MEK4350, fall 2016
Exercises XIII

Problem 1 — mathematics for surface tension

(a) Show by applying Stokes theorem

$$\oint_{\partial S} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{r} = \int_S \nabla \times \boldsymbol{v} \cdot \boldsymbol{n} \, \mathrm{d}\sigma$$

that the surface tension force can be written as

$$\boldsymbol{F}_\gamma = -\oint_{\partial S} \gamma \boldsymbol{n} \times \mathrm{d}\boldsymbol{r} = -\int_S \gamma \boldsymbol{n} \nabla \cdot \boldsymbol{n} \, \mathrm{d}\sigma$$

Hint: You may want to take the dot product of the entire expression with a constant arbitrary vector.

(b) Explain why the series expansion for $\nabla \cdot \boldsymbol{n}$ for small steepness $\epsilon = |\nabla \eta| \ll 1$, i.e. a power series expansion in ϵ, does not give any quadratic contributions in η, but will give cubic contributions in η.

(c) Show that for one horizontal dimension, say $z = \eta(x)$, we have $\nabla \cdot \boldsymbol{n} = -1/R$ where R is the radius of curvature of the surface.

Hint: Consult Wikipedia: "Radius of curvature" if you are not certain about what this is.

Problem 2 — the geometry of the dispersion relation

Recall that the dispersion relation is

$$\omega^2 = \left(gk + \frac{\gamma}{\rho}k^3\right)\tanh(kh).$$

Typical values for water and air at 20°C are $\gamma = 7.28 \cdot 10^{-2}$ N/m, $\rho = 998$ kg/m^3 and $g = 9.81$ m/s^2.

1. For infinite depth, find the wavelength for which the gravity and capillary terms have equal magnitude.

2. For infinite depth, compute the phase speed $c = \omega/k$ and the group velocity $c_g = \partial\omega/\partial k$, and find the wavelength for which they are equal. Plot the curves for c and c_g as a function of k. Also plot c_g/c as a function of k.

3. Show that the waves are approximately non-dispersive (ω proportional to k) for very long waves ($kh \ll 1$). Discuss what is sufficiently long.

4. Show that the waves are approximately non-dispersive for the special depth $h = \sqrt{\frac{3\gamma}{\rho g}}$ for sufficiently long waves. Compute this depth. Discuss what is sufficiently long.

   Hint: Consider the Taylor expansion $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \ldots$