Suggested solution, exam MEK4320, spring 2021

1. **Problem 1**

1.a)

$$k = \frac{2\pi}{\lambda} \approx 0.63\text{m}^{-1}$$

$$kh \approx 1.9, \qquad \frac{\sigma k^2}{\rho g} \approx 3.0 \cdot 10^{-6}, \qquad ka \approx 0.063, \qquad \frac{a}{h} = 0.033$$

This is a finite-depth gravity wave ($\tanh kh = 0.9549$ is not 1.0 with two digits of accuracy). It is weakly nonlinear.

1.b) Full dispersion relation

$$\omega^2 = (gk + \sigma k^3/\rho)\tanh kh, \qquad \omega \approx 2.4\text{s}^{-1}, \qquad T \approx 2.6\text{s}$$

Deep water without surface tension

$$\omega^2 = gk, \qquad \omega \approx 2.5\text{s}^{-1}, \qquad T \approx 2.5\text{s}$$

Finite depth without surface tension

$$\omega^2 = gk\tanh kh, \qquad \omega \approx 2.4\text{s}^{-1}, \qquad T \approx 2.6\text{s}$$

Shallow water without surface tension

$$\omega^2 = ghk^2, \qquad \omega \approx 3.4\text{s}^{-1}, \qquad T \approx 1.8\text{s}$$

Moreover, as the shallow water approximation is far off from the exact result, we may anticipate that “weakly dispersive” modifications of the pure shallow water approximation will not be useful. Clearly, we desire “finite depth without surface tension”

1.c)

$$c = \frac{\omega}{k} \approx 3.9\frac{\text{m}}{\text{s}}, \qquad c_g = \frac{g}{2\omega}\left(\tanh kh + \frac{kh}{\cosh^2 kh}\right) \approx 2.3\frac{\text{m}}{\text{s}}$$

(Wrong answers:
Deep water $c = 3.9$ m/s, $c_g = 2.0$ m/s. Shallow $c = c_g = 5.4$ m/s)

1.d)

$$E = \frac{1}{2}\rho g a^2 A \approx 4.9\text{kJ} \approx 1.4\text{Wh}, \qquad F = \frac{1}{2}\rho g a^2 c_g b \approx 1.1\text{kW}$$

(Wrong answers: Deep water $F = 0.97$ kW. Shallow $F = 2.7$ kW)

2. **Problem 2**

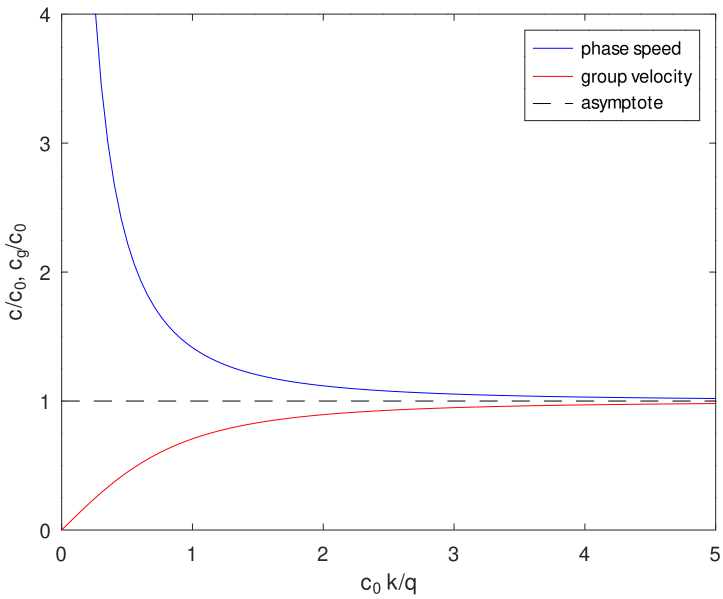
2.a)

$$\omega^2 = c_0^2 k^2 + q^2$$

2.b)

$$c = c_0 \sqrt{1 + \left(\frac{q}{c_0 k}\right)^2}, \qquad c_g = \frac{c_0}{\sqrt{1 + \left(\frac{q}{c_0 k}\right)^2}}$$

$c > c_0$ and $c_g < c_0$, both are asymptotic to c_0 as $k \to \infty$.
c_g goes to zero while c is asymptotic to infinity as $k \to 0$.



2.c) **Note: This is a one-dimensional version of the problem, not a two-dimensional version, therefore we do not introduce a general angle θ between the wavenumber vector $\boldsymbol{k}$ and the x-axis, rather the wavenumber vector is parallel to $\boldsymbol{U}$, either in the positive or negative x-direction.**

Stationary pattern for $U = c$ provided $U > c_0$ (this must be shown).

$$k = \frac{q}{\sqrt{U^2 - c_0^2}}, \qquad \omega = Uk = \frac{qU}{\sqrt{U^2 - c_0^2}}$$

Due to $c_g < c$ the stationary pattern will be behind the moving source.

3. **Problem 3**

3.a)

$$\omega = \Omega(k,t) \equiv \sqrt{g(t)k}$$

The ray equations are

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\partial \Omega}{\partial k} = \frac{g(t)}{2\omega} \\
\frac{\mathrm{d}k}{\mathrm{d}t} &= -\frac{\partial \Omega}{\partial x} = 0 \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\partial \Omega}{\partial t} = \frac{1}{2}\frac{\mathrm{d}g(t)}{\mathrm{d}t}\sqrt{\frac{k}{g(t)}}
\end{aligned}$$

3.b) First we note that the second ray equation implies k is constant, unaffected by the variation in $g(t)$.

After this observation, we recognize that the third ray equation becomes particularly easy to solve, since $g(t)$ is the only quantity on the right-hand side of the dispersion relation that depends on time, and the solution is given by the dispersion relation

$$\omega(t) = \sqrt{g(t)k} = \sqrt{(g_0 + \alpha \sin(\beta t))k}$$

At the highest position g is minimum, $\sin(\beta t) = -1$, and $\omega = \sqrt{(g_0 - \alpha)k}$.

At the lowest position g is maximum, $\sin(\beta t) = +1$, and $\omega = \sqrt{(g_0 + \alpha)k}$.

3.c) The governing equations, Taylor-expanded around equilibrium surface and linearized

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = 0$$

$$\frac{\partial \phi}{\partial t} + g(t)\eta = 0 \quad \text{at } z = 0$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{for } z < 0$$

$$\frac{\partial \phi}{\partial z} \to 0 \quad \text{as } z \to -\infty$$

Introduce slow coordinates $t_{\text{slow}} = \epsilon t_{\text{fast}}$ and $x_{\text{slow}} = \epsilon x_{\text{fast}}$, assuming that the time variation of $g(t)$ is precisely the slow time, we get

$$\epsilon \frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = 0$$

$$\epsilon \frac{\partial \phi}{\partial t} + g(t)\eta = 0 \quad \text{at } z = 0$$

$$\epsilon^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{for } z < 0$$

$$\frac{\partial \phi}{\partial z} \to 0 \quad \text{as } z \to -\infty$$

where we for brevity have omitted the index "slow" on time and horizontal position.

Assume a WKB-perturbation solution

$$\begin{aligned} \eta &= (A_0 + \epsilon A_1 + \ldots)\mathrm{e}^{\epsilon^{-1}\mathrm{i}\chi} \\ \phi &= (\hat{\phi}_0 + \epsilon \hat{\phi}_1 + \ldots)\mathrm{e}^{\epsilon^{-1}\mathrm{i}\chi} \end{aligned}$$

where $\partial\chi/\partial x = k$ and $\partial\chi/\partial t = -\omega$.

Problem at order $\mathcal{O}(\epsilon^0)$

$$-\mathrm{i}\omega A_0 - \frac{\partial \hat{\phi}_0}{\partial z} = 0 \quad \text{at } z = 0$$

$$-\mathrm{i}\omega \hat{\phi}_0 + g(t)A_0 = 0 \quad \text{at } z = 0$$

$$\frac{\partial^2 \hat{\phi}_0}{\partial z^2} - k^2 \hat{\phi}_0 = 0 \quad \text{for } z < 0$$

$$\frac{\partial \hat{\phi}_0}{\partial z} \to 0 \quad \text{as } z \to -\infty$$

We find $\hat{\phi}_0 = -\frac{\mathrm{i}g(t)}{\omega} A_0 \mathrm{e}^{kz}$ and the dispersion relation $\omega^2 = g(t)k$.

Problem at order $\mathcal{O}(\epsilon^1)$

$$-\mathrm{i}\omega A_1 - \frac{\partial \hat{\phi}_1}{\partial z} = -\frac{\partial A_0}{\partial t} \quad \text{at } z = 0$$

$$-\mathrm{i}\omega \hat{\phi}_1 + g(t)A_1 = -\frac{\partial \hat{\phi}}{\partial t} \quad \text{at } z = 0$$

$$\frac{\partial^2 \hat{\phi}_1}{\partial z^2} - k^2 \hat{\phi}_1 = -\mathrm{i}k \frac{\partial \hat{\phi}_0}{\partial x} - \mathrm{i}\frac{\partial}{\partial x}\left(k\hat{\phi}_0\right) \quad \text{for } z < 0$$

$$\frac{\partial \hat{\phi}_1}{\partial z} \to 0 \quad \text{as } z \to -\infty$$

The problem at order $\mathcal{O}(\epsilon^1)$ is an inhomogeneous (forced) version of the leading order homogeneous problem. As we have already insisted on the dispersion relation in order to have a non-trivial solution of the leading order problem, we know that this problem is singular. Therefore, according to the Fredholm alternative, we must insist that a solvability condition is applied on the forcing. We have seen several examples how this can conveniently be achieved by application of Green's theorem

$$\int_{-\infty}^{0} \hat{\phi}_1 \left(\frac{\partial^2 \hat{\phi}_0}{\partial z^2} - k^2 \hat{\phi}_0 \right) - \hat{\phi}_0 \left(\frac{\partial^2 \hat{\phi}_1}{\partial z^2} - k^2 \hat{\phi}_1 \right) \mathrm{d}z = \left[\hat{\phi}_1 \frac{\partial \hat{\phi}_0}{\partial z} - \hat{\phi}_0 \frac{\partial \hat{\phi}_1}{\partial z} \right]_{z=-\infty}^{0}$$

After some manipulation we arrive at the conservation law

$$\frac{\partial}{\partial t} \left(\frac{g(t) A_0^2}{\omega} \right) + \frac{\partial}{\partial x} \left(\frac{g(t)}{2\omega} \frac{g(t) A_0^2}{\omega} \right) = 0$$

which can be rewritten as

$$\frac{\partial}{\partial t} \left(\frac{E}{\omega} \right) + \frac{\partial}{\partial x} \left(c_g \frac{E}{\omega} \right) = 0$$

where $E = \frac{1}{2} \rho g(t) A_0^2$ and $c_g = \frac{g(t)}{2\omega}$ and the angular frequency ω is a solution of the ray equations expressed previously.