MEK4350, fall 2016
Exercises IV — suggested solutions

Exercise 6

We look at the three functions

$$f(x) = x \qquad g(x) = |x - \pi| \qquad h(x) = \cos(x) \qquad \text{for } 0 \le x < 2\pi$$

a) In order to explain how fast the Fourier coefficients $\hat{f}_n$, $\hat{g}_n$ and $\hat{h}_n$ are approach zero when $n \to \pm\infty$ we only have to count how many times we should differentiate the functions in order that a Dirac delta function appears. Respectively, once for $f(x)$, twice for $g(x)$, while $h(x)$ can be differentiated infinitely many times without the Dirac delta function ever appearing. Therefore, $\hat{f}_n \propto k_n^{-1}$, $\hat{g}_n \propto k_n^{-2}$ and $\hat{h}_n \propto k_n^{-\infty}$. We say that $\hat{h}_n$ goes to zero faster than algebraically.

Alternatively, it is a very recommendable exercise to compute the Fourier coefficients (remember that $k_n = n$ in this case):

$$\hat{f}_n = \begin{cases} \pi & \text{for } n = 0 \\ \frac{i}{n} & \text{otherwise} \end{cases}$$

$$\hat{g}_n = \begin{cases} \frac{1}{2} & \text{for } n = 0 \\ \frac{4}{n^2} & \text{for } n \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

$$\hat{h}_n = \frac{1}{2}\left(\delta_{n,-1} + \delta_{n,1}\right)$$

c–d) Notice that the curves for $|\hat{f}_n|$ and $|\hat{h}_n|$ should be nice and smooth as $n \to \pm\infty$. The curve for $|\hat{g}_n|$ will oscillate between 0 and positive values, these positive values are what we are interested in, therefore we should look at the upper envelope for the oscillations of $|\hat{g}_n|$ as $n \to \pm\infty$.

e) If we commit the error suggested in part b, then we are no longer considering $h(x)$ on a periodic interval of length 2π, rather we are considering the function

$$\mathcal{H}(x) = \begin{cases} \cos x & \text{for } 0 \le x < 2\pi \\ 1 & \text{for } 2\pi \le x < 2\pi(1 + \frac{1}{N}) \end{cases}$$

over a periodic interval of length $2\pi(1 + \frac{1}{N})$. Notice that the third derivative of $\mathcal{H}(x)$ brings out a Dirac delta function, therefore we should see $\hat{\mathcal{H}}_n \propto k_n^{-3}$.

There is indeed a considerable difference between slope -3 and infinite slope!