MEK4350, fall 2016
Exercises IV

In exercises 1–4 we let the Fourier transform be

$$\hat{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}\,dx$$

for functions on infinite interval, and

$$\hat{f}(k_n) \equiv \hat{f}_n = \frac{1}{L}\int_{0}^{L} f(x)e^{-ik_n x}\,dx \qquad \text{with} \qquad k_n = \frac{2\pi n}{L}$$

for functions on finite interval $0 \leq x \leq L$

Exercise 1

Show that if $f(x)$ is real then $\hat{f}(k) = \hat{f}^*(-k)$.

Exercise 2

Show that the Fourier transform of $\exp(ik_0 x)f(x)$ is $\hat{f}(k - k_0)$.

Exercise 3

Show that the Fourier transform of $\dfrac{d^m f(x)}{dx^m}$ is $(ik)^m \hat{f}(k)$, on the condition that either the limit $\dfrac{d^j f(x)}{dx^j} \to 0$ holds when $x \to \pm\infty$ for all $j < m$, or $\dfrac{d^j f(x)}{dx^j}$ are periodic over the finite interval for all $j < m$.

Convolution

The convolution $f(x) * g(x)$ of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

for functions on infinite interval, and

$$f(x) * g(x) = \int_{0}^{L} f(y)g(x-y)\,dy$$

for functions on finite interval $0 \leq x \leq L$. In the last case it does not matter that the argument of g extends outside the interval $0 \leq x \leq L$ since we suppose that the functions f and g repeat periodically outside the interval.

Exercise 4

Show that the Fourier transform of the convolution $f(x) * g(x)$ is proportional to $\hat{f}(k)\hat{g}(k)$.

Asymptotic behavior of Fourier coefficients

Exercise 5

Show that if we draw the graph of $y = ax^b$ in a doubly logarithmic coordinate system, then the result will be a straight line, and the values of a and b can be determined by respectively the point of intersection with the second axis and the slope of the line.

Choose suitable values for a and b and demonstrate that this actually works by plotting the graph on the computer with both `plot(x,y)` and `loglog(x,y)`.

Exercise 6

We shall look at the three functions

$$f(x) = x \qquad g(x) = |x - \pi| \qquad h(x) = \cos(x) \qquad \text{for } 0 \leq x < 2\pi$$

a) Plot the graphs of the three functions and explain how fast the Fourier coefficients $\hat{f}_n$, $\hat{g}_n$ and $\hat{h}_n$ are expected to approach zero when $n \to \pm\infty$.

We shall use the computer to show that the expectation above is correct. First we have to discretize x with N equally spaced values $x_j = 2\pi j/N$. Then the Fourier coefficients are computed by means of `fft(y)`, where y_j is the value of the function evaluated at x_j. Finally we plot `abs(fft(y))` in a doubly logarithmic coordinate system.

b) Explain why the command `x = linspace(0,2*pi,N)` is not correct, and show how it must be done correctly.

c) Use `plot` to plot the graphs of `abs(fft(y))` for the three functions in the same linear coordinate system. Also plot the graphs of `abs(fftshift(fft(y)))` in the same linear coordinate system. Explain why we only need to study half of the Fourier transform (see exercise 1 in this problem set).

d) Use `loglog` to plot the graphs of $\epsilon +$ `abs(fft(y))` in the same doubly logarithmic coordinate system. Also plot two straight lines, with the expected slopes for the two functions f and g (see part a above). Show that you achieve the expected result for all three functions!

Hint 1: The constant ϵ has been included to avoid that the computer complains about taking the logarithm of non-positive numbers. A useful value could be $\epsilon = 10^{-15}$.

Hint 2: In order to avoid aliasing N must be chosen sufficiently large, $N = 1024$ is probably sufficient.[1]

e) Show that if we commit the error suggested in part b above, then we ruin the nice behavior of $\hat{h}$.

[1] Traditional implementations of `fft` work fastest for N a power of 2, modern implementations work fastest for N a product of small primes.