MEK4350, fall 2016
Exercises I

First three exercises that are nice to review, but not necessary to do.

In these three exercises f and g and h are complex vectors, α and β are complex scalars, $\langle f, g\rangle$ is an inner product, and $\|f\| = \sqrt{\langle f, f\rangle}$ is the norm associated with the inner product.

Recall the requirements to be an inner product:
1) $\langle f, g\rangle = \overline{\langle g, f\rangle}$
2) $\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$
3) $\langle f, f\rangle \geq 0$
4) $\langle f, f\rangle = 0$ if and only if $f = 0$

Recall the requirements to be a norm:
1) $\|f + g\| \leq \|f\| + \|g\|$
2) $\|\alpha f\| = |\alpha|\,\|f\|$
3) $\|f\| \geq 0$
4) $\|f\| = 0$ if and only if $f = 0$

Exercise 1 (see section 1.1 in Løw & Winther)

Show that Cauchy–Schwartz inequality holds:

$$|\langle f, g\rangle| \leq \|f\|\,\|g\|$$

Hint: Start with $\langle f - \alpha g, f - \alpha g\rangle \geq 0$, where α is a complex scalar, and set $\alpha = \langle f, g\rangle/\langle g, g\rangle$.

Exercise 2

Show that the triangle inequality holds for the norm associated with the inner product $\|f\| = \sqrt{\langle f, f\rangle}$:

$$\|f + g\| \leq \|f\| + \|g\|$$

Hint: Start with $\|f + g\|^2 = \langle f + g, f + g\rangle$, recall that the real value of a complex number is smaller than or equal to the absolute value of the number, and use Cauchy–Schwartz inequality.

Exercise 3 (see theorem 3.1 in Løw & Winther)

Show that Bessel's inequality holds for a generalized Fourier series: If we represent f by a projection Pf into a subspace spanned by orthogonal basis vectors ϕ_n

$$f \sim Pf = \sum_n \hat{f}_n \phi_n \qquad \text{where the Fourier coefficients are} \quad \hat{f}_n = \langle f, \phi_n\rangle/\|\phi_n\|^2$$

then

$$\sum_n |\hat{f}_n|^2 \|\phi_n\|^2 \leq \|f\|^2$$

Hint: Start with $\|f - Pf\|^2 = \langle f - Pf, f - Pf\rangle \geq 0$.

Of the following exercises you should do 4, at least one of {5,6,7}, at least one of {8,9}, and 10:

As inner product for complex functions on the interval $a \le x \le b$ we use

$$\langle f, g \rangle = \int_a^b f(x) g^*(x)\, dx$$

and for complex sequences with index j we use

$$\langle f, g \rangle = \sum_j f(j) g^*(j)$$

Find out if the following vector sets are orthogonal, and compute the norm of the vectors.

Exercise 4

The functions $\{1, \cos(nx), \sin(nx); n = 1, 2, \ldots\}$ on the interval $0 \le x \le 2\pi$.

Exercise 5

The functions $\{1, \exp(inx); n = \pm 1, \pm 2, \ldots\}$ on the interval $0 \le x \le 2\pi$.

Exercise 6

The functions $\{\sin(nx); n = 1, 2, \ldots\}$ on the interval $0 \le x \le \pi$.

Exercise 7

The functions $\{1, \cos(nx); n = 1, 2, \ldots\}$ on the interval $0 \le x \le \pi$.

Exercise 8

The sequences $\{\exp \frac{2\pi i n j}{N}; n = 1, 2 \ldots N\}$ for $j = 1, 2, \ldots, N$.

Exercise 9

The sequences $\{\exp \frac{2\pi i n j}{N}; n = M, M+1 \ldots M+N-1\}$ for $j = J, J+1, \ldots, J+N-1$ where M and J are arbitrary integers.

Exercise 10

Let $f(x)$ be a 2π-periodic function defined by

$$f(x) = \begin{cases} 1 & 0 \le x < \pi \\ 0 & \pi \le x < 2\pi \end{cases}$$

Use the functions from exercise 4 to compute the truncated Fourier series

$$P_N f(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

Plot $f(x)$ and $P_N f(x)$ for $N = 1, 2, 3, 10, 100$ in the same coordinate system.