MEK4350, fall 2016
Exercises X

Problem 1

Suppose $A_n \sim N(\mu_n, \sigma_n^2)$ for $n = 1, 2, 3, \ldots$ are independent normally distributed stochastic variables with mean μ_n and variance σ_n^2, and suppose α_n are real constants for all n. Define a new stochastic variable as

$$B = \alpha_0 + \sum_{n>0} \alpha_n A_n.$$

Show that B is also Gaussian and determine its mean and variance.

Hint: Use characteristic functions.

For the remainder of these exercises we define the stochastic variables $X_n \sim N(0, \sigma^2)$ to be independent identically normally distributed with mean 0 and variance σ^2.

We define the new variables

$$Y = \sqrt{\sum_{n=1}^{N} X_n^2} \qquad \text{and} \qquad Z = \sum_{n=1}^{N} X_n^2.$$

Then $Y \sim \chi(N)$ has a chi distribution of N degrees of freedom and $Z \sim \chi^2(N)$ has a chi-square distribution of N degrees of freedom.

Problem 2

Find the probability density functions of the $\chi(1)$ and $\chi^2(1)$ distributions.

Problem 3

For two degrees of freedom we can use the transformation

$$X_1 = Y \cos \Theta$$

$$X_2 = Y \sin \Theta$$

to introduce the new variables Y and Θ (these two definitions of Y are in agreement). Show that Y and Θ are independent, and that Y is Rayleigh distributed

$$f_Y(y) = \begin{cases} \frac{y}{\sigma^2} \mathrm{e}^{-\frac{y^2}{2\sigma^2}} & \text{for } y \geq 0 \\ 0 & \text{for } y < 0 \end{cases}$$

and Θ is uniform on $[0, 2\pi)$.

Compute the mode, mean and variance of Y in this case.

Problem 4

Continuing from the previous problem, show that $Z = Y^2$ is exponentially distributed

$$f_Z(z) = \begin{cases} \frac{1}{2\sigma^2} \mathrm{e}^{-\frac{z}{2\sigma^2}} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

Compute the mode, mean and variance of Z in this case.

Problem 5

Using the toy model for the surface elevation at a point

$$\eta(t) = A \cos(\omega_p t) + B \sin(\omega_p t)$$

where A and B are independent identically distributed normal stochastic variables with mean 0 and variance σ^2, we have discussed that crest heights and wave heights are both Rayleigh distributed.

Sometimes one defines $H_{1/N}$ to be the mean of the $1/N$ highest waves, show that

$$H_{1/N} = \beta(N)\sigma$$

and plot the graph of $\beta(N)$.

Note: If we set $N = 3$ we recover the classical definition of significant wave height. The classical definition is complicated to employ in practice since $\beta(N)$ will depend on the shape of the distribution of η. The modern definition of significant wave height $H_s = 4\sigma$ is easier since it only depends on the single parameter σ.