MEK4350, fall 2016
Exercises II

First two exercises that are nice to review, but not necessary to do.

Exercise 1 — The sinc function $\operatorname{sinc} x = \frac{\sin x}{x}$

The sinc function is also known as the cardinal sine function. Our definition is the unnormalized sinc function adopted by Krogstad (2001) "Fouriertransformen — en innføring" (see his figure 3) and Mathematica. The alternative definition, the normalized sinc function, $\operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}$ is adopted by Python and Matlab and Octave and by DLMF. Please consult with Wikipedia which recognizes both definitions.

Interestingly, our definition coincides with the Spherical Bessel function of the first kind $j_0(x) = \frac{\sin x}{x}$, see DLMF.

a) Show using l'Hôpital's rule that $\operatorname{sinc} 0 = 1$.

b) Show that $\int_{-\infty}^{\infty} \operatorname{sinc} x \, dx = \pi$.

This can be done in several ways, one way is as follows:

1. Observe that the integrand is even, integrate only from 0 to ∞.
2. Rewrite as a double integral using $\int_0^\infty e^{-xt} dt = \frac{1}{x}$.
3. Reverse the order of integration.
4. Perform two integrations by parts.
5. Recognize the integral that defines $\arctan \infty = \pi/2$.

Exercise 2 — Justification that $\int_{-\infty}^{\infty} e^{ikx} \, dx = 2\pi\delta(k)$

Define $I(k, a) = \int_{-a}^{a} e^{ikx} \, dx = 2a \operatorname{sinc}(ka)$.

a) Plot $I(k, a)$ for small a, moderate a and large a.

b) Using the method of stationary phase (taught in MEK4320), argue that due to the extremely fast oscillations of $I(k, a)$ for large a there will be cancellations everywhere except near $k = 0$ such that

$$\begin{aligned}
\int_{-\infty}^{\infty} f(k) \int_{-\infty}^{\infty} e^{ikx} \, dx \, dk &= \lim_{a\to\infty} \int_{-\infty}^{\infty} f(k) I(k, a) \, dk \\
&\approx \lim_{a\to\infty} \int_{-\epsilon}^{\epsilon} f(k) I(k, a) \, dk \\
&\approx f(0) \lim_{a\to\infty} \int_{-\epsilon}^{\epsilon} I(k, a) \, dk \\
&\approx f(0) \lim_{a\to\infty} \int_{-\infty}^{\infty} I(k, a) \, dk \\
&= 2\pi f(0)
\end{aligned}$$

Here ϵ is small and positive, and the result from exercise 1 was used in the last step.

Note: The result of this exercise shows that even though the requirement $\delta(x) = 0$ for $x \neq 0$ is not satisfied, the essential behavior of $\int_{-\infty}^{\infty} f(x)\delta(x) \, dx = f(0)$ is still achieved. Therefore the requirement of zero everywhere except the origin is really not essential.

Do these three exercises:

The Dirac delta-function $\delta(x)$ is a "generalized" function with the properties

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0$$

and

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0)$$

where $f(x)$ is an "ordinary" function.

The Heaviside step function $H(x)$ is given by

$$H(x) = \begin{cases} 0 \quad \text{for} \quad x < 0 \\ 1 \quad \text{for} \quad x > 0 \\ \text{either } 0 \text{ or } \frac{1}{2} \text{ or } 1 \quad \text{for} \quad x = 0 \end{cases}$$

For continuous x it usually does not matter which finite value we select for $H(0)$.

We have seen that $H'(x) = \delta(x)$, and we have seen how to find $\delta'(x)$ by means of integration by parts.

Exercise 3

Let the function $h(x)$ be given by

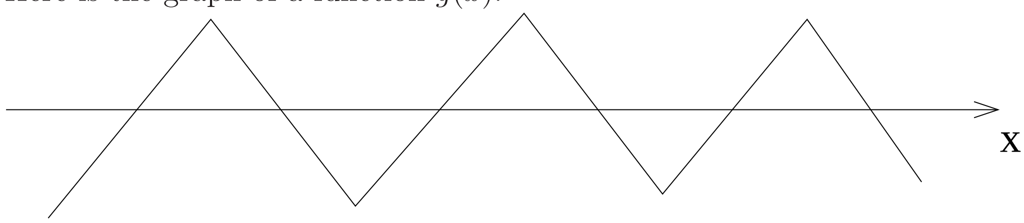
$$h(x) = \begin{cases} h_1 \quad \text{for} \quad x < a \\ h_2 \quad \text{for} \quad x \geq a \end{cases}$$

for arbitrary constants h_1, h_2 and a. Compute the derivative $h'(x)$.

Hint: Compute $h'(x)$ for $x \neq a$, and compute $\int_{-\infty}^{\infty} f(x)h'(x)\,dx$ for an "ordinary" function $f(x)$. Use integration by parts.

Exercise 4

Here is the graph of a function $g(x)$:



Sketch the graphs of the first derivative $g'(x)$ and the second derivative $g''(x)$.

Exercise 5

Compute $\delta''(x)$.

Hint: Integration by parts twice.