$

This dispersion relation is valid within a good approximation for seismic waves that propagate along the earth. We find that this dispersion relation implies

$$\frac{\partial W}{\partial x_j} = \frac{dc}{dr}\frac{x_j}{r}k, \qquad \frac{\partial W}{dk_j} = c\frac{k_j}{k}.$$

From equation (3.12) it follows therefore that

$$\frac{d\boldsymbol{r}}{dt} = c\frac{\boldsymbol{k}}{k}, \qquad \frac{d\boldsymbol{k}}{dt} = -\frac{dc}{dr}\frac{\boldsymbol{r}}{r}k,$$

where $\boldsymbol{r}$ is the coordinate for a point which moves along the ray with velocity c. We multiply the first of these equations vectoraly with $\boldsymbol{k}$ and the other with $\boldsymbol{r}$, we find that

$$\frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{k}) = 0.$$

This implies that

$$\boldsymbol{r} \times \boldsymbol{k} = constant$$

along the same ray. If the angle between the two vectors denotes i (see figure 3.4), we find that

$$p = \frac{r \sin i}{c} = constant$$

along the same ray. The quantity p is designated the wave parameter, and this parameter plays an important role in seismology.

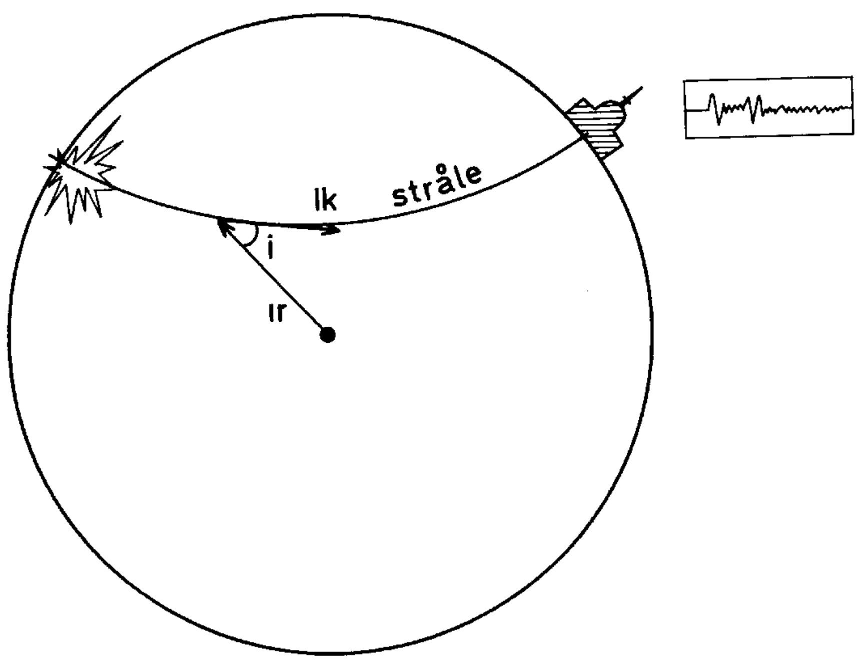

Figure 3.4: Seismic ray on a sphere.

3.4 Eikonal-equation and equations for amplitude variation along the rays

Previously we have assumed that there exists wave motion such that the wavenumber, wave amplitude, and angular frequency are slowly varying functions in space and time. Under these requirements we have derived the fundamental results of wave kinematics and ray theory.

Wave motion of this type is possible in inhomogeneous or non-stationary media where the physical properties change little over a distance comparable to a wavelength or a time comparable to a wave period. Once can therefore see that the requirements are fulfilled if the wavelength and period are sufficiently short.

As an example, we shall solve the wave equation (2.93) in the case that the propagation speed, $c_0 = \sqrt{gH}$, is a function of x and y. We assume that the scale for inhomogeneity in the wave field is equal to the scale for inhomogeneity for the propagation speed, c_0. We assume that the solution of (2.93) can be written in the form

$$\eta = A(x, y, t)\mathrm{e}^{\mathrm{i}\chi(x,y,t)} \tag{3.13}$$

where the frequency and the wavenumber vector are defined by

$$\omega = -\frac{\partial \chi}{\partial t}, \qquad \boldsymbol{k} = \nabla\chi.$$

The presumption of slowly varying wave fields means that the ratio between the wavelength, $\lambda = 2\pi/k$, and the scale for inhomogeneity, L, must be a small number. Correspondingly the ratio between a wave period, $2\pi/\omega$, and the scale for non-stationarity T must be a small number. We therefore introduce an ordering parameter such that

$$\epsilon = \frac{1}{kL} = \frac{1}{\omega T} \ll 1. \tag{3.14}$$

If we use the slowly varying scale for inhomogeneity and non-stationarity as references we can explicitly describe the rapid wave phase by the ordering parameter such that

$$\eta = A(x, y, t)\mathrm{e}^{\epsilon^{-1}\mathrm{i}\chi(x,y,t)}. \tag{3.15}$$

It will now be natural to assume that the amplitude, A, can be developed in a perturbation expansion in terms of the same ordering parameter

$$\eta = \big(A_0(x, y, t) + \epsilon A_1(x, y, t) + \epsilon^2 A_2(x, y, t) + \ldots\big)\, \mathrm{e}^{\epsilon^{-1}\mathrm{i}\chi(x,y,t)}. \tag{3.16}$$

Taking the time derivative of the series expansion gives

$$\frac{\partial \eta}{\partial t} = \left(-\mathrm{i}\epsilon^{-1}\omega A_0 - \mathrm{i}\omega A_1 + \frac{\partial A_0}{\partial t} + O(\epsilon)\right)\mathrm{e}^{\epsilon^{-1}\mathrm{i}\chi(x,y,t)}.$$

The series expansion (3.16) shall now be substituted into (2.93). It can be realized that the complex exponential function is common for all terms and can be factored out. The expression that remains must then be fulfilled to all orders in ϵ. This gives a hierarchy of equations for the determination of χ, A_0, etc.

The first of these equations is called *eikonal-equation*. In this case the equation is

$$\omega^2 = c_0^2 k^2 \tag{3.17}$$

where $k = |\boldsymbol{k}|$. We recognize this as the dispersion relation. If we can assume a stationary wave field we can write

$$\chi = S(x, y) - \omega t \tag{3.18}$$

and the phase line is determined by

$$S(x, y) - \omega t = const.$$

The wavenumber vector

$$\boldsymbol{k} = \nabla \chi = \nabla S$$

is oriented normal to the phase line.

The other equation is often called the *transport* or *amplitude* or *evolution* equation. It is a conservation equation for the so-called wave action, which is the wave energy divided by the so-called intrinsic frequency. It describes how the wave action is transported along a ray. In the present case, we have

$$\omega \frac{\partial A_0}{\partial t} + \frac{\partial}{\partial t}(\omega A_0) + c_0^2 \boldsymbol{k} \cdot \nabla A_0 + \nabla \cdot (c_0^2 \boldsymbol{k} A_0) = 0. \tag{3.19}$$

If we multiply the equation with A_0 and integrate once we find

$$\frac{\partial}{\partial t}(\omega A_0^2) + \nabla \cdot (\boldsymbol{c}_g \omega A_0^2) = 0. \tag{3.20}$$

As an exercis show that the by means of the ray equation for the frequency (3.8) this can be rewritten as a conservation equation for energy or for wave action, keeping in mind that the medium does not depend on time.

Equation (3.20) describes amplitude variation along the rays. Let us assume a stationary wave field and let us consider a domain (see figure 3.5) restricted to the rays a and b where the cross-sections σ_A and σ_B are normal to the ray segments which lie between the rays. We assume that rays a and b lay so near to each other that c_0, A_0 and ∇S can be set constant over the cross-sections σ_A and σ_B. We integrate equation (3.20) over the area and use Gauss' theorem. Since the group velocity is parallel with the wavenumber vector, the contribution to the integral disappears along the rays and we find

$$\sigma_A (c_0 A_0^2)\big|_A = \sigma_B (c_0 A_0^2)\big|_B \,. \tag{3.21}$$

A_0 represents the amplitude for the leading term in the series expansion (3.16), and the magnitude $c_0 A_0^2$ is therefore a measure of the energy flux. Equation (3.21) also expresses that in this case, the energy flux is constant along a ray. This is an important result, and this entails that there is no reflection of wave energy along the rays. This shows that the wave kinematics that we have developed above requires that the changes in the medium are so gradual that the waves are refracted without any of the energy being reflected.

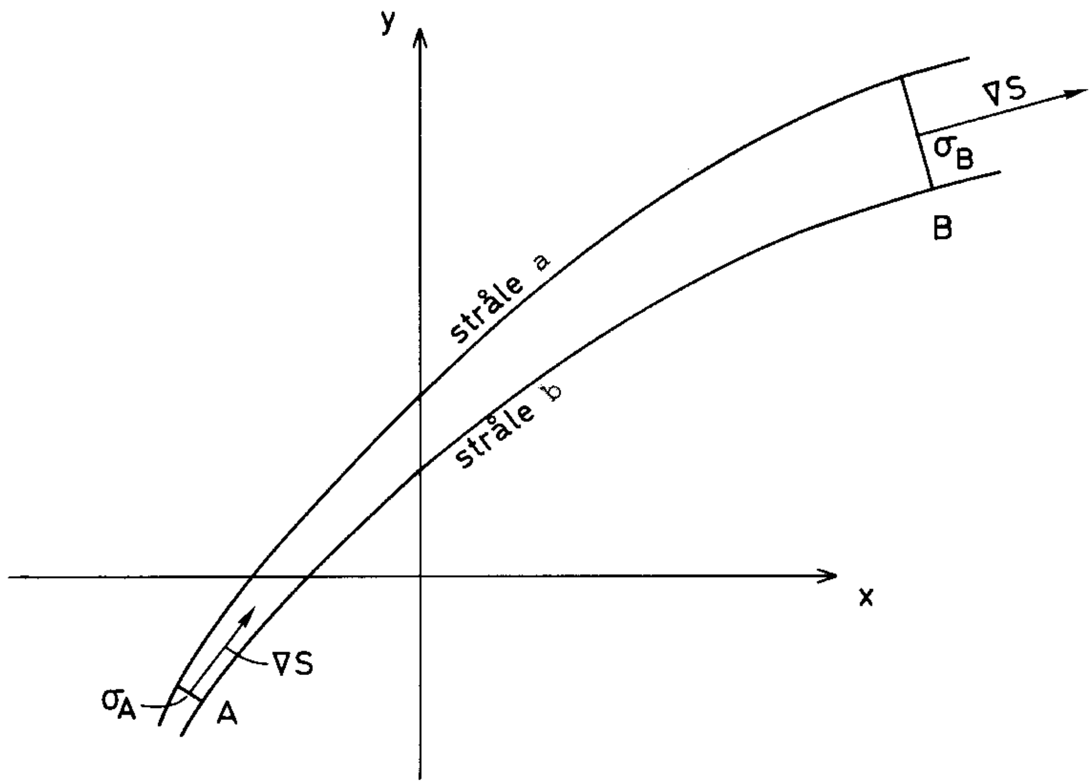

Figure 3.5: Sketch for the calculation of amplitude along a ray tube.

Notice that the application of Gauss' theorem was especially simple because the group velocity was parallel with the wavenumber vector in this case, and thereby the contribution vanishes when integrating along the rays. In the case where the group velocity is not parallel with the wavenumber vector, the contribution along the rays would be integrated. In such cases, it can be simpler to use wave paths instead of rays like those sketched in figure 3.5.

To end, we show an application of the results from this section. Consider surface waves propagating toward a coast with a coast line and bottom topography as sketched in figure 3.6. For a wave which has a straight phase line far out in deep water, the rays are indicated in the figure. We choose a ray tube $A - B$ outside of the headland and the ray tube $A' - B'$ outside of the inlet such that the cross-sections at A and A' are equally large. We furthermore assume that the wave amplitude is equally large at A and A'. Since the cross-sections for the ray tubes are smaller at B than at B', it follows from (3.21) that the wave amplitude is larger at B than at B'. The fact that wave energy is focused towards headlands and spreads in bays is often observed in Nature. This is best seen from the air, and the aerial photograph in figure 3.7 shows many such effects.

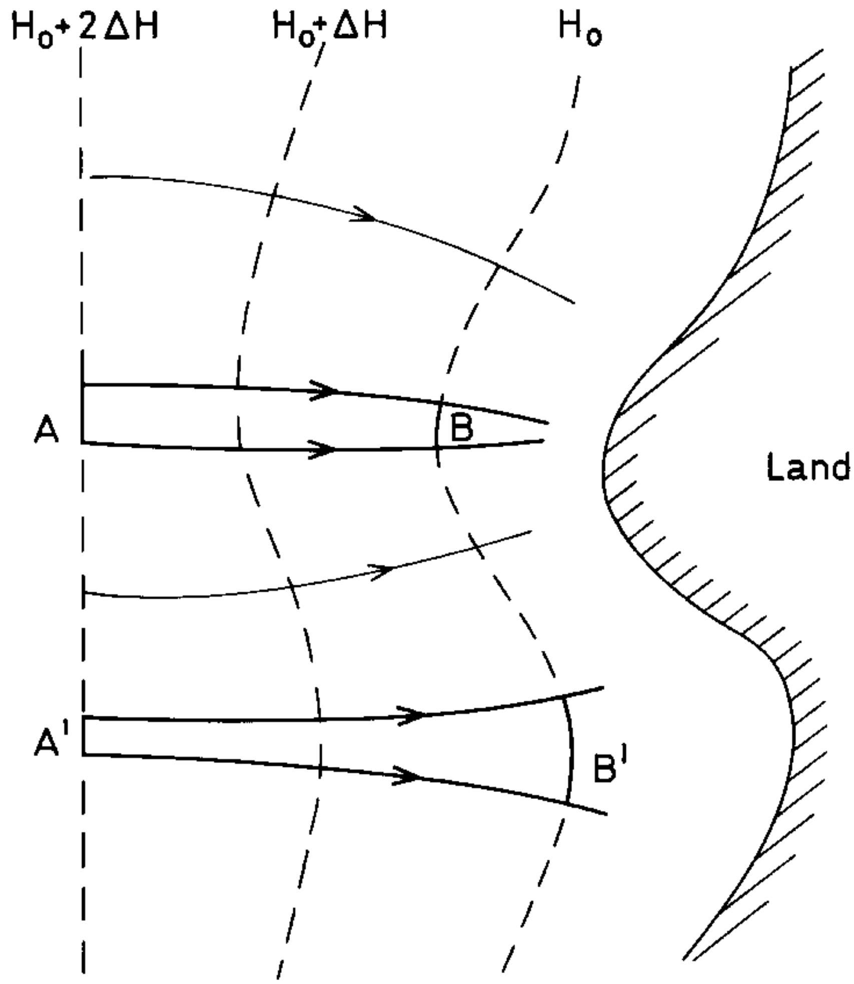

Figure 3.6: Sketch for waves approaching coast.

3.5 Amplitude variation in nearly peridoic wave trains

In the previous section, we integrated (3.20) with the help of Gauss' law for waves with fixed frequency (monochromatic waves). For nearly periodic wave

Figure 3.7: Aerial photograph from an area near Kiberg on the coast of Finnmark. The photo was taken 12. June 1976 by Fjellanger Widerøe A.S.

trains that arise from dispersion, the amplitude variation cannot be found by a simple application of Gauss' law, but rather equation (3.20) must be integrated in both space and time.

We know that for linear waves, the average energy flux, $\boldsymbol{F}$, is given by the average energy density, E, and the group velocity $\boldsymbol{c}_g$

$$\boldsymbol{F} = \boldsymbol{c}_g E.$$

Indeed, we have only seen this for a single wave component, but it is reasonable to assume that the same relation is valid, with a good approximation, in the case of slowly varying wave trains. If there is no energy input to the waves by outside influences such as wind, and the medium is stationary in time, we can write

$$\frac{\partial E}{\partial t} + \nabla \cdot (\boldsymbol{c}_g E) = 0. \tag{3.22}$$

For a non-stationary medium (waves interacting with currents, wave experiments carried out the the elevator) the conserved quantity is the wave action which is the energy divided by the so-called intrinsic frequency.

A generalized equation containing the source term and coupling term which respectively express the generation of waves caused by wind and the interaction between waves and currents, is often used for wave forecasting (Kinsman 1965).

The energy density is proportional to a^2 where a is the wave amplitude, and for plane gravity waves in deep water we have $E = \frac{1}{2}\rho g a^2$. From equation (3.22) we thus find

$$\frac{\partial a^2}{\partial t} + \nabla \cdot (\boldsymbol{c}_g a^2) = 0. \tag{3.23}$$

We illustrate this by studying two simple solutions.

For long surface waves which propagate over a smooth flat bottom with slope α, the group velocity is $c_g = \sqrt{gH} = \sqrt{g\alpha x}$ (see section 2.7). If the wave train is periodic in time such that $\frac{\partial a^2}{\partial t} = 0$, it follows from (3.23) that the energy flux is constant and we therefore must have that the wave amplitude is

$$a = const. \cdot x^{-\frac{1}{4}}.$$

This is in agreement with what we have found in section 2.10.

For waves generated by a concentrated disturbance at $t = 0$ and $x = 0$, the group velocity will be given by

$$c_g = \frac{x}{t}$$

where t is the transit time to a point at distance x. The wave amplitude can be found from (3.23)

$$a = const. \cdot x^{-\frac{3}{2}} t$$

By comparion with expression (2.74), the constant can be determined.

Exercises

1. We assume that the phase speed for seismic waves is a function of depth z, which can be written as $c(z)$. The speed at the surface $z = 0$ is c_0. Find the path for a wave with angular frequency ω_0 which goes out from a point O at the surface in a direction defined by an angle θ with the surface. We assume that the waves propagate towards a finite area such that the surface can be considered to be a plane. Find the distance from O to the point A where the rays come up to the surface. $c(z)$ is a monotonously increasing function of z. Determine the transit time as a function of the distance OA, and find the deepest point of the rays.

Figure 3.8: The figure is so far missing!

2. In a sea with uniform depth H_0 there is a shoal with depth given by

$$H = H_0(1 - \alpha e^{-\frac{x^2+y^2}{L^2}})$$

where α and L are constants. Find the rays for plane waves with wavelength $\lambda = \beta L$ (where β is a constant) coming in towards the shoal along the x-direction.

3. **Self-focusing wave generator**

   In one end of a wave channel, a wave is generated with frequency $f(t)$. The frequency is 10 Hz when the wave generator is turned on, and it is turned off 100 seconds later. We wish to concentrate as much wave energy as possible at a prescribed point in the wave channel. Use ray theory to find the optimal form of $f(t)$. Find the point where the energy concentrates and the time when this happens. Assume that the channel is long enough that there is no reflection from the opposite end and that it is much deeper than the length of the generated waves.

   Does ray theory work well here?

4. **Waves over a parabolic bottom**

   A bottom topography (bathymetry) is given by $h(x) = h_0(\frac{x}{l})^2$. We assume that linear, hydrostatic shallow water theory is valid and that the waves are monochromatic in time (only one frequency).

(a) Use ray theory to determine the phase function χ.

(b) Give an interpretation of the expression $\frac{1}{c}\frac{\partial c}{\partial x}\lambda$. Which requirement must we impose on $\frac{\omega l}{\sqrt{gh_0}}$ for ray theory to be valid?

(c) Find a solution of the shallow water equation of the form $\eta = \mathrm{e}^{-\mathrm{i}\omega t}x^q$. Compare this with the results we found with the help of ray theory.

5. **Slowly varying wave sources**

   We consider plane, nearly periodic waves in a homogeneous medium. We assume linear conditions, and the waves propagate in the positive x-direction. The dispersion relation is given by $\omega = \frac{1}{3}k^3$. At $t = 0$ the local wave length is given by $\lambda = a(2 + \tanh(bx))$.

   (a) We shall use ray theory, which conditions must then be imposed on the parameters a and b?

   (b) Use ray theory and the method of characteristics to estimate the local wave length for all x at a later time. A difficult algebraic equation will pop up here, and this should not be solved.

6. **Waves with oblique incidence**

   The given depth is independent of the y-direction, i.e. depth $h = h(x)$. We shall study how long harmonic waves behave over this bottom. The waves shall have an oblique incidence, which means that the wavenumber has a component in the y-direction.

   a Find the amplitude varies according to physical optics (the transport equation). Is it correct that a wave will always be amplified when it comes into shallower water?

   b Discuss what happens with the amplitude when the wave propagates into deeper water.

7. **Waves from an initial disturbance 2.** We consider the Klein-Gordon equation

$$\frac{\partial^2\eta}{\partial t^2} - \frac{\partial^2\eta}{\partial x^2} + q\eta = 0.$$

   At $t = 0$, assume a point disturbance at $x = 0$. Use ray theory to determine the phase function at large x and t. Compare with the stationary phase solution.

Chapter 4

TRAPPED WAVES

For surface waves on deep water the motion decays quite fast downwards. At a depth comparable to a wavelength or more, the motion is negligible. The wave energy is therefore associated with a zone close to the surface, and the energy is bound to this zone while the waves propagate forward at the surface. We may say that the waves are trapped at the surface and the surface acts as a wave-guide. Trapped waves and wave-guides play an important role in many types of wave motion. In the ocean and in the atmosphere the stratification of density may result in horizontal layers with low sound speed. Thereby this layer can act as a wave-guide, and sound of certain frequencies may propagate over long distances with little attenuation. When sound waves propagate in this manner along a horizontal layer, the energy will be spread over cylindrical shells of increasing radius r and the wave amplitude will attenuate as $1/\sqrt{r}$. This is significantly less attenuation than a spherical spreading for which the amplitude attenuates as $1/r$. Low geometric attenuation is a characteristic property of wave-guides. The mathematical descriptions of various wave-guides are rather similar. In the following we shall study some examples of trapped gravity waves.

4.1 Gravity waves trapped by bottom topography

As long gravity waves propagate faster on deep water than on shallow water, the waves will be refracted due to differences in water depth. At a straight coastline where the water depth is linearly increasing outwards, waves propagating towards the coast will be refracted toward the coast. If the waves are reflected at the shore, the waves will be refracted on their way out, and that refraction may be large enough that they turn and propagate towards the coast again. After a new reflection this phenomenon can of course happen again, and by periodic reflections and refractions along the coast the waves may be trapped. A sketch of the rays is shown in figure 4.1.

There exists a simple solution of Laplace equation (2.11) with boundary con-

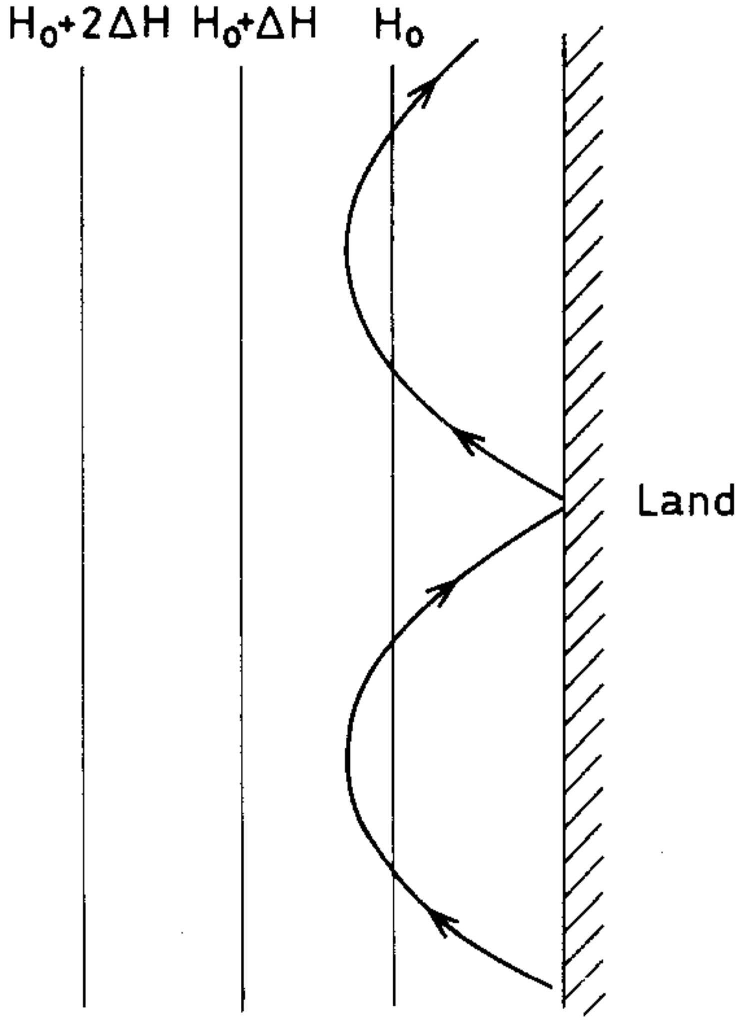

Figure 4.1: Rays for trapped waves.

ditions (2.12) and (2.13) (with $\sigma = 0$) that describes trapped gravity waves along a coastline where the depth increases linearly with the distance from the coast. The bottom topography and the choice of coordinate system are shown in figure 4.2.

Together with the boundary conditions (2.12) and (2.13) at the free surface, we have the kinematic bottom condition

$$\frac{\partial \phi}{\partial x} \tan\theta + \frac{\partial \phi}{\partial z} = 0$$

at the sea floor $z = -x\tan\theta$. The velocity potential is

$$\phi = -\frac{ag}{kc} \mathrm{e}^{-k(x\cos\theta - z\sin\theta)} \cos k(y - ct) \tag{4.1}$$

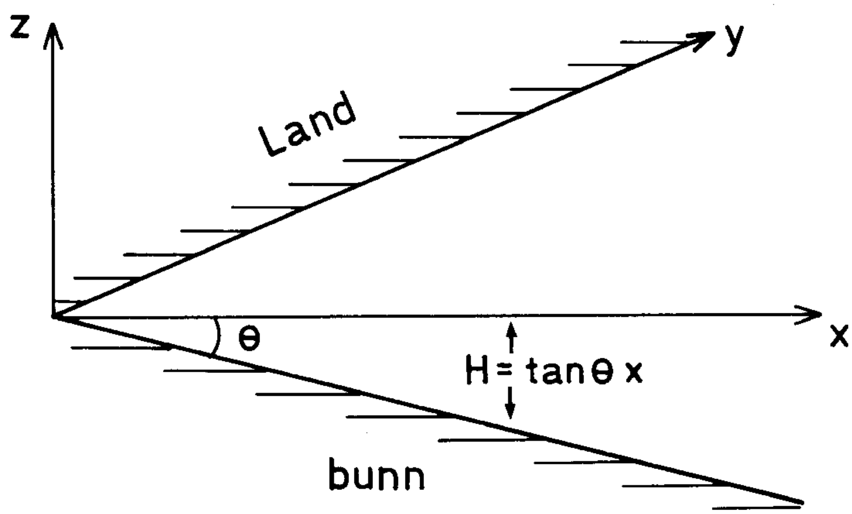

Figure 4.2: Geometry of the coast.

and

$$c = \pm(\frac{g}{k}\sin\theta)^{\frac{1}{2}}.$$

The corresponding surface elevation is

$$\eta = a e^{-kx\cos\theta}\sin k(y - ct).$$

We see that the waves propagate along the coast with a speed that depends on the inclination of the sea floor, and that the amplitude decays exponentially from the coast. The waves will be able to propagate in both directions along the coast. The coast and the bottom topography therefore constitute a two-way wave guide. This type of wave phenomenon can be observed for instance in channels with edges of uniform inclination. These waves are called edge-waves, and the particular solution (4.1) represents the first mode of several solutions (Ursell 1952). Similar simple solutions for other bathymetries are not known.

In order to study the phenomenon more carefully, we assume that the waves are sufficiently long that we can employ the shallow water equations. We put the coordinate axes as in figure 4.1, and let the depth H be an increasing function of x. At the shore we imagine there is a vertical wall and that the depth is much greater than the wave amplitude. We write the surface elevation η in the form

$$\eta = \hat{\eta}(x)\sin k(y - ct)$$

where $\hat{\eta}(x)$ is a function of x to be determined. From (2.93) we then have

$$\frac{d}{dx}(H\frac{d\hat{\eta}}{dx}) = k^2(H - \frac{c^2}{g})\hat{\eta}. \tag{4.2}$$

At the coast line $x = 0$ the volume flux in the x-direction is zero, and this implies

$$\frac{d\hat{\eta}}{dx} = 0 \qquad \text{for} \qquad x = 0. \tag{4.3}$$

For trapped wave we must have

$$\hat{\eta} \to 0 \qquad \text{as} \qquad x \to \infty. \tag{4.4}$$

Equation (4.2) with boundary conditions (4.3) and (4.4) is an eigenvalue problem of known type which for given values of k gives certain permissible values for c.

We introduce

$$\hat{\eta} = H^{-\frac{1}{2}}\psi$$

and substitute into (4.2). This gives

$$\frac{d^2\psi}{dx^2} = -k^2(W(x) - 1)\psi \tag{4.5}$$

where

$$W(x) = \frac{c^2}{gH} + \frac{1}{4}\frac{H'^2 - 2HH''}{(kH)^2}.$$

Here $'$ means differentiation with respect to x. The boundary conditions for ψ become

$$\begin{aligned} \frac{d\psi}{dx} &= \frac{1}{2}\frac{\psi}{H}\frac{dH}{dx} && \text{at } x = 0 \\ \psi &\to 0 && \text{as } x \to \infty. \end{aligned} \tag{4.6}$$

This eigenvalue problem is of similar type as those involving the solution of the Schrödinger equation in quantum mechanics. If H is a slowly varying function of x, then W can be approximated by

$$W = \frac{c^2}{gH}. \tag{4.7}$$

As the factor $1/(kH)^2$ appears in the term that is truncated, we expect that the approximation will be better the shorter the wavelength. In agreement with this approximation the first of the boundary conditions in (4.6) is modified as

$$\frac{d\psi}{dx} = 0 \qquad \text{at} \qquad x = 0. \tag{4.8}$$

The solution of (4.5) changes character according to $W > 1$ or $W < 1$. When $W > 1$ (i.e. $H < \frac{c^2}{g}$) the solution for ψ has oscillatory behavior, while for $W < 1$ (i.e. $H > \frac{c^2}{g}$) the solution is exponentially attenuated.

We now consider the case when H is a monotonically increasing function of x, and H approaches a constant value as $x \to \infty$ such that $W > 1$ for $0 < x < x_0$, and $W < 1$ for $x > x_0$. For certain values of k and c we can therefore have solutions for ψ behaving like trapped waves where the motion is exponentially damped for $x > x_0$. Depending on the number of nodes for the oscillations in the domain $x < x_0$ one can have different modes for the wave motion. In figure 4.3 we see a sketch of the 1st and 2nd modes for $\psi(x)$.

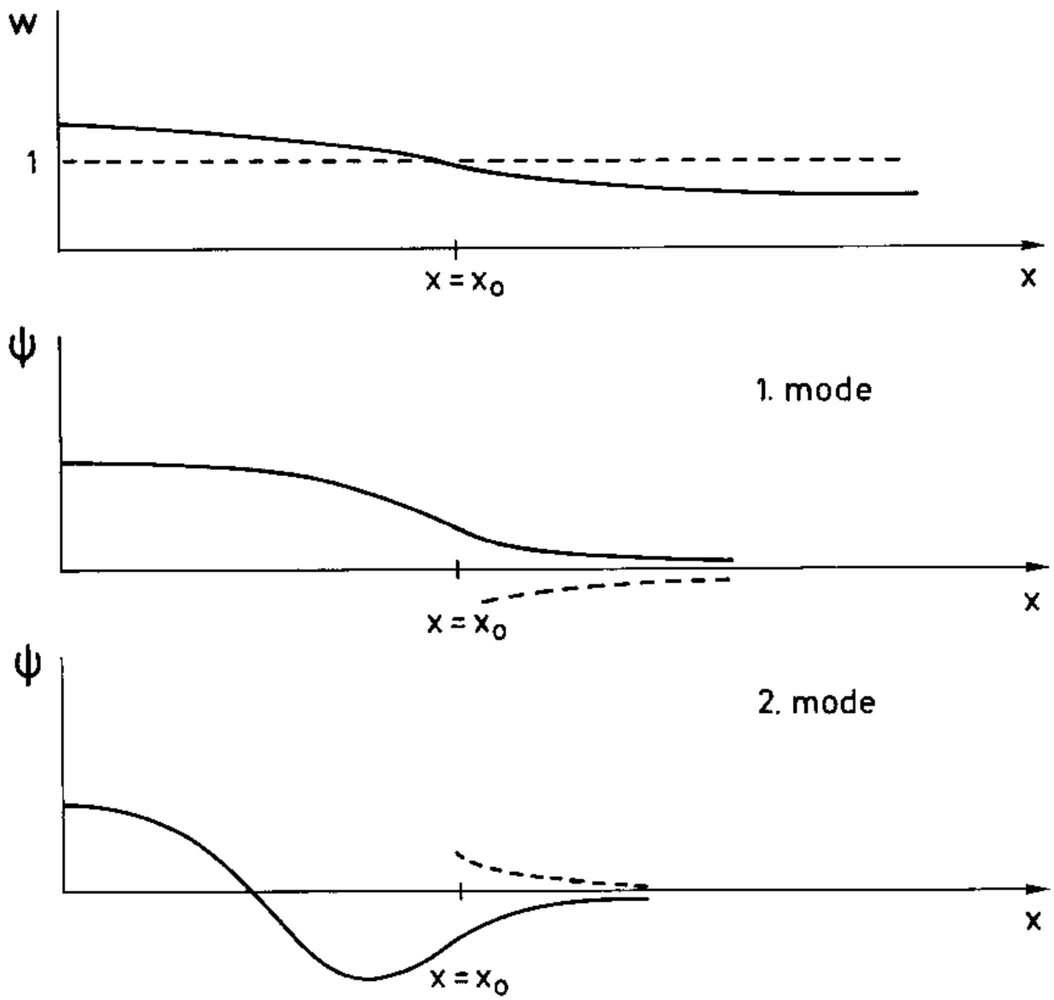

Figure 4.3: The two lowest trapped modes.

The phenomenon of trapped waves in harbors is especially important for long waves. Such waves with wavelength 100 km and more will be more or less unaffected by small irregularities in bathymetry and coastline and more affected by large scale modifications of bathymetry. Therefore the simple models we have discussed here often are applicable.

4.2 The WKB method

Without resorting to numerical methods it can be difficult to determine the solutions of (4.5) that satisfy the boundary conditions. We shall here show a method to find an approximate solution. We set

$$f(x) = W(x) - 1$$

where W is given by (4.7). Equation (4.5) can then be written

$$\frac{d^2\psi}{dx^2} = -k^2 f(x)\psi. \tag{4.9}$$

For simplicity we assume as before that H is a monotonously increasing function of x and that H goes to a constant value for $x \to \infty$. For large values of k we can find an approximate solution of (4.9) by using the same method as in (3.4). We therefore write ψ by an expansion

$$\psi = \mathrm{e}^{\mathrm{i}kS}[P_0(x) + \frac{P_1(x)}{\mathrm{i}k} + \cdots]$$

where S, P_0 and P_1 are functions of x. The expansion is substituted into (4.9), and the coefficient in front of each term in the resulting series should be zero. Thereby one gets

$$(\frac{dS}{dx})^2 = f(x),$$

$$\frac{d}{dx}(P_0^2\frac{dS}{dx}) = 0.$$

This leads to

$$S = \pm \int f D^{\frac{1}{2}}\, dx$$

and

$$P_0 = const./f^{\frac{1}{4}}.$$

If we consider only the leading term in the expansion then ψ can be written

$$\psi = Af^{-\frac{1}{4}}\sin(k\int f^{\frac{1}{2}}\, dx) + Bf^{-\frac{1}{4}}\cos(k\int f^{\frac{1}{2}}\, dx). \tag{4.10}$$

where A and B are constants. The solution (4.10) is known as the WKB solution, where the acronym is composed of the initials of Wentzel, Kramers and Brillouin who found the solution independently of each other around 1926. It is clear that (4.10) is not valid in the neighborhood of and at the point $x = x_0$ where $f = 0$. A solution of (4.9) valid in this region can be found by the following procedure. We write

$$f(x) \simeq -|f'(x_0)|(x - x_0) \tag{4.11}$$

where $f'(x_0)$ denotes the derivative of f at the point $x \simeq x_0$, and we introduce ξ as a new variable such that

$$x - x_0 = k^{-\frac{2}{3}} |f'(x_0)|^{-\frac{1}{3}} \xi.$$

Equation (4.9) can thereby be written in the form

$$\frac{d^2\psi}{d\xi^2} = \xi\psi. \tag{4.12}$$

This is the Airy differential equation, which by a special transformation can be brought into a Bessel equation. The Airy equation has two linearly independent solutions $\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$ known as Airy functions. One of these, $\mathrm{Ai}(\xi)$, oscillates for $\xi < 0$, while for $\xi > 0$ it is exponentially damped such that $\mathrm{Ai}(\xi) \to 0$ for $\xi \to \infty$. (See Abramowitz and Stegun 1972, or the more modern Digital Library of Mathematical Functions). We notice that since k is supposed to be large, then $|\xi|$ can be large even if $|x - x_0|$ is so small that the expansion (4.11) is valid. Since we assume that $\psi \to 0$ for $x \to \infty$, we must select as the solution of (4.12)

$$\psi = D\mathrm{Ai}(\xi) \tag{4.13}$$

where D is a constant. For large values of ξ we have the following asymptotic expression for $\mathrm{Ai}(\xi)$

$$\mathrm{Ai}(\xi) \simeq \frac{1}{2\sqrt{\pi}} \xi^{-\frac{1}{4}} \mathrm{e}^{-\frac{2}{3}\xi^{\frac{3}{2}}} \qquad \text{for} \qquad \xi > 0$$

and

$$\mathrm{Ai}(\xi) \simeq \frac{1}{\sqrt{\pi}} |\xi|^{-\frac{1}{4}} \sin(\frac{2}{3}|\xi|^{\frac{3}{2}} + \frac{\pi}{4}) \qquad \text{for} \qquad \xi < 0.$$

If the expressions (4.10) and (4.13) for ψ match for large values of $|\xi|$, this implies that $A = B$ and $k \int f^{\frac{1}{2}}\, dx = \frac{2}{3}|\xi|^{\frac{3}{2}}$. This leads to the following WKB solution for $x < x_0$

$$\psi = A f^{-\frac{1}{4}} \sin(k \int_x^{x_0} f^{\frac{1}{2}}\, dx + \frac{\pi}{4}) \tag{4.14}$$

where A is an arbitrary constant.

Since f has been assumed to be a slowly varying function of x, we have

$$\frac{d\psi}{dx} = -A\, k\, f^{\frac{1}{4}} \cos(k \int_x^{x_0} f^{\frac{1}{2}}\, dx + \frac{\pi}{4})$$

in the approximation being used here. The boundary condition at $x = 0$ given by (4.8) implies that

$$k \int_x^{x_0} f^{\frac{1}{2}}\, dx + \frac{\pi}{4} = (2n+1)\frac{\pi}{2} \tag{4.15}$$

where $n = 0, 1, 2$. Equations (4.15) determine the eigenvalues for k or c. It turns out that the WKB method often gives eigenvalues with good approximation even for moderate or even small values of k. It is tempting to mention a quote by a well known applied mathematician: A good formula is characterized by being useful far beyond its domain of validity.

4.3 Waves trapped by rotational effects. Kelvin waves

The Coriolis force is important for very long gravity waves and this force will modify the wave motion. These waves are called gravity-inertia waves and have different properties than ordinary gravity waves. In this section we will study how the Coriolis force is able to trap waves even if they propagate on constant depth. We will only investigate the wave motion in a fluid layer limited by a horizontal bottom at $z = -H$ and a straight coastline $x = 0$. (See figure 4.4).

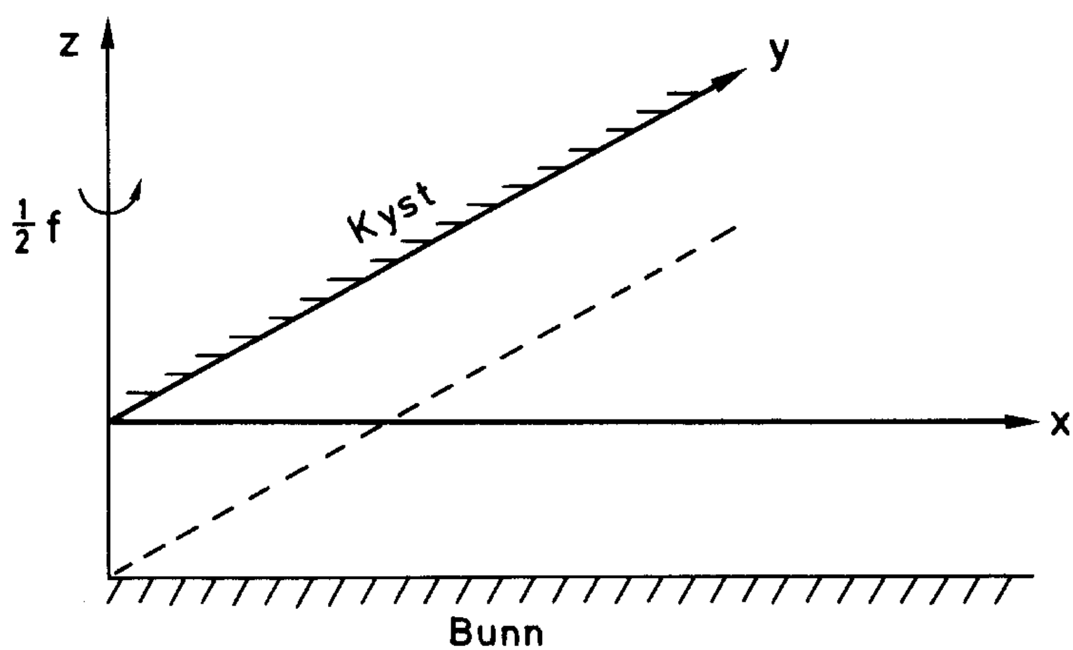

Figure 4.4: Coastal geometry.

The fluid layer rotates with a constant angular velocity $\frac{1}{2}f$ around the vertical axis. For sufficiently long waves, such that the pressure is essentially hydrostatic, the wave motion is described by equations similar to (2.86) and (2.92). However, the Coriolis force must be included in the equations of motion together with the pressure force. If the elevation and volume flux in the x- and y-directions are still denoted η, U and V, we get the following system of equations:

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= -\frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} \\
\frac{\partial U}{\partial t} - fV &= -c_0^2 \frac{\partial \eta}{\partial x} \\
\frac{\partial V}{\partial t} + fU &= -c_0^2 \frac{\partial \eta}{\partial y}
\end{aligned}$$

These equations have a solution of the form

$$\begin{aligned}
\eta &= a\mathrm{e}^{-\frac{f}{c_0}x} \sin k(y + c_0 t) \\
U &= 0 \\
V &= -c_0 a\mathrm{e}^{-\frac{f}{c_0}x} \sin k(y + c_0 t)
\end{aligned}$$

This is a wave with amplitude a and wavenumber k that propagates in negative y-direction. The motion decays exponentially from the coast, the e-folding distance being c_0/f. Waves of this type are called *Kelvin waves.* Note that with a choice of direction of rotation that is made here, (see figure 4.4), the Kelvin waves only propagate such that the coast is to the right compared to the direction of propagation. The coast is therefore a one-way wave guide for the Kelvin waves.

The tidal waves close to the coast are usually Kelvin waves, but in this case the wave motion is often modified by the shape of the coast and bottom topography. The simple model we have studied has only the purpose to illustrate the physical principles of this wave motion. Trapped waves like those discussed in this chapter have many applications; e.g. tidal motion, flood waves and tsunamis. For further reading see Mysak and Le Blond (1978) and Gill (1982). Kelvin and edge waves and their importance for flooding and tides along the Norwegian coast are discussed in Martinsen, Gjevik and Røed (1979).

Exercises

1. **Trapped waves on a ridge**

   Consider the effect of a submarine ridge given by the bottom $z = -h(x)$ with depth contours parallel to the y-axis. Let the depth be slowly varying in comparison to a local wavelength.

   For this problem we neglect earth-rotational effects and capillary effects.

   The geometry of the submarine rigde may be given by the bell shape

   $$h(x) = h_0 + (h_m - h_0)\mathrm{e}^{-\frac{x^2}{l^2}}$$

   where h_0 is the uniform depth far away from the ridge and $h_m < h_0$ is the minimum depth on the top of the ridge.

   On top of such a ridge we can have trapped waves.

a Write down the ray equations. Explain why the angular frequency ω and the y-component of the wavenumber vector are constants of motion.

b For a given wavenumber component in the y-direction, find the limitation on possible frequencies ω for trapped modes.

c Find the corresponding locations of the turning points and sketch the rays.

Chapter 5

WAVES ON CURRENTS. DOWNSTREAM AND UPSTREAM WAVES. SHIPWAKES. WAVE RESISTANCE.

In applications one often encounters the situation that the wave motion occurs at the same time as there is a current in the liquid. The wave motion can in such cases be modified by the current, and current can give rise to special wave phenomena.

A stationary current past some kind of obstacle, for example a rock on the bottom of a river, can give rise to stationary wave patterns on the downstream or upstream side of the obstacle. These stationary wave patterns are called downstream waves or upstream waves. Similar wave phenomena can occur when a disturbance, for example a ship or an insect, is moving with constant velocity along the liquid surface. In the following we shall first treat down- and upstream waves in the case that the current is uniform and stationary. In that case there is a direct analogy between the wave pattern generated by a fixed obstacle in the current and the waves generated by a disturbance that moves with constant velocity along the water surface.

In those cases where the current is not stationary or uniform, the current alone can modify the wave motion. Waves that propagate into a region with horizontal shear currents can for example have their wavelength and amplitude modified due to the effects of the current. This phenomenon can often be observed for example near river estuaries. Similarly, vertical shear currents can modify wave motion, and the effect will depend on the wavelength. Wave motion that is affected by non-uniform currents is difficult to treat mathematically, and we shall in the following suffice by studying a simple example. The effect of currents on wave motion is an active and interesting field of research. Peregrine (1976) has in a review article given a good introduction to mathematical methods and problems within this field.

5.1 Downstream and upstream waves

In Norwegian these are called "le-bølger" and "lo-bølger".

For simplicity we start by considering two-dimensional wave motion generated by some disturbance that moves along the surface with constant speed U. Similar wave patterns will arise if the liquid moved with uniform and stationary speed U past a fixed obstacle.

The dispersive properties of gravity–capillary waves on deep water can be summarized in the following diagram (figure 5.1) where c_m is the smallest phase speed that can occur and λ_m the corresponding wavelength (see section 2.2). If $U > c_m$ the we have two wave components with wavelengths λ_k and λ_t, respectively, and phase speed $c = U$. These wave components can therefore compose a stationary wave pattern in front of or behind the disturbance. We know that for wavelengths greater than λ_m the group velocity is less than the phase speed ($c_g < c$), but the opposite ($c_g > c$) is the case for wavelengths shorter than λ_m. We also know that the energy of the waves propagates with the group velocity.

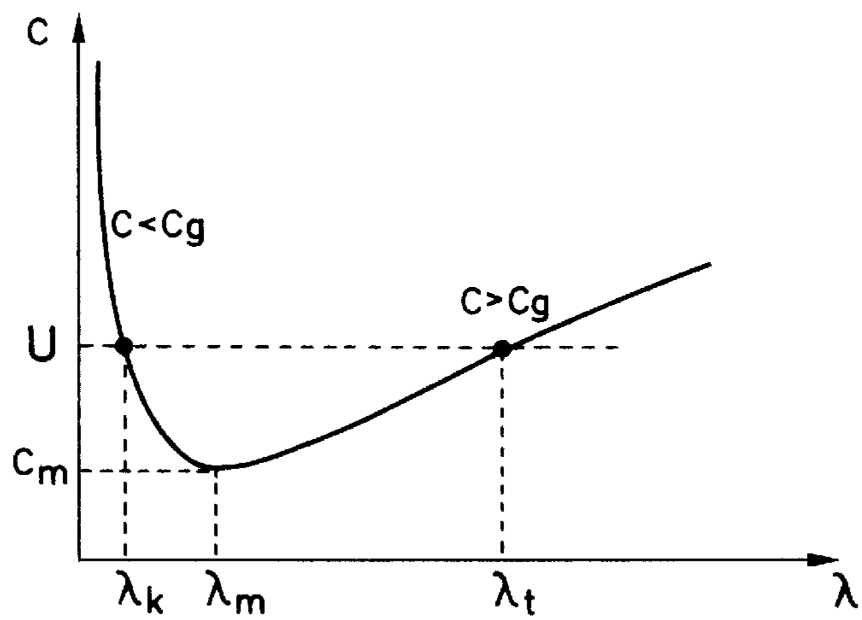

Figure 5.1: Dispersive properties for gravity–capillary waves.

The wave energy for waves with length $\lambda_t > \lambda_m$ will therefore not keep up with the disturbance, while wave energy for waves with length $\lambda_k < \lambda_m$ will run away from the disturbance. We will therefore get two stationary wave trains as shown in figure 5.2, gravity waves with wavelength λ_t behind the disturbance (downstream waves), and capillary waves with wavelength λ_k in front of the disturbance (upstream waves). If $U < c_m$ then no waves can follow the disturbance, and in this case we will not have either downstream or upstream waves associated with the disturbance. Similarly, downstream waves will not exist if U exceeds the velocity $\sqrt{gH}$ where H is the water depth. The amplitudes for downstream and upstream waves will depend on the disturbance. Even though both types of waves may be possible, it may occur that some disturbance only provokes one type with appreciable amplitude while the other type has insignificant amplitude.

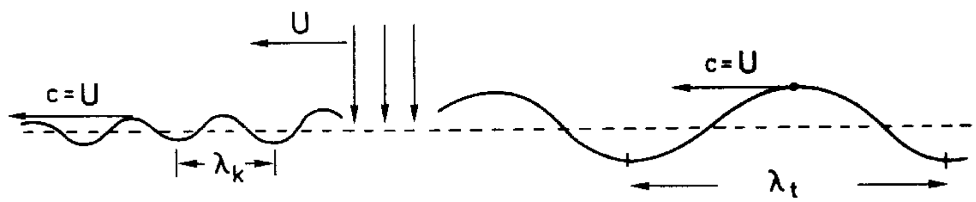

Figure 5.2: Downstream and upstream waves.

5.2 The amplitude for two-dimensional downstream waves

When we compute the amplitude of the downstream and upstream waves, we may end up with mathematical difficulties if we seek the stationary wave motion directly. The equations for the stationary wave motion will usually lead to ambiguity, such that it is not immediately clear which of the waves are on the downstream and upstream side of the disturbance. If we let the wave motion develop gradually into a stationary state, we may avoid this difficulty. Here we shall for simplicity carry out the analysis by letting the disturbance develop in a special manner (Whitham 1974). We consider a pressure disturbance

$$p_0(x,t) = f(x)\mathrm{e}^{\varepsilon t} \qquad \varepsilon > 0 \tag{5.1}$$

on the surface of a liquid layer of thickness H, which flows with uniform and stationary velocity U in the direction of the x-axis over a plane horizontal bottom $z = -H$, see figure 5.3.

We let the pressure disturbance be concentrated around $x = 0$ such that $p_0 \to 0$ for $x \to \pm\infty$, and such that the pressure has developed from zero at $t = -\infty$ to the value p_0 at $t = 0$. After a certain time $t > 0$ it is clear that we will have an unrealistically high pressure such that the linearizations we are

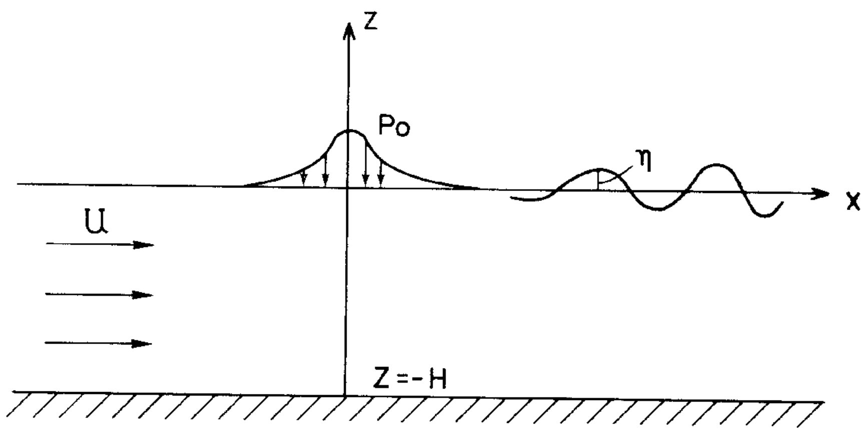

Figure 5.3: Downstream waves.

going to do are not valid. For sufficiently small values of p_0 the solution will be valid during a limited time $t = O(\frac{1}{\varepsilon})$ after the initial time $t = 0$. Letting $\varepsilon \to 0$ we will find the stationary wave motion provoked by the pressure disturbance p_0.

We assume the liquid is inviscid, homogeneous and incompressible and that the motion is irrotational. We shall furthermore neglect surface tension such that we limit to downstream waves. The velocity potential ϕ satisfies the Laplace equation

$$\nabla^2\phi = 0.$$

The linearized boundary conditions at the surface can be written

$$\begin{aligned} \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} &= \frac{\partial\Phi}{\partial z} \\ \frac{\partial\Phi}{\partial t} + U\frac{\partial\Phi}{\partial x} + g\eta &= -\frac{p_0}{\rho} \end{aligned} \tag{5.2}$$

for $z = 0$. We have redefined the potential in the usual way such that the constant $\frac{\rho}{2}U^2$ in Euler's pressure equation has been merged into the potential together with the function $f(t)$ (see section 2.1). The boundary condition at the bottom $z = -H$ is

$$\frac{\partial\Phi}{\partial z} = 0.$$

We seek a solution of these equations of the form

$$\begin{aligned} \Phi(x, z, t) &= \phi(x, z)\mathrm{e}^{\varepsilon t} \\ \eta(x, z) &= \eta_0(x)\mathrm{e}^{\varepsilon t} \end{aligned} \tag{5.3}$$

If we substitute these expression into the Laplace equation and the boundary conditions (5.2), the factor e^{ε} can be eliminated, and we are left with a set of equations for ϕ and η_0. We assume that the Fourier transform in x for the functions $f(x)$, $\phi(x,z)$ and $\eta_0(x)$ exist, and we denote these by $\tilde{f}$, $\tilde{\phi}$ and $\tilde{\eta_0}$ (see section 2.7). We get the solution

$$\tilde{\eta_0} = \frac{\tilde{f} k \tanh kH}{\rho[(kU - \mathrm{i}\varepsilon)^2 - gk \tanh kH]}.$$

By the inverse transform (2.59) we find that the surface deformation due to the pressure disturbance can be written

$$\eta(x,t) = \frac{\mathrm{e}^{\varepsilon t}}{2\pi\rho} \int_{-\infty}^{\infty} \frac{\tilde{f} k \tanh kH \mathrm{e}^{\mathrm{i}kx}}{(kU - \mathrm{i}\varepsilon)^2 - gk \tanh kH} dk. \tag{5.4}$$

We shall compute the integral (5.4) in the special case that the depth is infinite, and we have that

$$k \tanh kH \to |k|$$

for $H \to \infty$. We shall also for simplicity choose a pressure disturbance such that

$$f(x) = Q\delta(x)$$

where $\delta(x)$ is the Dirac delta function previously discussed in section 2.7, and Q is a constant. This implies that

$$\tilde{f}(k) = Q$$

such that $\tilde{f}$ can be placed outside the integral in (5.4). Then we can write the integral in (5.4) as a sum of two integrals such that

$$\eta(x,t) = \frac{Q\mathrm{e}^{\varepsilon t}}{2\pi\rho}[I_1 + I_2] \tag{5.5}$$

where

$$I_{1,2} = \int_0^{\infty} \frac{\mathrm{e}^{\pm \mathrm{i}kx}}{kU^2 \mp 2\mathrm{i}\varepsilon U - g} dk.$$

Index 1 and 2 correspond respectively to the upper and lower sign choices in the integrand. Assuming that ε is a small quantity we have eliminated contributions of order ε^2. In the complex k-plane the integrals $I_{1,2}$ have poles for

$$k_{1,2} = \frac{g}{U^2} \pm \frac{2\varepsilon}{U}\mathrm{i}$$

respectively.

In order to compute the integral I_1 we must distinguish between the cases $x > 0$ and $x < 0$. In the first case ($x > 0$) we integrate along the real k-axis to $k = \infty$, then along a circular path at infinity where the integrand $\to 0$, to the imaginary axis, and then back to the origin along the latter axis (see figure 5.3). This closed curve contains the pole, and the integral along the curve is therefore equal to $2\pi\mathrm{i}$ times the residue. In the other case $x < 0$, we choose a corresponding integration path in the fourth quadrant. This will therefore not contain the pole. In figure 5.4 the poles are indicated, and the two different integration paths for I_1 are sketched. We compute the integral I_2 in a similar manner, but with the path in the first quadrant for $x < 0$ and in the fourth quadrant for $x > 0$.

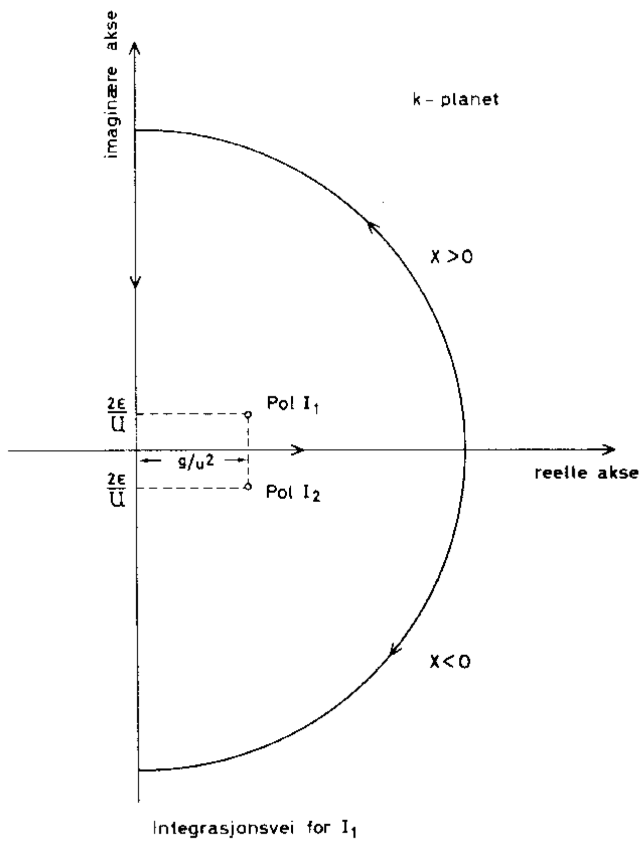

Figure 5.4: Integration contour.

The result of these computations gives for $\varepsilon \to 0$

$$I_1 + I_2 = -\frac{4\pi}{U^2}\sin k_s x + \frac{2}{U^2}\int_0^\infty \frac{k\mathrm{e}^{-kx}}{k^2 + k_s^2}\,dk \qquad x > 0$$

and

$$I_1 + I_2 = \frac{2}{U^2}\int_0^\infty \frac{k\mathrm{e}^{kx}}{k^2 + k_s^2}\,dk \qquad x < 0$$

where the wavenumber is $k_s = g/U^2$. We note that this wave has phase speed $c = U$ such that it is stationary with respect to the disturbance. If we go far from the disturbance the integrals in the expression for I_1+I_2 will only give small contributions, and we will have a stationary surface shape $\eta = \eta_s(x)$ which is well approximated by

$$\begin{aligned} \eta_s(x) &= -\frac{2Q}{\rho U^2}\sin k_s x \qquad & x > 0 \\ \eta_s(x) &\simeq 0 & x < 0 \end{aligned} \tag{5.6}$$

Finally we note that the introduction of the quantity ε was needed to get a unique solution for $\varepsilon \to 0$. If we had set $\varepsilon = 0$ in (5.5), the poles would be on the integration path and then there would be ambiguity in the solution. This ambiguity would imply that the stationary wave could be upstream or downstream of the disturbance. There are also other ways to avoid the ambiguity, one approach is to introduce a small artificial friction proportional to the velocity (Lamb 1932, p. 242). The problem of ambiguity is discussed by Palm (1953) in a paper on downstream waves in the atmosphere.

Exercises

1. Determine the stationary surface shape caused by a point pressure disturbance in the case that H is finite. Discuss in particular the cases $U < \sqrt{gH}$ and $U > \sqrt{gH}$.

2. Determine similarly the stationary surface shape for capillary waves on deep water.

5.3 Ship wake

Up to now we have studied the wave pattern resulting from disturbances with infinite extension in the transversal direction. If the disturbance has a finite transversal extension, there will be a two-dimensional wave pattern on the surface. Such stationary wave patterns can be observed behind a ship moving with steady velocity (figure 5.5), or in connection with a twig touching the surface of a brook flowing with constant velocity. Satellite images have shown similar wave patterns in the atmosphere when wind blows past an isolated mountain. It has recently been discovered that Beerenberg at Jan Mayen creates such wave patterns (figure 5.6).

The most characteristic property of the ship wake pattern is that the wave motion only appears inside a certain sector radiating out from the ship. If the ship moves on deep water, the angle of this sector is 39°, independent of the speed of the ship.

Suppose that a ship moves with constant velocity U along a straight course. After a time t the ship has moved a distance Ut from position B to position A (see figure 5.7).

Figure 5.5: Ship wake pattern in the Geiranger fjord. (Photo Røstad).

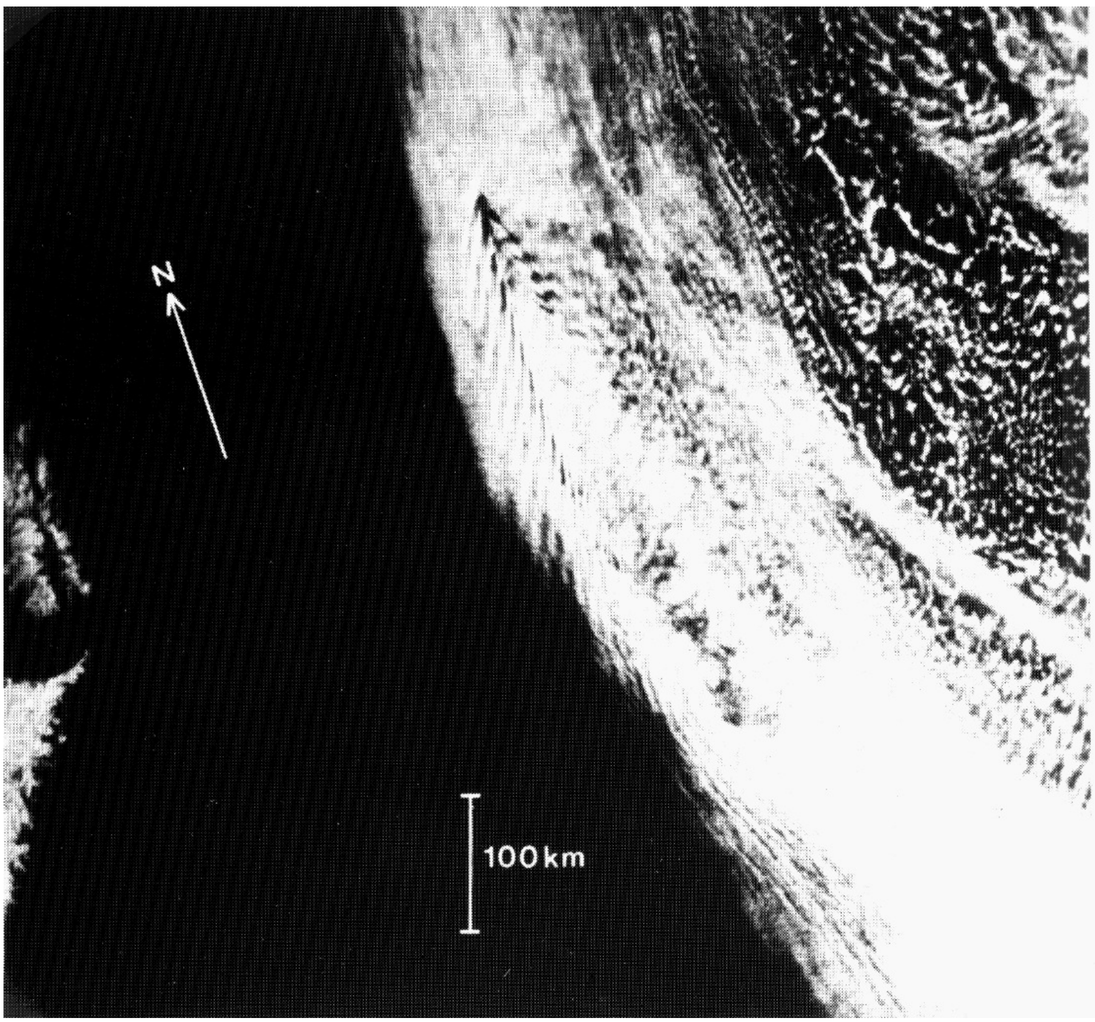

Figure 5.6: Satellite image showing a “ship wake pattern” in the cloud cover over Jan Mayen on the 1st of September 1976.

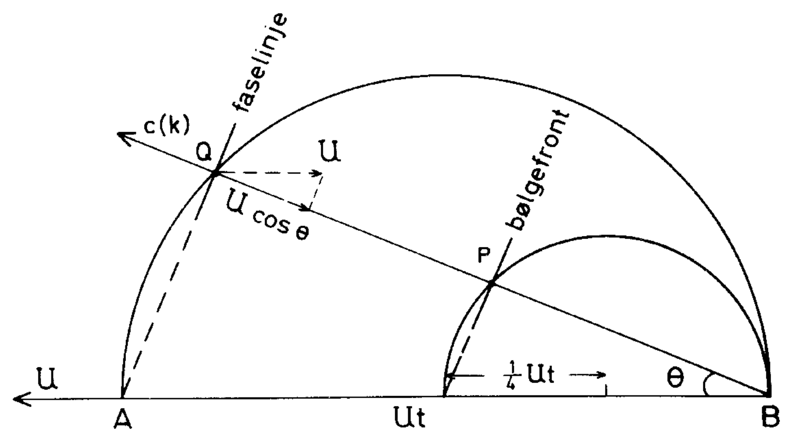

Figure 5.7:

At each location along its course the ship creates waves that propagate out in all directions and that interfere with each other. We seek a wave pattern that appears stationary as observed from the ship. We therefore consider waves sent out when the ship was at B along a ray BQ forming an angle θ with AB. The condition for these waves to be stationary is that

$$c(k) = U\cos\theta \tag{5.7}$$

where $c(k)$ is the phase speed for a wave component with wavenumber k. The energy propagates with the group velocity c_g, and for gravity waves we have $c_g \leq c$. Waves propagating from B along the ray BQ are therefore to be found somewhere along the ray between the points B and Q. For gravity waves on deep water we have $c_g = \frac{1}{2}c$, thus these waves will have arrived at a point P half way between Q and B. The stationary waves sent out from B will therefore lie on a circle through P and B with radius $\frac{1}{4}Ut$.

We can construct similar circles characterizing the wave fronts of stationary waves radiating from all points along AB. This is sketched in figure 5.8.

The circles delimit a sector where the half opening angle θ_s is determined by

$$\sin\theta_s = \frac{\frac{1}{4}Ut}{\frac{3}{4}Ut} = \frac{1}{3}.$$

The stationary wave pattern behind a ship moving with constant velocity on deep water therefore lies inside a sector with opening angle $2\theta_s = 39°$. If the ship moves on shallow water then

$$\frac{1}{2}c < c_g < c$$

and this causes the angle θ_s to achieve a value between $19.5°$ and $90°$ such that $\theta_s \to 90°$ when $c_g \to c$.

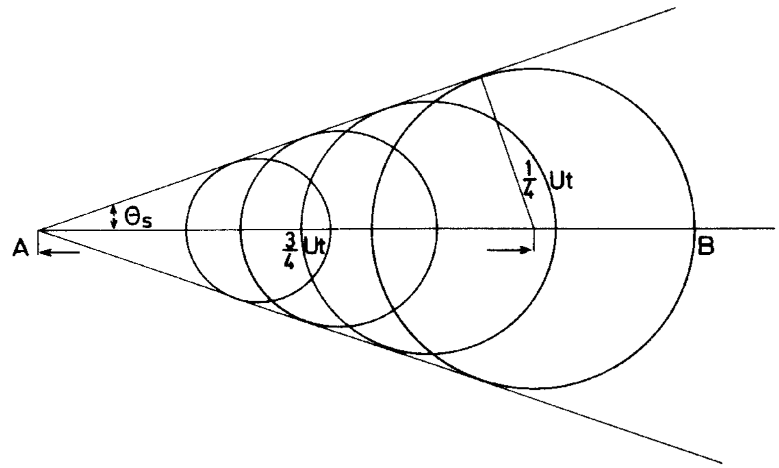

Figure 5.8:

The above simple explanation gives no information about the visual impression of the waves or their amplitudes. It turns out to be a formidable task to determine this for a ship hull of arbitrary shape. In the following we limit the discussion to wake pattern generated by a point disturbance on the surface. In the following we present different possible methods, otherwise the reader is referred to Newman (1977).

Method I

Let the disturbance (S) be at rest and let the water be moving with constant velocity U in the x-direction (figure 5.9). The wave pattern is described by a phase function ϕ where a curve of constant phase, $\phi = constant$, could for example be a wave crest. The wavenumber vector is then

$$\boldsymbol{k} = \nabla\phi. \tag{5.8}$$

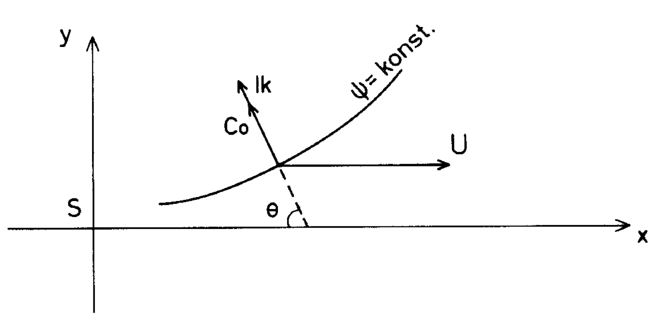

Figure 5.9:

A wave component, which in still water would have propagated with phase speed $c_0(k)$, will be swept by the current such that the phase speed relative to a fixed frame becomes

$$c = c_0(k) + \boldsymbol{U} \cdot \frac{\boldsymbol{k}}{k}. \tag{5.9}$$

For stationary waves we have $c = 0$, and by setting (5.8) into (5.9) we get a differential equation for ϕ. We will solve this differential equation by introducing characteristic curves such that the wavenumber components $k_x = \frac{\partial \phi}{\partial x}$ and $k_y = \frac{\partial \phi}{\partial y}$ are constants along these curves. This method is described by Whitham (1974). With $c = 0$ equation (5.9) becomes an equation for k_x and k_y which can be written

$$F(k_x, k_y) = 0 \tag{5.10}$$

where F denotes a function of the wavenumber components. This means that k_x can be expressed by a function of k_y, and we write

$$k_x = f(k_y). \tag{5.11}$$

From equation (5.8) it follows that $\nabla \times \boldsymbol{k} = 0$ such that

$$\frac{\partial k_y}{\partial x} - \frac{\partial k_x}{\partial y} = 0.$$

Substituting k_x from (5.11) the last equation can be written

$$(\frac{\partial}{\partial x} - f'(k_y)\frac{\partial}{\partial y})k_y = 0.$$

This means that k_y is constant on a characteristic curve $y = y(x)$ such that

$$\frac{dy}{dx} = -f'(k_y).$$

It follows from (5.11) that also k_x is constant along this characteristic curve. We integrate the last equation and choose the constant of integration such that the characteristic curve goes through the origin

$$y = -f'(k_y)x. \tag{5.12}$$

By differentiating (5.10) with respect to k_y we find

$$\frac{\partial F}{\partial k_y} + \frac{\partial F}{\partial k_x} f'(k_y) = 0$$

such that (5.12) can be written

$$\frac{y}{x} = \frac{\partial F}{\partial k_y} / \frac{\partial F}{\partial k_x}. \tag{5.13}$$

By means of (5.10) we can find k_x and k_y as functions of x and y. By integrating $\nabla\phi$ along the characteristic curves where k_x and k_y are constants, we find that the phase function is $\phi = k_x x + k_y y$. Curves of constant phase are therefore given by

$$\phi = k_x x + k_y y = -A \tag{5.14}$$

where A is a constant. The choice of sign for A and for k_x determines if the wave pattern is upstream or downstream.

As an example of how this theory can be used, we shall determine the phase lines for gravity waves on deep water. In this case we can write

$$F(k_x, k_y) = Uk_x + \sqrt{gk} = 0.$$

We introduce the angle θ (see figure 5.8) defined by

$$\cos\theta = -\frac{k_x}{k} \qquad \sin\theta = -\frac{k_y}{k}$$

and limit $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Thereby we find that

$$k = \frac{g}{U^2\cos^2\theta}$$

and

$$\frac{\partial F}{\partial k_x} = U(1 - \frac{1}{2}\cos^2\theta), \qquad \frac{\partial F}{\partial k_y} = \frac{U}{2}\sin\theta\cos\theta.$$

Inserting into (5.13) and (5.14) we find a parameter representation for the lines of constant phase

$$\begin{aligned} x &= A\frac{U^2}{g}\cos\theta(1+\sin^2\theta) \\ y &= A\frac{U^2}{g}\cos^2\theta(1+\sin\theta) \end{aligned} \tag{5.15}$$

where A is a constant. If we choose $A = 2\pi n$ where $n = 1, 2, \ldots$, we find a collection of phase lines behind the ship separated by a distance corresponding to the local wavelength. Such a collection of phase lines are shown in figure 5.10. The phase lines form a cusp for values of θ such that y, for a given A, has a maximum or a minimum. At these points we have $\sin^2\theta = \frac{1}{3}$, and therefore

$$|\frac{y}{x}| = \frac{1}{4}\sqrt{2}.$$

We have $\arctan(\frac{1}{4}\sqrt{2}) = 19.5°$, thus the wave pattern is inside a sector with opening angle 39°.

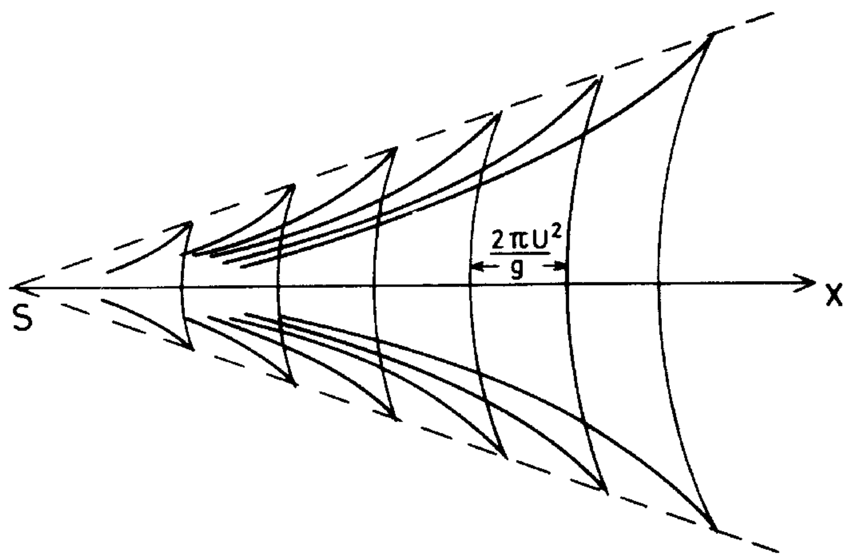

Figure 5.10:

The wave pattern consists of two wave systems. One has crests oriented in the transversal direction of the velocity of the ship. These are known as transversal waves.[1] The wavelength of the transversal waves behind the ship is $\lambda = 2\pi U^2/g$. The crests of the other wave system forms a fan that spreads out from the ship. These waves are called diverging waves.[2]

The method we have employed to compute the phase lines for the ship wake pattern is general and can be applied to other wave types. Gjevik and Marthinsen (1978) employed the same method to compute the phase lines for downstream waves in the atmosphere. They found good agreement comparing the computed wave pattern with satellite images (figure 5.6).

For a point disturbance it is also possible by relatively simple methods to determine the amplitude of the waves. Such computations for ship wakes can be found in Lamb (1932) and Newman (1977).

Method II

A point disturbance moving at the surface with velocity $-\boldsymbol{U} = -U\boldsymbol{i}$ generates a stationary and slowly varying wave train. This can be treated by ray theory as described in chapter 3. In a frame moving with the liquid we have an isotropic dispersion relation with corresponding phase speed $c = c_0(k)$ where $k = |\boldsymbol{k}|$. We change coordinate system such that the disturbance is at rest. We then get the

[1] Norwegian: tverrbølger, hekkbølger

[2] Norwegian: divergerende bølger, baugbølger

frequency

$$\omega = c_0(k)k + \boldsymbol{U} \cdot \boldsymbol{k} \equiv W(k_x, k_y) \tag{5.16}$$

where k_x and k_y are components of the wavenumber vector. W defined as above corresponds to F employed earlier. We note that (5.16) gives *anisotropic* dispersion due to the last term which is often denoted Doppler shift. The wave pattern being stationary means that in this coordinate system $\omega = 0$. The dispersion relation (5.16) immediately gives an equation for k_x and k_y

$$W(k_x, k_y) = 0. \tag{5.17}$$

The group velocity is now given by

$$\boldsymbol{c}_g = \frac{\partial W}{\partial k_x}\boldsymbol{i} + \frac{\partial W}{\partial k_y}\boldsymbol{j}. \tag{5.18}$$

As the medium is uniform, the ray equations give

$$\frac{d\boldsymbol{k}}{dt} = \boldsymbol{0}, \quad \frac{d\boldsymbol{r}}{dt} = \boldsymbol{c}_g \tag{5.19}$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \boldsymbol{c}_g \cdot \nabla. \tag{5.20}$$

It immediately follows that $\boldsymbol{k}$, and therefore $\boldsymbol{c}_g$, are constants along the characteristic curves, such that these become straight lines. Only those characteristic curves that go through the disturbance can carry energy. Without loss of generality we place the disturbance at the origin. Each ratio y/x therefore corresponds to a characteristic curve and everywhere in the medium we have the equation

$$\frac{y}{x} = \frac{\frac{\partial W}{\partial k_y}}{\frac{\partial W}{\partial k_x}} \tag{5.21}$$

which together with (5.17) implicitly defines k_x and k_y as functions of x and y. In order to find explicit expressions we may proceed in different ways. One possibility is to employ (5.17) to express the components of the wavenumber vector in terms of c_0. Substitution into (5.21) gives a biquadratic equation for c_0. Another possibility is to introduce the angle θ between $\boldsymbol{k}$ and the negative x direction. We can then set

$$k_x = -k\cos\theta, \quad k_y = k\sin\theta. \tag{5.22}$$

From (5.17) we immediately have

$$k = \frac{g}{U^2\cos^2\theta}. \tag{5.23}$$

We notice that this is a rewriting of $U = c_0/\cos\theta$. Substitution into (5.21) followed by elimination of k gives

$$\frac{y}{x} = \frac{\cos\theta\sin\theta}{1+\sin^2\theta} \equiv f(\theta). \tag{5.24}$$

The right hand side, $f(\theta)$, has a maximum for $\sin\theta = \sqrt{1/3}$, i.e. $\theta = \theta_c \approx 35.3°$. This corresponds to $y/x = \sqrt{2}/4$ which is identical to the outer limit of the wave pattern found earlier. For $y/x < \sqrt{2}/4$ we find two solutions for θ, one smaller than and one greater than θ_c. These two branches define two different wave fields. The first defines the so-called *transversal waves (hekkbølger)*, while the second gives the *diverging waves (baugbølger)*. Equation (5.24) can easily be written as a biquadratic equation for $\sin\theta$,

$$\left(1+\left(\frac{y}{x}\right)^2\right)\sin^4\theta + \left(2\left(\frac{y}{x}\right)^2 - 1\right)\sin^2\theta + \left(\frac{y}{x}\right)^2 = 0, \tag{5.25}$$

which has four solutions for $(y/x)^2 < 1/8$, corresponding to the two wave fields defined for positive and negative y.

For the phase function we can set up the integral

$$\chi(\boldsymbol{r}) = \chi_0 + \int\limits_{C(\boldsymbol{r})} \boldsymbol{k}\cdot d\boldsymbol{r} \tag{5.26}$$

where χ_0 is the phase at the origin and $C(\boldsymbol{r})$ denotes an arbitrary curve starting at the origin and ending in $\boldsymbol{r}$. In this case we get a trivial computation if we integrate along the characteristic curves

$$\chi(\boldsymbol{r}) = \chi_0 + k_x x + k_y y \tag{5.27}$$

where k_x and k_y are known functions of x and y. Each value of χ defines a phase line. We express the wavenumbers by means of θ also in (5.27). The equations (5.21) and (5.27) can then be solved with respect to x and y and we find a parameter representation of the phase lines ($\chi = -A = constant$)

$$x = \frac{(A+\chi_0)U^2}{g}\cos\theta(1+\sin^2\theta), \tag{5.28}$$

$$y = \frac{(A+\chi_0)U^2}{g}\cos^2\theta\sin\theta. \tag{5.29}$$

For $|\theta| < \theta_c$ we get transversal waves, while for $|\theta| > \theta_c$ we get diverging waves. As these two fields are independent, they do not need to have the same value for χ_0. The ray theory breaks down at the outer edge of the wave pattern, for $\theta \approx \pm\theta_c$.

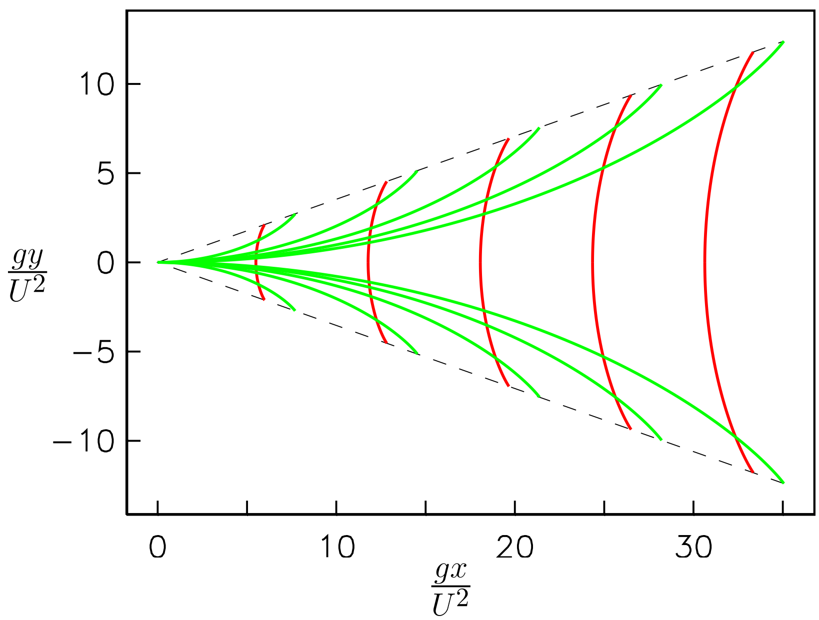

Figure 5.11: Phase diagram for the two wave systems behind a point disturbance on infinite depth. We have sketched the phase lines corresponding to $\chi = -2n\pi$, $n = 1, 2, \ldots, 5$.

Ray theory does not tell us how to determine the phase χ_0. For a point disturbance it can be shown that, e.g. using the method of stationary phase after Fourier transform, that $\chi_0 = \frac{1}{4}\pi, -\frac{1}{4}\pi$ for respectively transversal and diverging waves (see e.g. Newman (1977)). These values are used in figure 5.11. In figure 5.3 we have also sketched the geometrical location reached by the group velocity from a chosen point. The figure also shows the "energy transport paths" up to the crossing points with the phase lines.

Exercises

1. Determine the phase lines for stationary capillary waves on deep water generated by a point disturbance moving with constant velocity.

2. Determine the phase lines for gravity waves on water of finite depth generated by a point disturbance moving with constant velocity. Sketch the phase lines in the two cases $U < \sqrt{gH}$ and $U > \sqrt{gH}$.

5.4 Wave resistance

When a disturbance moves along the surface such that a stationary wave pattern arises, then the wave field will be constantly added energy in such a way that the

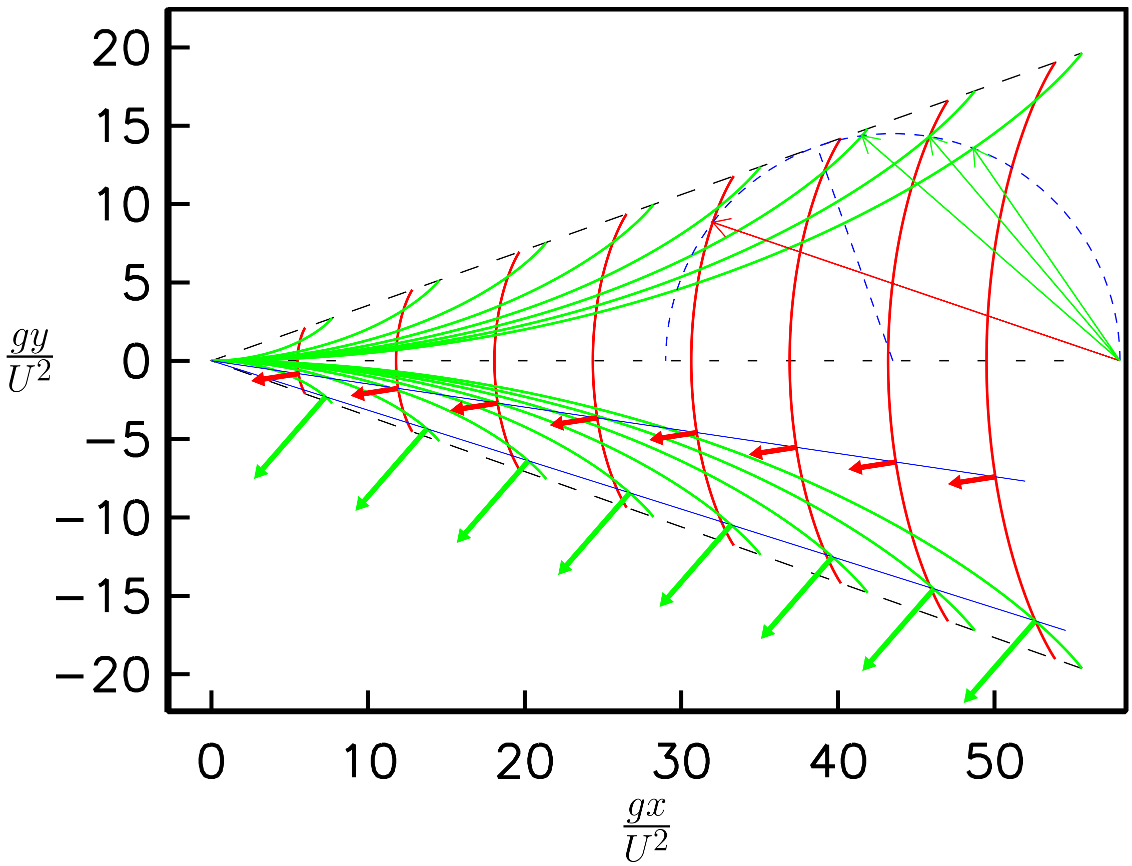

Figure 5.12: Phase diagram for Kelvin ship wake. The dashed half-circle denotes the geometric location where the group velocity brings us from the location where the half-circle crosses the x-axis toward the right. This means that the energy contributed by the disturbance when it passes this point now is along the dashed circle (and its mirror image under the y-axis). The arrows from the point to the wave crest intersection with the circle then corresponds to wave paths in the coordinate system of the moving disturbance. We note that the wave paths are at right angles to the crests.
In the lower half plane two characteristic curves have been sketched. Along the upper (scaled) wavenumber vectors corresponding to transversal waves is indicated, while along the lower we have sketched wavenumber vectors for diverging waves.

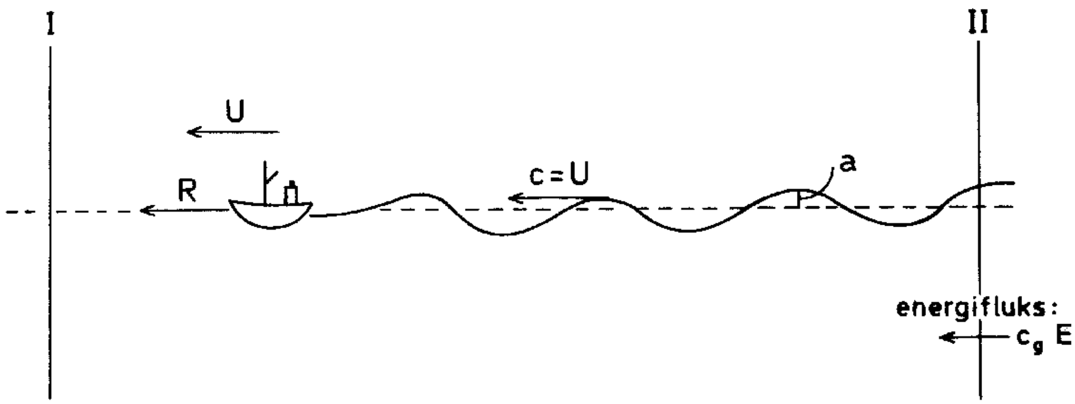

Figure 5.13:

size of the region affected by the wave motion increases. This energy must be transferred by the work done by the disturbance, and this has the consequence that one has to exert a force R to move the disturbance with constant velocity. This force is denoted *wave resistance*. We shall find an expression for this force for plane (two-dimensional) wave motion behind a disturbance that moves with constant velocity U. The stationary wave pattern has amplitude a, and according to (2.34) the average wave energy for gravity waves per unit of length along the surface is

$$E = \frac{1}{2}\rho g a^2$$

where ρ is the water density. In front of and behind the disturbance we construct vertical control planes, respectively I and II (figure 5.13). We then look at the increase in energy within the liquid volume delimited by the planes I and II within a time interval Δt. The work done by the propulsion force R is $RU\Delta t$. The increase in wave energy within the liquid volume due to the wave train becoming longer, is $Ec\Delta t$. There is also a flux of wave energy $c_g E\Delta t$ into the volume at the plane II. The energy balance for the liquid volume within the control planes is therefore

$$RU\Delta t + c_g E\Delta t = Ec\Delta t.$$

Thereby we find that the wave resistance on the disturbance can be written

$$R = \frac{c - c_g}{c} E.$$

For gravity waves on deep water $c_g = \frac{1}{2}c$ and

$$R = \frac{1}{4}\rho g a^2.$$

If one uses the computed amplitude for a stationary wave behind a point pressure disturbance (section 5.2), we find that in this case the wave resistance decreases with increasing velocity.

Methods for computing wave resistance when one accounts for the three-dimensional structure of the ship wake pattern are described by Newman (1977).

5.5 Surface waves modified by variable currents

We shall here briefly show how surface waves are modified when they propagate into a region with gradual changes in current velocity in the horizontal direction. This phenomenon can be observed in Nature where bathymetry or river estuaries lead to horizontal variations of current velocity. Figure 5.14 suggest part of this situation where a wave is propagating along the x-direction on a current with velocity $U(x)$ which is a function of x.

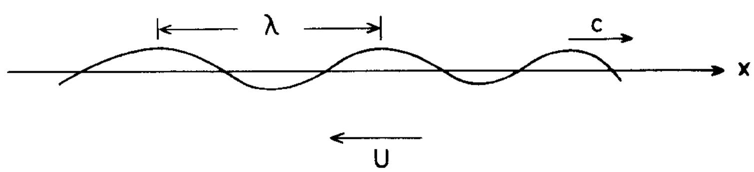

Figure 5.14:

Notice that the figure does not suggest the full story, since a variable current can produce up-welling or down-welling which can cause the surface displacement ζ due to the current to not be horizontal. Our goal is to describe the surface displacement η due to a wave disturbance relative to the background surface displacement due to the current. If we assume the current to be stationary, we can recall the Bernoulli equation for a stationary current

$$\frac{p}{\rho} + gz + \frac{1}{2}U^2 = \text{constant along a streamline}$$

Choosing the streamline to be the free surface in the absence of waves, but in the presence of the current, then the surface elevation $z = \zeta(x)$ is given by

$$\zeta = C - \frac{U^2}{2g}$$

where C is a constant.

We assume the current is slowly varying such that

$$|\frac{\lambda}{U}\frac{dU}{dx}| \ll 1$$

where λ is the wavelength. We can then apply ray theory and WKB theory to compute the details of the wave evolution.

Under quite general conditions it is found that the dispersion relation becomes

$$\omega = \sigma + \boldsymbol{k} \cdot \boldsymbol{U}$$

where σ is the so-called *intrinsic frequency*, i.e. the angular frequency we would have had in the absence of the current. On the left-hand side we have the *physical frequency* ω. The last term is called the *Doppler shift*. Taking the analysis to the next level in the WKB theory it can be shown that the *wave action* defined with the intrinsic frequency is conserved

$$\frac{\partial}{\partial t}\left(\frac{E}{\sigma}\right) + \nabla \cdot \left((\boldsymbol{c}_g + \boldsymbol{U})\frac{E}{\sigma}\right) = 0$$

where $\boldsymbol{c}_g = \partial\sigma/\partial\boldsymbol{k}$ is the group velocity due to the intrinsic frequency.

Weak frequency modulation of long swell due to time-varying tides has been observed in Nature, see figure 5.15.

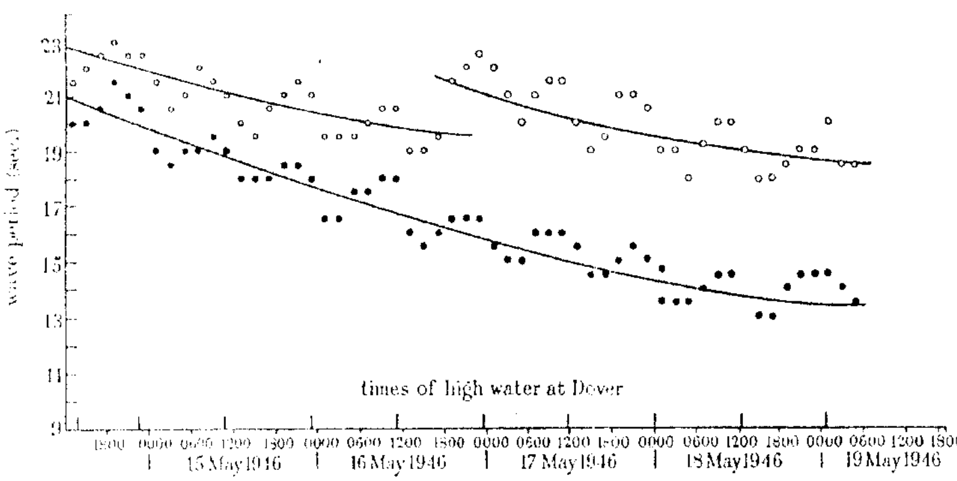

Figure 5.15: Modulations of the period of swell due to tidal currents. Three episodes observed at the coast of Cornwall, England. (After Barber 1949)

Exercises

1. Determine the frequency modulation for gravity waves that propagate against a uniform current $U(t) = U_0 \sin \Omega t$. The period $2\pi/\Omega$ is assumed to be much longer than the average period T_0 of the waves. Find a simple expression for the period when the current velocity U_0 is much smaller than the phase speed c_0 of the waves in quiescent water. Employ the data in figure 5.15 to determine the strength of the tidal currents.

2. For waves on a slowly varying background current, $\boldsymbol{U}$, we can assume that the dispersion relation is of the form

$$\omega = \sigma + \boldsymbol{k} \cdot \boldsymbol{U}$$

where σ is the frequency we would have had without current, the so-called *intrinsic frequency*. The last term is called the *Doppler shift*.

In this problem we look at long, linear surface waves on shallow water and assume the background current is given by $\boldsymbol{U} \equiv U(x)\boldsymbol{j}$ where $\boldsymbol{j}$ is a unit vector in the y-direction. Further we assume U has a bell shape

$$U = -U_m \mathrm{e}^{-\frac{x^2}{l^2}}$$

In such a current we can have trapped waves.

a Give a physical explanation of the trapping mechanism.

b Suppose that we have a given wavenumber in the y-direction. Find limitations on possible ω for trapped modes by ray theory.

c Set up the full linear eigenvalue problem determining the eigenmodes.

3. From point (c) in the previous problem we have that the wave motion in a unidirectional background current is governed by the equation

$$\left(\frac{\hat{\eta}_x}{P}\right)_x + (1 - \frac{k_y^2}{P})\hat{\eta} = 0; \quad P \equiv (\omega - Uk_y)^2/(gh)$$

where the background current U varies with x, and where $\hat{\eta}$ is related to the surface according to

$$\eta(x, y, t) = \mathrm{Re}\ \left(\hat{\eta}(x) e^{\imath(k_y y - \omega t)}\right)$$

Use the WKBJ method to find approximations to $\hat{\eta}$. Where are they valid? Does the result correspond to the conservation of energy of the wave motion?

Chapter 6

INTERNAL GRAVITY WAVES

Internal gravity waves can appear in the atmosphere, the ocean and in lakes, and are due to the force of buoyancy due to the vertical density stratification. Even though the density stratification can be due to different reasons, such as variations of temperature, salinity or concentration of dissolved substances, many of the properties of the internal waves are similar. The primary reason for the internal waves is the force of buoyancy, but the wave motion will in many cases be modified for other reasons, primarily compressibility and diffusion processes. Long internal waves in Nature will often be substantially affected by the Coriolis force. For relatively short internal waves in water or the ocean the liquid can be considered incompressible, and the wave motion can be practically unaffected by variations of density due to slow diffusion of heat or salt.

There is in particular one parameter which has fundamental importance for the description of internal waves. This is the buoyancy frequency or the Väisälä–Brunt frequency. We shall start by defining this quantity and give it a simple physical interpretation. At equilibrium the density is ρ_0 constant along horizontal surfaces, and ρ_0 is a function of the vertical coordinate z which may represent the depth.

A liquid particle which is displaced in the vertical direction a distance Δz from equilibrium, see figure 6.1, will be affected by a force of buoyancy per volume

$$-g[\rho_0(z+\Delta z)-\rho_0(z)] \simeq -g\frac{d\rho_0}{dz}\Delta z$$

which tries to move the particle back to the equilibrium position. The mass per volume times the acceleration is

$$\rho_0\frac{d^2}{dt^2}(\Delta z).$$

If we neglect the pressure force, the motion of the particle can be written

$$\frac{d^2}{dt^2}(\Delta z) = -N^2\Delta z \tag{6.1}$$

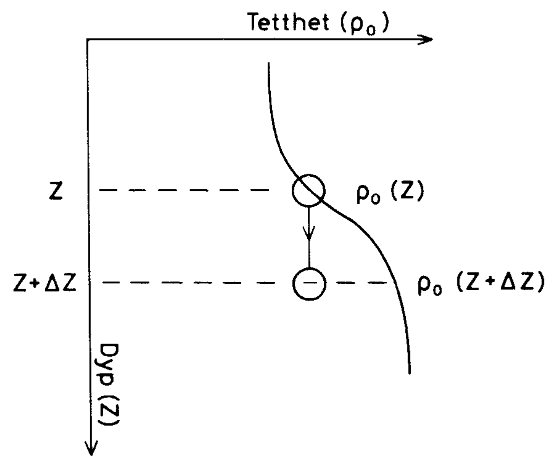

Figure 6.1:

where the quantity

$$N = (\frac{g}{\rho_0}\frac{d\rho_0}{dz})^{\frac{1}{2}} \tag{6.2}$$

is the *buoyancy frequency*. Equation (6.1) says that the liquid particle performs vertical oscillations with frequency N. Otherwise this simple model, neglecting pressure forces, does not give information about the properties of the wave motion.

6.1 Internal waves in incompressible liquids where diffusion can be neglected. Dispersion and particle motion.

The wave motion leads to velocity perturbations that can be expressed by the vector $\boldsymbol{v}$ with components u, v and w respectively along the x-, y- and z-axes. The former two axes are in the horizontal plane, while the z-axis denotes the depth as mentioned above. The motion causes the density to be modified in comparison to its value at equilibrium. We denote the change in density by ρ, and the total density at a particular location by $\rho_0 + \rho$. We assume the changes of density are only due to motion of fluid particles associated with the wave motion, and we thus neglect changes of density due to diffusion. We furthermore assume that the wave amplitude is sufficiently small that we can linearize the equations describing the motion. Since ρ depends on the wave amplitude, the linearization also implies that we omit products between for instance ρ and $\boldsymbol{v}$. Under these conditions the equation of motion, the equation of state and the continuity equation can be written

$$\frac{\partial \boldsymbol{v}}{\partial t} = -\frac{1}{\rho_0}\nabla p + \frac{\rho}{\rho_0}\boldsymbol{g} \tag{6.3}$$

$$\frac{\partial \rho}{\partial t} = -w\frac{d\rho_0}{dz} \tag{6.4}$$

$$\nabla \cdot \boldsymbol{v} = 0 \tag{6.5}$$

where p is the change of density due to the wave motion, and the vector $\boldsymbol{g}$ oriented in the direction of the z-axis denotes the acceleration of gravity.

Between equations (6.3)–(6.5) we can eliminate p, u, v and ρ such that we are left with an equation for the vertical velocity

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) + \frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{\partial^3 w}{\partial t^2 \partial z} = -N^2\nabla_h^2 w \tag{6.6}$$

where

$$\nabla_h^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In order to derive (6.6) we can proceed as follows: The vector equation (6.3) gives three equations for the velocity components

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{\rho}{\rho_0}g
\end{aligned}$$

By differentiating the first two, respectively, with respect to x and y, and add, we get

$$\frac{\partial}{\partial t}(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = -\frac{1}{\rho_0}\nabla_h^2 p.$$

Due to (6.5) this can be written

$$\frac{\partial}{\partial t}(\frac{\partial w}{\partial z}) = \frac{1}{\rho_0}\nabla_h^2 p.$$

Differentiating with respect to z gives

$$\frac{\partial}{\partial t}(\frac{\partial^2 w}{\partial z^2}) = -\frac{1}{\rho_0}\nabla_h^2 p(\frac{1}{\rho_0}\frac{d\rho_0}{dz}) + \frac{1}{\rho_0}\nabla_h^2\frac{\partial p}{\partial z}.$$

If we now apply the operator ∇_h^2 on the last of the three component equations, we can eliminate the pressure from the equation above such that we get an equation where only w and ρ appear. Using (6.4) we can also eliminate ρ. Thus we arrive at equation (6.6).

We expect that the most important effect of the density stratification is associated with the buoyancy which is now represented by the term on the right-hand side of equation (6.6). When the density modifications from equilibrium are small, the second term on the left-hand side in the same equation can be omitted. This corresponds to keeping the dynamic effect of the density stratification, but omitting the kinematic effect. This is known as the Boussinesq approximation, and it is in many cases a good approximation. The validity is discussed later in this section. In light of this we can therefore write (6.6)

$$\frac{\partial^2}{\partial t^2}(\nabla^2 w) = -N^2 \nabla_h^2 w. \tag{6.7}$$

For constant N we seek wave solutions of the form

$$w = A \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t) \tag{6.8}$$

where A is a constant. The corresponding velocity components in the horizontal direction are

$$u = -\frac{k_x k_z}{k_x^2 + k_y^2} A \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$$

and

$$v = -\frac{k_y k_z}{k_x^2 + k_y^2} A \sin(\boldsymbol{k} \cdot \boldsymbol{x} - \omega t)$$

where k_x, k_y and k_z denote respectively the x-, y- and z-components of the wavenumber vector.

Setting the expression in (6.8) into equation (6.7) we find the dispersion relation

$$\omega^2 = N^2 \frac{k_x^2 + k_y^2}{k^2} \tag{6.9}$$

This shows us that internal gravity waves of the form (6.8) have an angular frequency

$$\omega \leq N$$

and that the buoyancy frequency is an upper limit for the feasible frequencies. Setting $\omega = N$ causes the wave motion to be independent of z and that the vertical component of the wavenumber vector to vanish $k_z = 0$. The liquid particles move in this case in vertical planes ($u = v = 0$), and the oscillation period is according to what we found at the beginning of this chapter. For given values of ω and N the wavenumber vector has an angle

$$\theta = \arcsin(\frac{\omega}{N})$$

with the vertical. When the motion is periodic in x, y and z, it follows from the continuity equation (6.5) that

$$\boldsymbol{k} \cdot \boldsymbol{v} = 0$$

This shows that the particle motion for the internal waves is oriented normally to the wavenumber vector. The particle motion therefore happens in planes parallel to the phase planes, see figure 6.2.

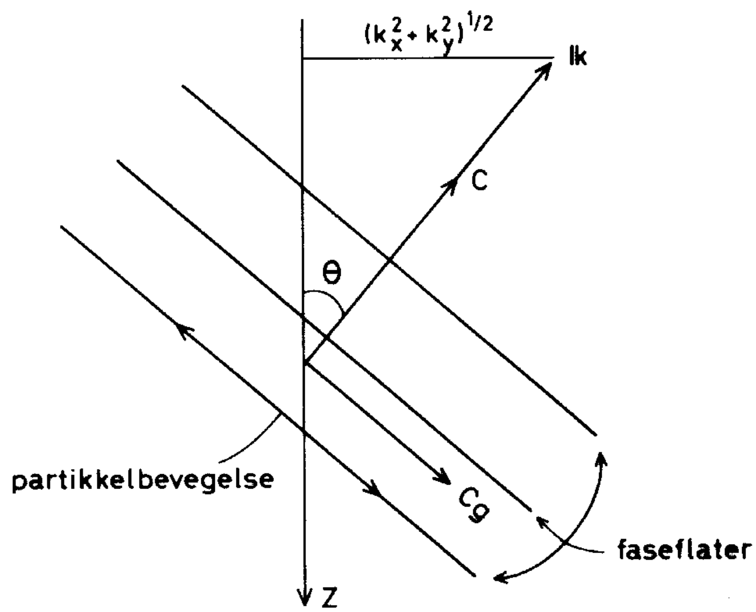

Figure 6.2:

In the dispersion relation (6.9) there is reference to the magnitude and the direction of the wavenumber vector. This means that we have anisotropic dispersion and that the propagation velocity depends on the direction. For anisotropic dispersion the group velocity is not along the direction of the wavenumber vector, and for internal gravity waves describe by the dispersion relation (6.9), the group velocity is oriented along the phase planes. The group velocity has components

$$\boldsymbol{c}_g = (\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z})$$

respectively along the three coordinate directions x, y and z. With the relation (6.9) we find

$$\boldsymbol{c}_g = \frac{c}{k}\{k_x \cot^2\theta, k_y \cot^2\theta, -k_z\} \tag{6.10}$$

We find

$$\boldsymbol{c}_g \cdot \boldsymbol{k} = 0.$$

This shows that the group velocity is oriented normally to the wavenumber vector. From (6.10) we see that the horizontal components of $\boldsymbol{c}_g$ and $\boldsymbol{k}$ have the same direction and that the vertical components are oriented in opposite directions. This means that the vertical energy transport due to internal gravity waves is oriented in the opposite direction of the vertical displacement of the phase planes. In figure 6.2 corresponding directions for wavenumber vector, group velocity and particle motion are indicated.

We are now able to describe under which conditions we can omit the second term on the left-hand side of equation 6.6. By means of 6.8 we find that the ratio between the second and the first terms in the equation is

$$|\frac{1}{\rho_0}\frac{d\rho_0}{dz}\frac{k_z}{k^2}|$$

This shows that the second term can be omitted in comparison to the first when the relative variation of density over a distance corresponding to k_z/k^2 is sufficiently small.

6.2 Internal waves associated with the stratification layer

In the ocean and in lakes one often find situations with stratification of water in a surface layer and a bottom layer where the density is approximately uniform in each of the layers. In a thin layer between the layers the density changes relatively fast in the vertical direction. This layer is called the *pycnocline*,[1] and arises through mixing of the different waters from the surface and the bottom layers. The variation of density in the pycnocline is due to differences in salinity or temperature. In fjords there is often a pycnocline in situations with large influx of fresh water at the surface, and in lakes it appears due to heating of the surface layer.

Figure 6.3 shows a characteristic temperature distribution in the upper part of a deep lake during the summer season. The temperature shown in the figure is the mean value for a given range of time. We see that the surface layer is 15–20 m thick and that the temperature is about 14°. The thermocline extends from 20 to 30 meter depth, and the water under the thermocline shows little variation of temperature with the depth. Based on these temperature observations one can compute the density ρ_0, the buoyancy frequency N and the period $2\pi/N$. The latter two quantities are also shown as function of depth in figure 6.3. In

[1] The English word *cline* corresponds to the Norwegian word *sprangsjikt*, meaning that some property changes rapidly across a thin layer. This can be a *pycnocline* if we refer to density, a *halocline* if we refer to salinity, a *thermocline* if we refer to temperature, or a *chemocline* if we refer to some chemical substance.

the surface layer and the bottom layer the buoyancy frequency is small, while the pycnocline is associated with a large value for N. Under such conditions a strong wind in the longitudinal direction of the lake can transport the warm surface layer in the direction of the wind such that the pycnocline is lifted up at the up-wind end and lowered at the down-wind end. When the wind calms, this displacement of the pycnocline can propagate as internal waves. Observations of such waves in Mjøsa are described by Mørk, Gjevik and Holte (1980). Similar phenomena are known from many other lakes, including Loch Ness in Scotland (Thorpe *et al.* 1972).

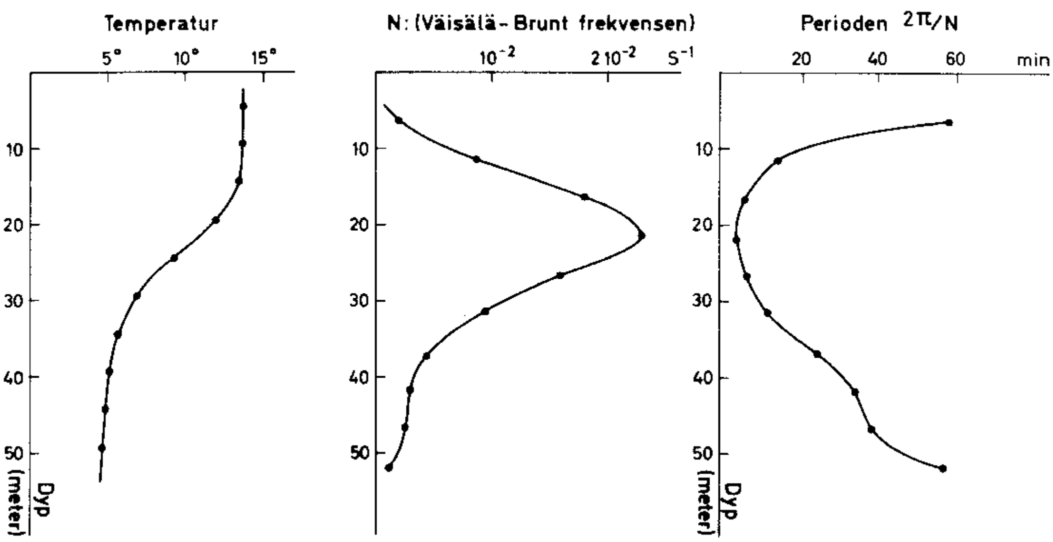

Figure 6.3: Measurement from Hamarbukta in Mjøsa 4–9 September 1974.

In fjords the variations in tidal currents over fjord thresholds can give rise to internal waves. This was observed in measurements of the Herdlefjord west of Bergen by Fjeldstad already during the 30's. The observations were subsequently published in Fjeldstad (1964). Internal tidal waves have been observed at several locations and under different circumstances. See for instance the survey paper by Farmer and Freeland (1983). Stigebrand (1979) found that such waves are generated at the Drøbak threshold in the Oslo fjord.

In the atmosphere one can sometimes find weather situations with temperature inversions or isothermal layers, while the layers over and under are in neutral equilibrium. The buoyancy frequency as a function of the height over the ground changes under such conditions in a similar way as the pycnocline in the ocean. This can give rise to internal gravity waves, and the ship wake pattern near Jan Mayen (figure 5.6) typically arises in such situations. In the atmosphere we also have to take into account the compressibility and variation of winds with the height. These problems are treated by Miles (1969), Gill (1983) and in GARP publication series (1980).

We shall here show some characteristic properties for internal waves associated with the pycnocline. We suppose N is a function of z and seek solutions of (6.7) describing plane waves propagating horizontally

$$w(x, z, t) = \hat{w}(z)\sin(kx - \omega t) \tag{6.11}$$

and we find

$$\frac{d^2\hat{w}}{dz^2} = -k^2\left(\frac{N^2}{\omega^2} - 1\right)\hat{w}. \tag{6.12}$$

Solutions of this equation for different models of the pycnocline are described by Krauss (1966) and Roberts (1975). The equation is of the same form as (4.5), and for large k the WKB method can be employed to find approximate solutions. For $\omega < N$ the solution of (6.12) is similar to harmonic oscillations, while for $\omega > N$ the solution will correspond to exponential attenuation (or increase). In the pycnocline we can therefore have wave motion with vertical velocity as indicated in figure 6.4.

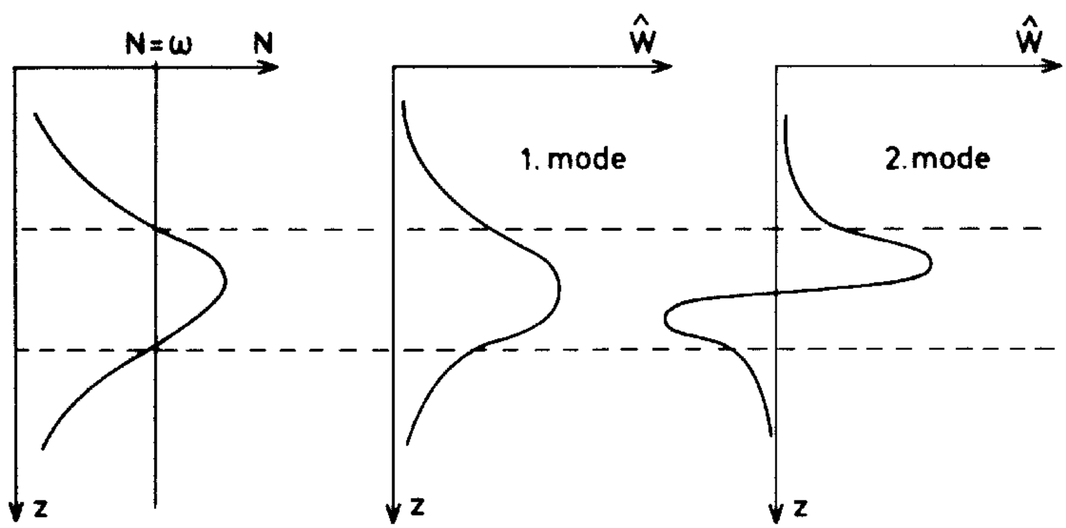

Figure 6.4:

The higher modes (mode 3, 4 etc.) are characterized by multiple oscillations within the region where $\omega < N$. For a certain value of ω there is a certain value of the wavenumber (eigenvalue) such that the lowest mode has the smallest wavenumber and the largest wavelength. The wavelength decreases gradually for the other modes. Waves in mode 1 have the largest propagation velocity in the horizontal direction. To modes 1 and 2 there there is a corresponding deformation of the pycnocline as suggested in figure (6.5).

The largest vertical velocities in the wave motion can be found near the pycnocline, and the vertical velocity is negligible at the surface. The displacement

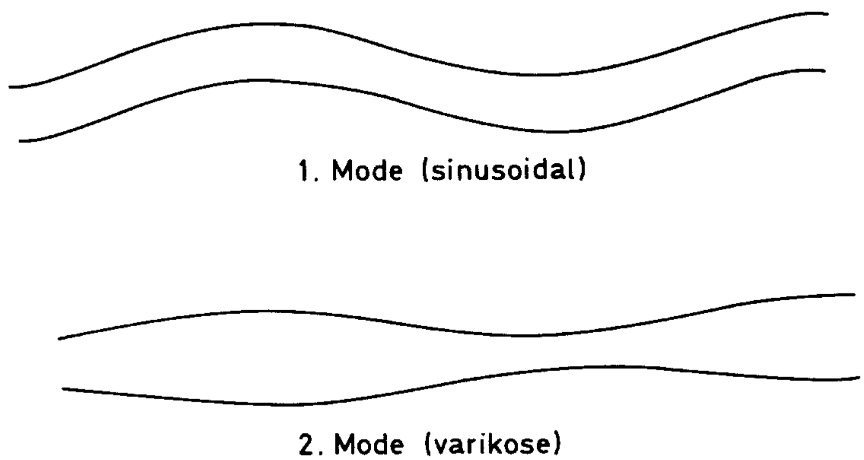

Figure 6.5:

of the surface will therefore be negligible even though the displacement of the pycnocline can be several meters.

In cases of a shallow pycnocline the internal waves can provoke relatively strong current variations on the surface. Such currents can modify short surface waves, with the consequence that under light winds the small waves have a tendency to gather in bands suggesting the internal waves below. These bands can sometimes be visible for the bare eye. Since the the reflection of radar waves from the sea surface strongly depends on the roughness of the surface, these bands are especially easy to see by radar. An example of such a radar image showing internal waves can be seen in figure (6.6).

Exercises

1. In cases with strong pycnocline of small vertical extent, the layered structure can be approximated by a two-layer model with constant density $\rho_0 - \Delta\rho_0$ and ρ_0 respectively in the upper and lower layer. The thickness of the upper and lower layers are denoted by h and H, respectively. Determine the dispersion relation and the velocity field for plane waves in this model, and show that the motion has two modes: a surface wave mode and an internal mode associated with the internal interface. Show that for long waves the phase speed for the surface mode and the internal mode are, respectively,

$$c = \sqrt{g(H+h)} \qquad \text{and} \qquad c = \sqrt{\frac{g^* H h}{H+h}}$$

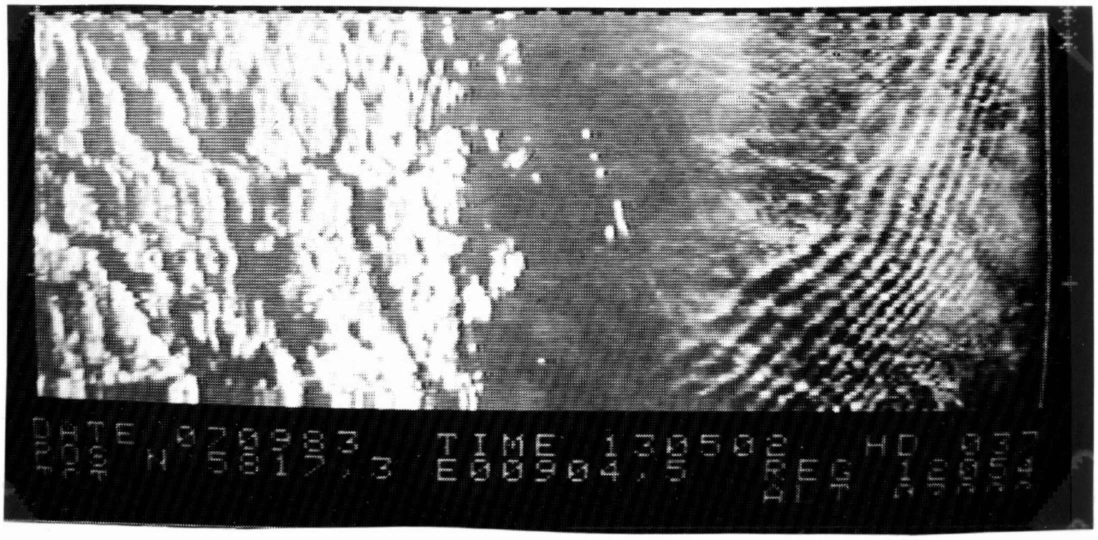

Figure 6.6: Radar echo from the sea surface in an area 10–15 km from shore south-east of Ryvingen between Grimstad and Arendal. The picture was taken on the 7th of September 1983 by a plane used for surveillance of oil pollution. The band structure suggests internal waves with wavelengths from 0.5 to 1 km. Further discussion is found in a paper by Gjevik and Høst (1984). (Photo Fjellanger Widerøe A/S).

where

$$g^* = \frac{\Delta\rho}{\rho_0} g$$

2. Determine the vertical velocity, and find the dispersion relation for internal gravity waves in a horizontal liquid layer with uniform depth H. The density at equilibrium increases exponentially with the depth (z)

$$\rho_0(z) = \rho_s e^{-\beta z}$$

   where ρ_s is the density at the surface. We suppose that the vertical velocity is zero both at the surface and at the bottom.

3. **Long waves on a pycnocline**

   We have a frictionless liquid with two immiscible layers. We suppose that the pressure in the liquid is hydrostatic, with zero pressure at the free surface, and that the motion happens in a vertical plane. At $t = 0$ the horizontal component of the velocity is independent of the vertical coordinate. We let h_1 and h_2 be the thicknesses of the upper and lower layers at equilibrium. η_1 and η_2 are the vertical displacements of the free surface and the internal interface, while u_1 and u_2 are the horizontal components of the velocities in the upper and lower layers. The acceleration of gravity is g, and the relative density difference between the layers is $\epsilon = \frac{\rho_2 - \rho_1}{\rho_2}$. The motion will then be governed by the following equations:

$$
\begin{aligned}
\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x} &= -g \frac{\partial \eta_1}{\partial x}, \\
\frac{\partial u_2}{\partial t} + u_2 \frac{\partial u_2}{\partial x} &= -g(1-\epsilon) \frac{\partial \eta_1}{\partial x} - g\epsilon \frac{\partial \eta_2}{\partial x}, \\
\frac{\partial}{\partial t}(\eta_1 - \eta_2) &= -\frac{\partial}{\partial x}\left((h_1 + \eta_1 - \eta_2) u_1\right), \\
\frac{\partial \eta_2}{\partial t} &= -\frac{\partial}{\partial x}\left((h_2 + \eta_2) u_2\right).
\end{aligned}
$$

(a) Derive these equations. What is the implication of the assumption of hydrostatic pressure?

In the rest of this problem we assume the bottom is flat and the displacements from equilibrium are small.

(b) Find the harmonic wave solutions of the set of equations. Show that they fall in two classes, and discuss the properties of these two modes.

(c) The set of equations have solutions that are pulses that propagate with permanent shape. Show that we have two modes of these pulses, corresponding to the modes in (b). Show by means of simple sketches how the displacement of the surface and the internal interface depend on the parameters of the problem. Illustrate graphically how the wave velocity depends on these parameters.

Chapter 7

NONLINEAR WAVES

Until now we have focused on linear wave solutions, which are found by assuming that the amplitude, a, is small. Previously we have found that for periodic deep water waves the nonlinear terms are small provided the wave steepness, ak, is small. For long waves in shallow water, on the other hand, the requirement for removing nonlinear terms is $a/H \ll 1$, where H is a typical equilibrium water depth. If the nonlinear terms are taken into account some features are changed and new features added in relation to the linear solutions.

1. The wave celerity will depend on the amplitude.

2. We may have steepening and, eventually, breaking of waves.

3. Even periodic waves may produce a net volume transport in the direction of wave advance.

4. The principle of super-positioning of waves is no longer valid. Different harmonic components may interact and exchange energy.

Nonlinear wave theory is a large field which is still evolving. Below we will study a few nonlinear wave solutions and effects.

7.1 Nonlinear waves in shallow water. Riemann's solutions. Wave breaking.

If the Ursell parameter is large, $\frac{a}{H}(\frac{\lambda}{H})^2 \gg 10$, say, nonlinear effects will dominate over dispersion and we may employ the nonlinear shallow water equations (NLSW). We will transform the NLSW equations to characteristic form (explained below) and to this end we introduce the new dependent variable

$$c^2 = g(H + \eta). \tag{7.1}$$

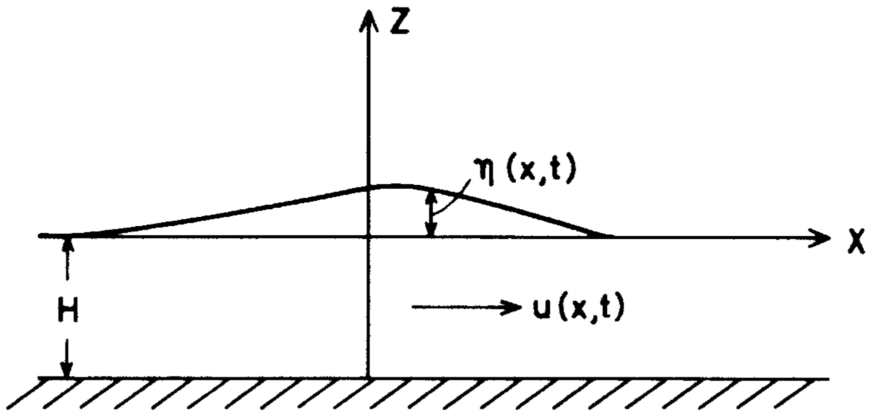

Figure 7.1:

Replacing η in (2.90) and (2.91) by c, and assuming constant depth we obtain

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -2c\frac{\partial c}{\partial x},$$

$$\frac{\partial}{\partial t}(2c) + c\frac{\partial u}{\partial x} + u\frac{\partial}{\partial x}(2c) = 0.$$

Adding and subtracting these equations we find

$$[\frac{\partial}{\partial t} + (c+u)\frac{\partial}{\partial x}](2c+u) = 0 \tag{7.2}$$

$$[\frac{\partial}{\partial t} - (c-u)\frac{\partial}{\partial x}](2c-u) = 0 \tag{7.3}$$

These equations are expressed in terms of temporal differentiation along curves in the (x, t) plane, as explained below. This is an example of Riemann's method. Riemann (1892) applied a similar transformation to the nonlinear acoustic equations, which are of the same form as (7.2) and (7.3).

Equation (7.2) expresses that

$$P \equiv 2c + u = \text{constant} \tag{7.4}$$

along *characteristics*, C^+, in the (x,t) plane defined according to

$$C^+: \quad \frac{dx}{dt} = c + u. \tag{7.5}$$

Correspondingly, (7.3) implies that

$$Q \equiv 2c - u = \text{constant} \tag{7.6}$$

along characteristics defined according to

$$C^- : \quad \frac{dx}{dt} = -(c - u) \tag{7.7}$$

The quantities P and Q are generally denoted as Riemann invariants.

Any point in the (x, t) plane corresponds to an intersection of one C^+ and one C^- characteristic. If u and η were given as initial conditions and we knew where the two characteristics passing through a point (x_s, t_s) crossed $t = 0$ we would know the values of P and Q along C^+ and C^-, respectively, and hence the values of P and Q at (x_s, t_s). Then u and η at the point are readily found from P and Q. Unfortunately, before the characteristics reach (x_s, t_s) their path is influenced by all other characteristics they encounter for $t < t_s$. Hence, the intersections with the x-axis are not easy to find.

The characteristics C^+ and C^- describe wave propagation in positive and negative x-direction, respectively. Thus, for a unidirectional wave information of deviation from equilibrium is conveyed along only one type of characteristics. In this case the description simplifies substantially, as described in the following. We assume that at $t = 0$ a disturbance is localized to a confined region

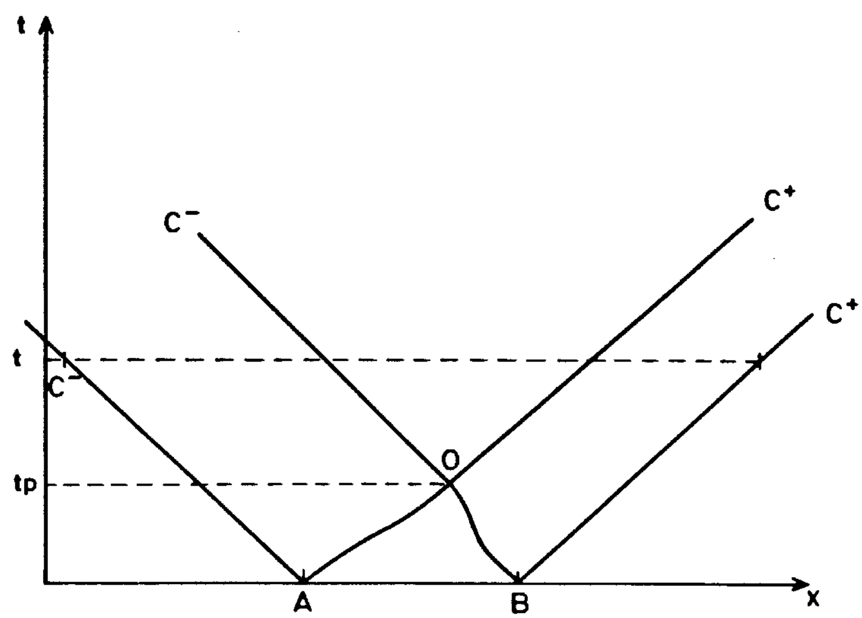

Figure 7.2:

AB. In this case the characteristics passing through A and B can be depicted as in figure 7.2. Naturally, there are an infinite number of other characteristics which are not depicted. At t_p the C^- characteristic from B intersects the C^+ characteristic from A. Hence, for $t > t_p$ the waves are split into two unidirectional waves propagating in the positive and negative x-direction, respectively.

We now look closer at the wave propagating in the positive direction. Information from the initial condition is then transported along the C^+ characteristics which are between those originating from A and B. For $t > t_p$ these will meet C^- characteristics originating from points at the x-axis with initial equilibrium. Therefore, these C^- characteristics all carry the equilibrium value of Q, namely $Q_0 = 2c_0 = 2\sqrt{gH}$. As a consequence Q equals Q_0 everywhere in the region of right-going waves. That yields a relation between u and c

$$Q = Q_0 = 2c_0 \quad \Rightarrow \quad c = c_0 + \frac{1}{2}u,$$

that may be inserted into P and the equation for the C^+ characteristics

$$C^+ : \quad \frac{dx}{dt} = c_0 + \frac{3}{2}u, \qquad P = 2c_0 + u = \text{constant.},$$

Which means that u is constant along C^+ which in turn implies that the characteristic is a straight line as indicated in figure 7.2. Moreover, since $Q = Q_0$ it also follows that c and η are constant along C^+. Using $Q = Q_0$, the characteristic speed may be rewritten

$$\frac{dx}{dt} \equiv c_f = c_0 + \frac{3}{2}u = 3c - 2c_0 = c_0[3\sqrt{1 + \eta/H} - 2]. \tag{7.8}$$

If $\eta/H \ll 1$ we then have

$$c_f \simeq c_0[1 + \frac{3}{2}\frac{\eta}{H}]. \tag{7.9}$$

Equations (7.8) and (7.9) show that the wave propagation speed increases with η/H. When $\eta/H > 0$ the wave peak will propagate faster than the rest. As a consequence the wave will steepen at the front as illustrated in figure 7.3. At a time, t_b, the tangent at the wave front will be vertical. This can be taken as an indication of wave breaking. For $t > t_b$ the front will be multi-valued corresponding to intersection of two, or more, C^+ characteristics. Naturally, the solution is then invalid. We should also be aware that a very steep front, prior to breaking, may challenge the long wave assumption. In fact, if the amplitude is small, $\eta_{\max}/H < 0.35$, say, dispersion will check the steepening and an *undular bore* will evolve. In section 7.3 we will investigate the waveform that will appear for larger amplitudes, namely a hydraulic shock.

7.2 Korteweg-de Vries equation. Nonlinear waves with permanent form. Solitons.

Important properties of long waves in shallow water can be summarized by three simple model equations. We choose to look at waves propagating in the x-axis

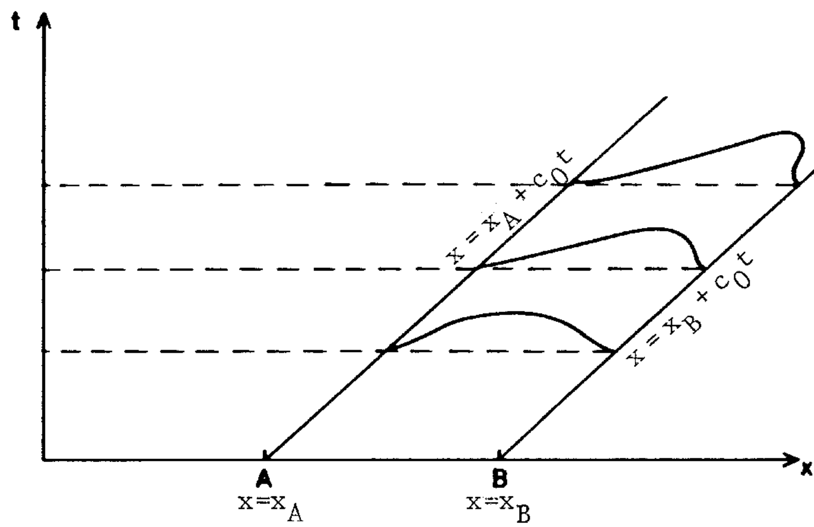

Figure 7.3:

direction. Then the equation

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} = 0 \tag{7.10}$$

where $c_0 = \sqrt{gH}$, describes linear non-dispersive waves. The equation has the solution $\eta = f(x - c_0 t)$ where f is an arbitrary function. The wave moves with constant velocity and unchanging form. The equation

$$\frac{\partial \eta}{\partial t} + c_0 \frac{\partial \eta}{\partial x} + \frac{c_0 H^2}{6} \frac{\partial^3 \eta}{\partial x^3} = 0 \tag{7.11}$$

describes in the first approximation, linear dispersive waves. The equation has the solution $\eta = Ae^{ik(x-ct)}$ where $c = c_0(1 - \frac{(kH)^2}{6})$. This is in agreement with what we have found in section 2.1 for long waves in shallow water after having corrected for the deviation from the hydrostatic pressure distribution. Further, the equation

$$\frac{\partial \eta}{\partial t} + c_0(1 + \frac{3}{2}\frac{\eta}{H}) \frac{\partial \eta}{\partial x} = 0 \tag{7.12}$$

will in the first approximation describe nonlinear waves propagating with a velocity which depends on amplitude in correspondence with what we have found in section 7.1. Equation (7.12) expresses therefore that the wave form changes and develops a steep front if $\eta/H > 0$. By combining equations (7.11) and (7.12), in the first approximation, one gets an equation that incorporates both dispersion and non-linear effects. This is the Korteweg-de Vries equation (KdV equation).

$$\frac{\partial \eta}{\partial t} + c_0(1 + \frac{3}{2}\frac{\eta}{H}) \frac{\partial \eta}{\partial x} + \frac{c_0 H^2}{6} \frac{\partial^3 \eta}{\partial x^3} = 0. \tag{7.13}$$

In section 2.11.2 the KdV equation was derived by a formal expansion in the parameters a/H and $(H/\lambda)^2$. A differential equation of the KdV type will be valid for all wave types where nonlinearity and dispersion are of the special form described here. The KdV equation for waves in shallow water was first derived by Korteweg and de Vries (1985). It has been also shown that the equations can emerge from long waves in other media such as plasma in a magnetic field.

In section 2.10 we found that the dispersion term which comes from deviations from the hydrostatic pressure distribution is of the same order of magnitude as the dominant nonlinear term if the Ursell parameter $\frac{a}{H}(\frac{\lambda}{H})^2$ is around 10. In this case, we can expect that nonlinearity and dispersion may balance each other such that nonlinear waves can propagate with constant velocity and permanent form. To investigate this we seek a solution of the KdV equation of the form

$$\eta = H\zeta(\psi)$$

where ζ is a function of $\psi = (x - Ut)/H$. Setting into the KdV equation we find

$$\zeta''' + [6(1 - \frac{U}{c_0}) + 9\zeta]\zeta' = 0$$

where the mark $'$ denotes the differentiation with respect to ψ. By integrating

$$\zeta'' + 6(1 - \frac{U}{c_0})\zeta + \frac{9}{2}\zeta^2 = \text{constant}$$

we shall now assume that ζ, ζ' and $\zeta'' \to 0$ for $\psi \to \infty$. The constant of integration can be set to zero, and by multiplying the last equation with ζ' and integrating we find

$$\zeta'^2 = 3\zeta^2(\alpha - \zeta) \tag{7.14}$$

where

$$\alpha = 2(\frac{U}{c_0} - 1) \tag{7.15}$$

Equation (7.14) has the solution

$$\zeta = \alpha\,\mathrm{sech}^2[(\frac{3\alpha}{4})^{\frac{1}{2}}\psi] \tag{7.16}$$

where $\mathrm{sech} z = 1/\cosh z$. The surface displacement corresponding to (7.16) is sketched in figure 7.4 for $\alpha = 0.5$. The figure shows a symmetric wave form established by a single wave crest. The wave has the amplitude αH, and moves with velocity

$$U = c_0(1 + \frac{\alpha}{2}).$$

Because of this, waves of this form are called *solitary waves*. They can be generated in a wave channel, propagate with permanent form for long distances,

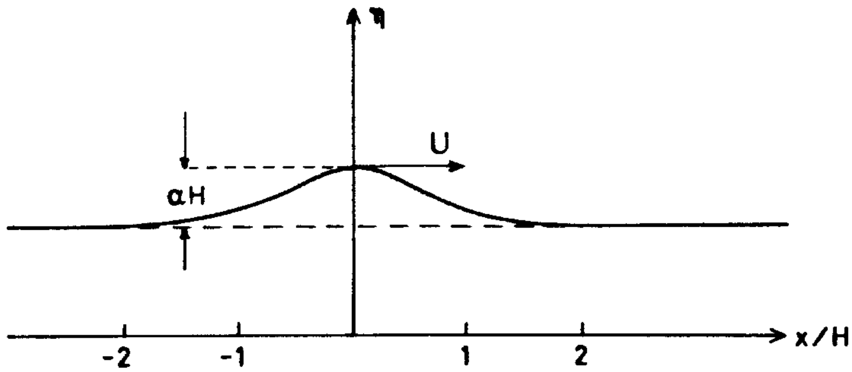

Figure 7.4:

and reflect from the end wall in the channel. The KdV equation also has a periodic wave solution propagating with permanent form and constant velocity. These waves are called *cnoidal waves.* They consist of steep wave crests separated by extended wave troughs.

The solitary wave and the cnoidal wave have been known for a long time, both as solutions of the KdV equation and from experiments. Recently there have been a series of discoveries that have renewed interest in these wave phenomena. Through theoretical work, the KdV equation has been successfully transformed into a form which makes it possible to discuss other solutions of the equation. One has, among another discoveries, that two solitary waves propagating toward each other will collide and then reappear as solitary waves moving from each other. There are other important wave phenomena described by the KdV equation within quantum mechanics, optics, and in crystal lattices. Also for equations different from the KdV equation, solitary waves similar to the one described above have been found. The common designation for such waves is *soliton.*

7.2.1 More about solitons

Simple and idealized wave solutions play an important role in wave theory. This is partly because these are building blocks for more complex wave patterns. Another reason is that the study of special solutions can give general insight about the behavior of waves and the physical mechanisms involved.

In the literature wave of *permanent form* are important. As the name suggests these are waves propagating with permanent form and constant velocity. Such solutions are exact only in uniform media, but often they give a good approximation when the medium is slowly varying. The simplest and most important example of waves of permanent form is the simple harmonic mode (the sine wave) which we find in linear theory. There are also some nonlinear waves of permanent form, such as the Stokes wave which is a generalization of the harmonic mode, shocks of finite extent in diffusive media and solitary waves which we will discuss here. As the name suggests, a solitary wave consists of a single wave crest. Strictly speaking it does not have infinite extent, but the strength (deviation from equilibrium) disappears when we move away from the crest. If solitary waves satisfy certain interaction relations they are called solitons. These relations typically require that the solitons survive collisions without loss of identity or total energy, etc.

Solitary waves were first described by J. Scott Russel in his groundbreaking investigation "Report on Waves" which was written upon request from The Royal Society of London in 1842. Scott Russel was a British waterway engineer and did many observations of waves in canals, including observations of a single (solitary) wave crest propagating over long distances without noticeable change of shape or splitting. He managed to recreate such a wave in the laboratory and measured it velocity to be $\sqrt{g(h+A)}$ where g is the acceleration of gravity, h is the depth at equilibrium and A is the amplitude.

The first who gave a complete theoretical description of the solitary wave was J. Boussinesq in 1871. It was in connection with this work that he derived the original Boussinesq equation. Later more accurate expressions for the solitary wave have been found from perturbation expansions applied directly on the full set of equations for an ideal liquid. Totally different types of solitary waves have also been found. Those solitary wave that are of the same kind as the long surface waves are today often called Boussinesq solitons.

Research on solitary waves was popular during the 60's. Based on the simple KdV equation a comprehensive theory on such waves was developed. Remarkable interaction properties between solitary waves were found, analogous to collision between particles, which gave a connection to quantum mechanics. By the so-called inverse scattering theory one could predict much about how solitons could develop from initial conditions of general form. Later one has found some other special phenomena where solitary waves are, or can be, involved. We will not treat these topics here.

7.3 Bores. Hydraulic jumps.

After breaking, a steep front with strong turbulence and aeration may evolve. This front may propagate as a wave or be steady on a current. The former may be observed for tsunamis entering shallow water while the latter is often seen in rivers or canals with rapid flow and are the often generated by a sill or boulder at the bottom.

We will outline a model of a simple, stationary *bore*, or *hydraulic jump*. The jump is idealized as a discontinuity between to different flow depths, namely h_1 and h_2, where we have chosen $h_1 < h_2$, as shown in figure 7.6.

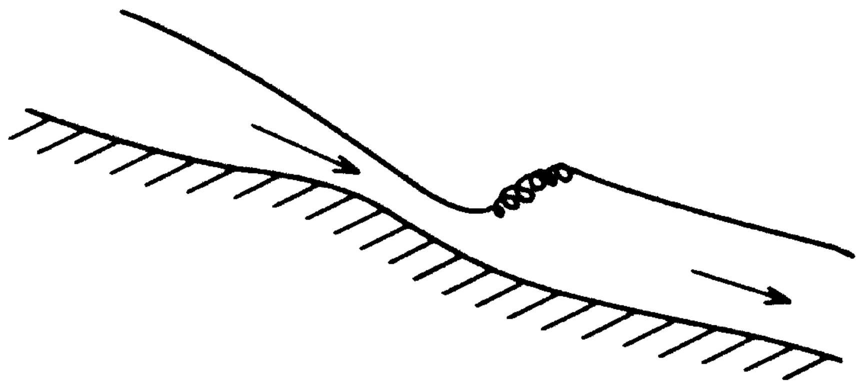

Figure 7.5:

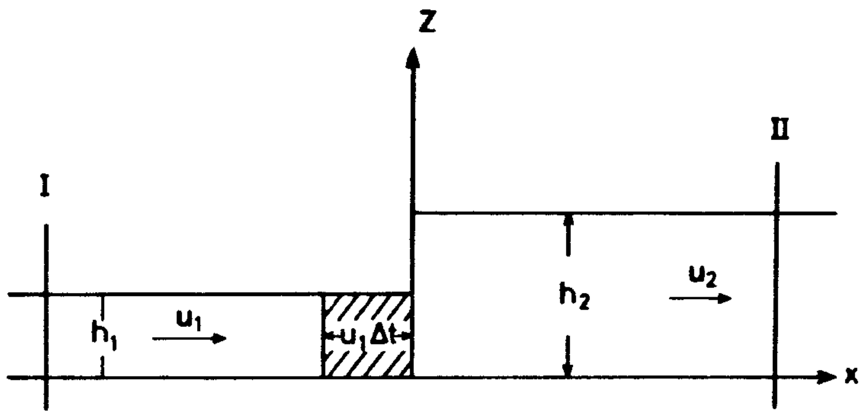

Figure 7.6:

At each side of the jump we assume constant flow depths and velocities, denoted by u_1 and u_2, respectively. We assume that nonlinear shallow water theory is valid on each side of the jump. Then u_1 and u_2 are independent of z and the volume flux becomes

$$Q = u_1 h_1 = u_2 h_2. \tag{7.17}$$

Next we invoke conservation of horizontal momentum in the volume between the two vertical transects I and II, as marked in figure 7.6. The net out-flux of momentum from the volume and the net pressure force at the transects must then counterbalance each other. Due to the hydrostatic pressure distribution we obtain

$$\rho u_2^2 h_2 - \rho u_1^2 h_1 = -\frac{1}{2}\rho g h_2^2 + \frac{1}{2}\rho g h_1^2,$$

where ρ is the density of the fluid. This relation may be rewritten by means of

(7.17). We find

$$Q(u_2 - u_1) = \frac{g}{2}(h_1^2 - h_2^2). \tag{7.18}$$

From (7.17) and (7.18) we find Q, u_1 and u_2 expressed in terms of h_1 and h_2,

$$\begin{aligned} Q^2 &= \frac{g}{2}h_2 h_1(h_1 + h_2), \\ u_1 &= c_1\sqrt{\frac{1}{2}\left(\frac{h_2}{h_1}\right)\left(1+\frac{h_2}{h_1}\right)}, \\ u_2 &= c_2\sqrt{\frac{1}{2}\left(\frac{h_1}{h_2}\right)\left(1+\frac{h_1}{h_2}\right)}, \end{aligned} \tag{7.19}$$

where $c_1 = \sqrt{gh_1}$ and $c_2 = \sqrt{gh_2}$. From (7.19) it follows that $h_1 < h_2$ implies $u_1 > c_1$ and $u_2 < c_2$. The upstream velocity is larger than the linear shallow water speed (supercritical), whereas the downstream velocity behind the jump is smaller that the shallow water speed (subcritical). A hydraulic jump occurs when a supercritical flow changes to a subcritical flow.

Next we investigate the energy budget of the hydraulic shock. In terms of the volume flux, Q, the power (work per time) exerted by the pressure at the transects I and II are, respectively,

$$W_I = \frac{1}{2}\rho g h_1 Q \quad \text{and} \quad W_{II} = \frac{1}{2}\rho g h_2 Q.$$

The mechanical energy advection, per time, into the control volume at I is

$$f_I = (\frac{1}{2}\rho u_1^2 + \frac{1}{2}\rho g h_1)Q,$$

whereas the corresponding flux rate at II is

$$f_{II} = (\frac{1}{2}\rho u_2^2 + \frac{1}{2}\rho g h_2)Q.$$

In the present context the zero level for potential energy in the gravity field is at $z = 0$. This choice is now convenient, but is different from the zero level applied in the sections on wave energy. We may the express the net energy supplied to the control volume, per time, as

$$\dot{E} = W_I - W_{II} + f_I - f_{II} = \frac{\rho}{2}Q[u_1^2 - u_2^2 - 2g(h_2 - h_1)].$$

Rewriting this expression by means of (7.19) we obtain

$$\dot{E} = \frac{\rho g}{4}\frac{Q}{h_1 h_2}(h_2 - h_1)^3. \tag{7.20}$$

Hence, $h_2 > h_1$ implies $\dot{E} > 0$. Since, the flow is stationary there is no change of kinetic energy in the control volume. Thus $\dot{E}$ is the total dissipation rate due to the turbulence associated with the jump. It is noteworthy that the simple model of the hydraulic shock enables us to determine the total dissipation rate, even if we have no detailed description of the bore front or the turbulence distribution there.

In fact, production of hydraulic shocks in waterways may be an efficient tool for current reduction in waterways.

From (7.20) we observe that for $h_1 > h_2$ mechanical energy must be supplied to the fluid to sustain the jump. If this is not possible then this type of hydraulic shock cannot exist.

If we enter a frame of reference where the fluid in the front of the shock is at rest and the shock propagates, we must add $-u_1$ to all velocities in the above description. The hydraulic jump will then move with celerity u_1, in the negative x-direction.

Finally we mention that the hydraulic shocks is a shallow water analogy to shock waves in gases.

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