MEK4350, fall 2016
Exercises III

Exercise 1 — Discrete Fourier Transform (DFT)

We consider complex sequences with N elements, $f(j)$ for $j = 1, 2, \ldots, N$. Define the inner product as $\langle f, g\rangle = \sum_{j=1}^{N} f(j)g^*(j)$ and use the following as a basis $\phi_n(j) = e^{\frac{2\pi i n j}{N}}$ for $n = 1, 2, \ldots, N$ and $j = 1, 2, \ldots, N$. We have already shown that $\phi_n(j)$ and $\phi_m(j)$ are orthogonal when $n \neq m$. In the following we write $f(j) = f_j$ for simplicity. We can now represent f_j as

$$f_j = \sum_{n=1}^{N} \tilde{f}_n e^{\frac{2\pi i n j}{N}}$$

Show that the Fourier coefficients are

$$\tilde{f}_n = \frac{1}{N} \sum_{j=1}^{N} f_j e^{-\frac{2\pi i n j}{N}}.$$

Derive Parseval's equation

$$N \sum_{n=1}^{N} |\tilde{f}_n|^2 = \sum_{j=1}^{N} |f_j|^2.$$

Exercise 2 — cyclic permutation and fftshift

Let us consider that the finite complex sequences, $\{f_j\}$ for $j = 1, 2, \ldots, N$, and $\{\tilde{f}_n\}$ for $n = 1, 2, \ldots, N$, are extended periodically to infinite sequences by $f_j = f_{j+N}$ and $\tilde{f}_n = \tilde{f}_{n+N}$ for all j and n. With cyclic permutation we mean that we can start with an arbitrary element with index r in these sequences and count N subsequent elements, rather than to start with index 1.

Show that for an arbitrary integer r

$$f_j = \sum_{n=r}^{r+N-1} \tilde{f}_n e^{\frac{2\pi i j n}{N}}$$

and

$$\tilde{f}_n = \frac{1}{N} \sum_{j=r}^{r+N-1} f_j e^{-\frac{2\pi i j n}{N}}.$$

On the computer we have the functions `fftshift` and `ifftshift` that perform cyclic permutations, the first one moves the first element of an array to the middle, the second one does the opposite. How do `fftshift` and `ifftshift` behave for even and odd N? When are they equal? When is `fftshift` the inverse of itself? Do this on computer systems like Matlab, Octave, Python or Mathematica.

Exercise 3 — fft and ifft on your computer

On computer we have the the functions `fft` and `ifft` that perform the DFT and its inverse. These are defined in various ways, usually something like this:

$$\text{ifft:} \qquad f_j = A \sum_{n=\alpha}^{N+\alpha-1} \tilde{f}_n e^{\pm \frac{2\pi i (n-\alpha)(j-\alpha)}{N}}$$

$$\text{fft:} \qquad \tilde{f}_n = B \sum_{j=\alpha}^{N+\alpha-1} f_j e^{\mp \frac{2\pi i (n-\alpha)(j-alpha)}{N}}$$

Here α is not a cyclic permutation, but the index of the first element of an array, usually $\alpha = 0$ (Python, C) or $\alpha = 1$ (Matlab, Octave, FORTRAN).

Determine the values of A and B and α and the signs on your computer system!

For the pair {`fft`, `ifft`} to be exact inverses it is necessary that $ABN = 1$. Be warned! This turns out not to be satisfied on all computer systems! Is it satisfied on your system?

Hint: Set N to a small value, say $N = 4$, and let $\{f_j\}$ have mostly zero values.

Exercise 4

Show that

$$\sum_{j=r}^{r+N-1} e^{\frac{2\pi i j n}{N}} = N \sum_{l=-\infty}^{\infty} \delta_{n,lN}$$

where r and n are arbitrary integers.

Note: The result is an infinite row of Kronecker delta functions.

Show that if we had limited consideration to values of n within N subsequent values, $n = M, M+1, \ldots, M+N-1$, for an arbitrary integer M, then we have left only one of these Kronecker delta functions.

Exercise 5 — Folding or aliasing

Let the function $f(x)$ be periodic with period L and have Fourier series

$$f(x) \sim \sum_{n=-\infty}^{\infty} \hat{f}_n e^{ik_n x}, \qquad \hat{f}_n = \frac{1}{L} \int_0^L f(x) e^{-ik_n x} \, dx,$$

where $k_n = 2\pi n / L$.

We shall sample $f(x)$ at the collocation points $x_j = Lj/N$ for $j = 0, 1, \ldots, N-1$. We shall represent $f_j \equiv f(x_j)$ by means of a DFT

$$f_j = \sum_{n=0}^{N-1} \tilde{f}_n e^{ik_n x_j}, \qquad \tilde{f}_n = \frac{1}{N} \sum_{j=0}^{N-1} f_j e^{-ik_n x_j}.$$

Show that

$$\tilde{f}_n = \sum_{l=-\infty}^{\infty} \hat{f}_{n+lN}$$

Note: The last formula describes what we call folding or aliasing. This result shows that if N is sufficiently big, at the same time as $\hat{f}_n$ goes sufficiently fast to zero when $n \to \pm\infty$, then $\tilde{f}_n$ will be approximately equal to $\hat{f}_n$ for indices n close to where $|\tilde{f}_n|$ attains its maximum.