MEK4350, fall 2016
Exercises XII

The exceedance probability of a stochastic variable X is defined as $Pe(x) = P\{X > x\} = 1 - F(x)$ where $F(x)$ is the cumulative distribution of X.

Problem 1

For our toy model for the surface elevation at a point

$$\eta(t) = A\cos(\omega_p t) + B\sin(\omega_p t)$$

where A and B are independent identically distributed normal stochastic variables with mean 0 and variance σ^2, we have found that the estimator for the spectrum $\tilde{S}(\omega_l) = |\hat{\eta}(\omega_l)|^2$ has an exponential distribution.

Compute the exceedance probability for the estimator $\tilde{S}$, and in particular compute the probability that $\tilde{S}$ is greater than its expected value and greater than twice its expected value.

Plot the exceedance probability with linear and logarithmic axis for the probability.

Problem 2

Let us modify our toy model to describe waves that propagate in the x-direction

$$\eta(x,t) = A\cos(k_p x - \omega_p t) + B\sin(k_p x - \omega_p t)$$

where A and B are independent identically distributed normal stochastic variables with mean 0 and variance σ^2.

- Find the expectation and the autocorrelation function of η.
- Check that η is weakly stationary/homogeneous.
- Compute the 2D two-sided spectrum of η.
- Compute the 2D one-sided spectrum of η.

Hint: Suppose our preference is that waves that propagate in the positive x-direction should be located in the first quadrant of the (k,ω) plane. Suppose that both k_p and ω_p are positive, so that our toy model waves indeed propagate in the positive x-direction.

Read Phillips (1958) — all of it, even the parts that you don't understand!

Recall that the dispersion relation for linear gravity–capillary waves on arbitrary depth is

$$\omega^2 = \left(gk + \frac{T}{\rho} k^3 \right) \tanh kh \tag{1}$$

where g is the acceleration of gravity, T is the surface tension between water and air, ρ is the density of water (we assume that the density of air is negligible) and h is the depth.

Problem 3

Show that in the limit of deep water gravity waves, equation (1) can be approximated by

$$\omega^2 = gk$$

which was employed in Phillips (1958).

Problem 4

Show that in the limit of deep water capillary waves, equation (1) can be approximated by

$$\omega^2 = \frac{T}{\rho} k^3.$$

Repeat the argument of sharp crests to arrive at the appropriate asymptotic power law for the spectrum for large wavenumbers and for large frequencies.

Can you arrive at the same result by dimensional arguments?

Problem 5

Show that in the limit of long shallow water gravity waves, equation (1) can be approximated by

$$\omega = \sqrt{gh} k.$$

Repeat the argument of sharp crests to arrive at the appropriate asymptotic power law for the spectrum for large wavenumbers and for large frequencies.

Can you arrive at the same result by dimensional arguments?

This exercise suffers from a serious error, or at least a serious conceptual problem. Can you explain what this problem is?