MEK4350, fall 2016
Exercises V

We shall look at the two functions

$$f(x) = x \qquad \text{and} \qquad g(x) = \cos(x) \qquad \text{for } 0 \le x < 2\pi$$

Let the truncated Fourier series be

$$P_N f(x) = \sum_{n=-N}^{N} \hat{f}_n \mathrm{e}^{\mathrm{i}k_n x}$$

a) Compute $\hat{f}_n$ and $\hat{g}_n$ analytically

b) Demonstrate with computer graphics that $P_N f(x)$ suffers from Gibbs phenomenon as $N \to \infty$, therefore there is no uniform convergence of $P_N f(x)$ to $f(x)$.

Hint: If you are clever you can figure out how to produce this computer graphics with `fft` and/or `ifft` rather than brute force computation of the sum.

c) Demonstrate analytically that $P_N g(x)$ does not suffer from Gibbs phenomenon and does indeed converge uniformly to $g(x)$.

d) Compute analytically the derivative $\frac{\mathrm{d}}{\mathrm{d}x}(P_N f(x))$ by termwise differentiation of the sum. Compare this with the analytically computed truncated Fourier series $P_N \frac{\mathrm{d}f}{\mathrm{d}x}$.

e) Repeat for $g(x)$.

If there is any difference between $\frac{\mathrm{d}}{\mathrm{d}x} P_N$ and $P_N \frac{\mathrm{d}}{\mathrm{d}x}$ make sure you investigate with computer graphics!

We have previously shown that the Fourier transform of $\dfrac{d^m f(x)}{dx^m}$ is $(ik)^m \hat{f}(k)$ for any positive integer m. We can now generalize this defining the

$$\textbf{fractional derivative } \frac{d^\alpha f(x)}{dx^\alpha}$$

for any number α (it could be rational or real or even complex) in terms of the Fourier transform $(ik)^\alpha \hat{f}(k)$.

f) Let us amuse ourselves by computing and plotting $\dfrac{d^\alpha f(x)}{dx^\alpha}$ as α changes continuously from 0 to 1.

g) Repeat with $g(x)$.