MEK4350, fall 2016
Exercises VIII

Complex stochastic variables

Notice that we have not defined the concepts "smaller than" or "greater than" for complex numbers!

A complex stochastic variable, $Z = X + \mathrm{i}Y$ is understood to consist of the two simultaneous real stochastic variables X and Y. In order to discuss the distribution of a complex stochastic variable we therefore discuss the joint distribution of the real and imaginary parts.

The cumulative distribution function of Z is

$$F_Z(z) = F_{XY}(x, y) = P\{X \le x, Y \le y\}.$$

The probability density function of Z is

$$f_Z(z) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}.$$

The expected value of Z is

$$\mu_z = E[Z] = E[X] + \mathrm{i}E[Y] = \mu_x + \mathrm{i}\mu_y.$$

The variance of Z is

$$\mathrm{Var}[Z] = E[(Z - \mu_z)(Z - \mu_z)^*] = E[(X - \mu_x)^2] + E[(Y - \mu_y)^2] = \mathrm{Var}[X] + \mathrm{Var}[Y].$$

There is also a different kind of variance, called the complementary variance, given by

$$\begin{aligned} E[(Z - \mu_z)^2] &= E[(X - \mu_x)^2] - E[(Y - \mu_y)^2] + 2\mathrm{i}E[(X - \mu_x)(Y - \mu_y)] \\ &= \mathrm{Var}[X] - \mathrm{Var}[Y] + 2\mathrm{i}\,\mathrm{Cov}[X, Y]. \end{aligned}$$

Notice that the variance is real and non-negative, while the complementary variance is complex.

The covariance of two complex stochastic variables A and B is

$$\mathrm{Cov}[A, B] = E[(A - \mu_a)(B - \mu_b)^*]$$

The covariance of two real stochastic variables X and Y this reduces to

$$\mathrm{Cov}[X, Y] = E[(X - \mu_x)(Y - \mu_y)]$$

Statistically independent and uncorrelated variables

Two real stochastic variables X and Y are said to be statistically independent if their joint probability density function can be factored

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Two stochastic variables A and B are said to be uncorrelated if their covariance is zero, $\mathrm{Cov}[A, B] = 0$.

Problem 1

Show that if the real stochastic variables X and Y are statistically independent then they are uncorrelated.

Problem 2

Show that if the complex stochastic variable Z has uncorrelated real and imaginary parts then the complementary variance is real.

Problem 3

Let X and Y be statistically independent uniformly distributed real stochastic variables on the domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$.

Let $Z = X + \mathrm{i}Y$ be a complex stochastic variable.

Find the probability density function, the cumulative distribution function, the expected value, the variance, and the complementary variance, of Z.