MEK4350, fall 2016
Exercises VI

We have learnt that if X is a random variable, and $F(x)$ is the cumulative distribution function for X, then the probability density function is given by

$$f(x) = \frac{\mathrm{d}F}{\mathrm{d}x},$$

the mode is the value for x where $f(x)$ achieves its maximum, the median is the value for x where $F(x) = 0.5$, the mean (or expected value) is

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx,$$

the variance is

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx,$$

the standard deviation is

$$\sigma = \sqrt{\sigma^2}$$

the skewness is

$$\gamma = \frac{E[(X-\mu)^3]}{\sigma^3}$$

and the kurtosis is

$$\kappa = \frac{E[(X-\mu)^4]}{\sigma^4}.$$

A random variable X has moments

$$m_n = E[X^n]$$

and central moments

$$\mu_n = E[(X-\mu)^n].$$

The the characteristic function

$$\phi(k) = E[\mathrm{e}^{\mathrm{i}kX}]$$

can be a useful tool to compute the moments. Either $\phi(k)$ can be Taylor-expanded such that the moments appear as coefficients in subsequent terms, or derivatives of $\phi(k)$ can be evaluated at certain values of k in order to extract the moments.

Problem 1 — moments and central moments

Show how the variance σ^2, skewness γ and kurtosis κ can be expressed only in terms of the moments m_n, and only in terms of the central moments μ_n.

Problem 2 — characteristic function and moments

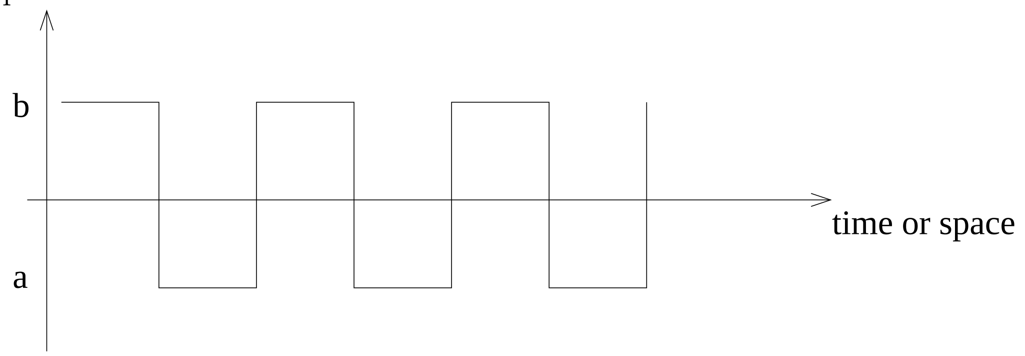
Show how the moments m_n and the central moments μ_n can be obtained by differentiation of $\phi(k)$ and subsequent evaluation at certain values of k.

Problem 3 — toss heads and tails of a fair coin

Let the outcome "heads" have value a and the outcome "tails" have value b. The characterization that the coin is "fair" means that each outcome is equally probable. The probability density function is

$$f(x) = \frac{\delta(x-a) + \delta(x-b)}{2}.$$

Convince yourself that this is the probability density distribution for the surface displacement of this wave:

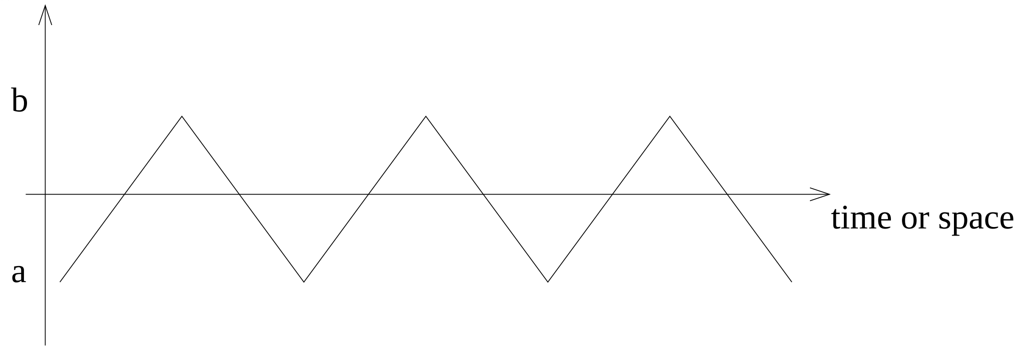


Compute the cumulative distribution function, mode, median, mean, variance, standard deviation, skewness, kurtosis and characteristic function.

Problem 4 — uniform distribution between a and b, for $a < b$

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for} \quad a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

Convince yourself that this is the probability density function for the surface displacement of this wave:



Compute the cumulative distribution function, mode, median, mean, variance, standard deviation, skewness, kurtosis and characteristic function.