MEK4350, fall 2016
Exercises IX

We have learnt that if $X(t)$ is a (complex) stochastic process, in this case as a function of time t, then the mean (or expected value) of the process is

$$\mu(t) = E[X(t)]$$

and the autocorrelation function is

$$R(t_1, t_2) = E[X(t_1)X^*(t_2)].$$

If the mean is independent of time, $\mu(t) = \mu$, and the autocorrelation function only depends on the relative time $\tau = t_1 - t_2$ and not the absolute time, $R(t+\tau, t) = R(\tau)$, then we say the process is weakly stationary.

For a weakly stationary process we define the spectrum $S(\omega)$ as the Fourier transform of the autocorrelation function $R(\tau)$

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)\mathrm{e}^{\mathrm{i}\omega\tau}\,\mathrm{d}\tau$$

with inversion

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega)\mathrm{e}^{-\mathrm{i}\omega\tau}\,\mathrm{d}\omega$$

We shall insist on the normalization criterion for the spectrum, that the integral of the spectrum shall be equal to the autocorrelation function evaluated at the origin, and this quantity shall be known as the mean power

$$R(0) = \int_{-\infty}^{\infty} S(\omega)\,\mathrm{d}\omega$$

NOTE: Different textbooks have different traditions regarding the first or second X being conjugated, regarding τ being added to t for the first or second X, and regarding the sign of the exponent in the Fourier transform.

In problems 1–9 assume that $X(t)$ is a complex weakly stationary process:

Problem 1

Show that $R(\tau)$ could equivalently have been written as $R(\tau) = E[X(t)X^*(t-\tau)]$.

Problem 2

Show that $R(-\tau) = R^*(\tau)$.

Problem 3

Show that if $X(t)$ real then $R(\tau)$ is even.

Problem 4

Show that $R(0) = E[|X|^2] \geq 0$.

Problem 5

Show that $|\text{Re}\{R(\tau)\}| \leq R(0)$.

Hint: Consider $E[(aX(t)+X(t+\tau))(aX^*(t)+X^*(t+\tau))] \geq 0$ for a real number a. Solve the second degree equation for a corresponding to the limit that the inequality becomes an equality, and discuss the properties of the radicand.

Problem 6

Given the result in the previous problem, show that if $X(t)$ is real then $|R(\tau)| \leq R(0)$.

Problem 7

Show that if $X_n(t)$ are mutually independent, and $Z(t) = \sum_n X_n(t)$, then $R_{ZZ}(\tau) = \sum_n R_{X_n X_n}(\tau)$.

Problem 8

Show that $S(\omega)$ is real.

Problem 9

Show that if $X(t)$ is real then $S(\omega)$ is even.

Problem 10

Let $X(t) = \cos(\omega_0 t + \Theta)$ where ω_0 is a real constant and Θ is uniformly distributed from 0 to 2π. Compute the mean and autocorrelation function of $X(t)$, and determine if $X(t)$ is weakly stationary. In the case that $X(t)$ is weakly stationary then compute the spectrum.

Problem 11

Let $X(t) = A\cos(\omega_0 t)$ where ω_0 is a real constant and A is uniformly distributed from 0 to 1. Compute the mean and autocorrelation function of $X(t)$, and determine if $X(t)$ is weakly stationary. In the case that $X(t)$ is weakly stationary then compute the spectrum.