UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MEK 4320 — Hydrodynamic wave theory.

Day of examination: Wednesday 2 June 2021.

Examination hours: 09:00 – 13:00.

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: All non-living aids.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

In this problem we seek numerical values given in decimal form with physical units clearly indicated. We want answers with two digits of accuracy. You can employ any formula from the compendium without re-derivation, but you should explain which formulas you use.

Consider an ocean basin with uniform depth $h = 3$ m, filled with monochromatic long-crested surface waves of wavelength $\lambda = 10$ m and amplitude $a = 0.1$ m. The density of the water is $\rho = 10^3$ kg/m^3, the acceleration of gravity is $g = 9.8$ m/s^2 and the surface tension is $\sigma = 7.4 \cdot 10^{-2}$ N/m.

1a

Compute the wavenumber k and the values of the four non-dimensional quantities

$$kh, \qquad \frac{\sigma k^2}{\rho g}, \qquad ka, \qquad \frac{a}{h}.$$

Discuss if these waves are deep-water or shallow-water or finite-depth waves. Discuss if they are gravity or capillary or gravity–capillary waves. Discuss if they are linear or nonlinear waves.

For the rest of this problem we limit to linear wave theory:

1b

Show the simplest form of the dispersion relation that allows to determine the angular frequency ω with two digits of accuracy. Compute the corresponding wave period T.

(Continued on page 2.)

1c

Compute the phase speed and the group velocity.

1d

Compute the total wave energy within a basin of horizontal area $A = 100\ \mathrm{m}^2$. Also compute the integrated energy flux that arrives through a $b = 10$ m wide vertical wall normal to the wavenumber vector.

Problem 2

The Klein–Gordon equation is given by

$$\frac{\partial^2 \eta}{\partial t^2} - c_0^2 \frac{\partial^2 \eta}{\partial x^2} + q^2 \eta = 0$$

where c_0 and q are constants, t is time and x is the position along a spatial axis.

2a

Find the dispersion relation.

2b

Find the phase speed c and the group velocity c_g. Plot both in the same coordinate system as functions of the wavenumber.

2c

Suppose that a point disturbance, moving at constant velocity U in the positive x-direction, produces a stationary wave pattern in the moving frame of reference of the point disturbance. Find the wavenumber and the angular frequency of the stationary wave pattern. Discuss if there are constraints for the value of U for which this can happen. Discuss if the wave pattern is ahead of or behind the moving point disturbance.

Problem 3

To have fun, some people installed a wave tank in the elevator in order to study the effect of a time-varying acceleration of gravity, $g(t)$, on the behavior of gravity waves.
Assume the experiment deals with gravity waves on deep water, the fluid is inviscid and the flow field is irrotational, the acceleration of gravity has slow variation compared to the wave frequency, and the wave field is slowly varying in space and time.

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3a

The wave field can be described by the so-called ray equations, describing the time evolution of the position $x(t)$ and the wavenumber $k(t)$ and the angular frequency $\omega(t)$ of a "wave particle". Write down the ray equations.

3b

Suppose that when the elevator is at rest the background acceleration of gravity is g_0 and the waves have wavenumber k_0 and angular frequency ω_0. Departing from this background state, we let the elevator move up and down in a sinusoidal fashion such that the acceleration of gravity felt by the waves is $g(t) = g_0 + \alpha \sin(\beta t)$ where α and β are constants.
Describe how the wavenumber k and angular frequency ω are modified when the elevator moves up and down in the sinusoidal fashion. Describe in particular how k and ω are affected when the elevator is at its highest and lowest positions.

3c

The wave action is defined as the wave energy divided by the angular frequency. Show that the wave action is conserved when the elevator moves up and down.
(Do this for a general slowly varying acceleration of gravity, $g(t)$, without limiting to the sinusoidal motion mentioned above.)

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