The Gaussian Model of Wind-Generated Waves

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Inversion of radar remote sensing images and deterministic prediction of ocean waves.

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Contents (I)

Contents (II)

- Statistical and spectral estimation of sea state parameters
  - Temporal analysis of the sea surface elevation (1D):
    - Sea state parameters zero-up and -down crossing statistics
    - Sea state parameters derived from the wave spectra
  - Spatio (2D) and spatio-temporal (3D) analysis of the sea surface elevation.
- Wave grouping
- DFT estimation of the Three-dimensional spectrum

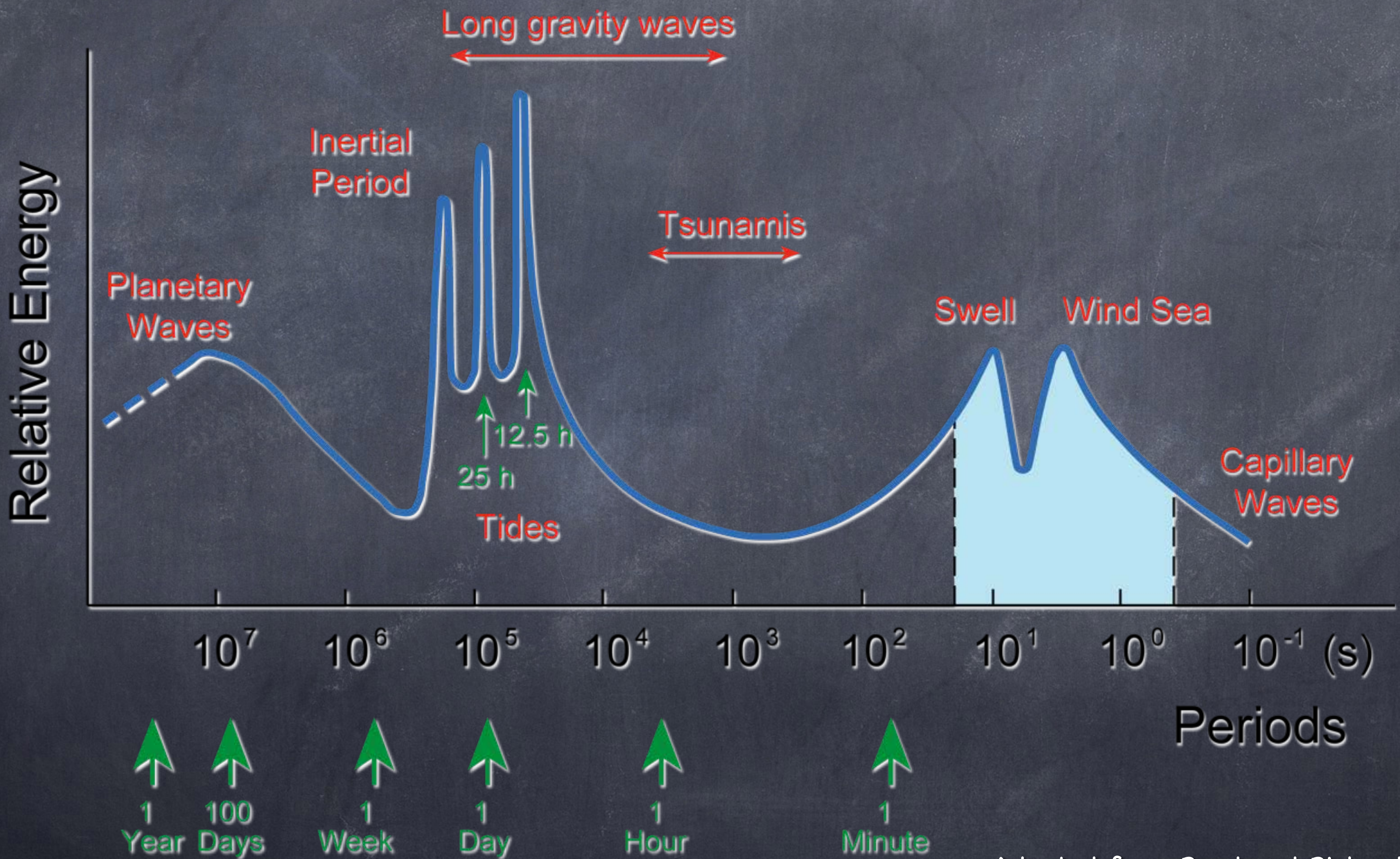
Wind-Generated Waves

Wind-Generated Waves

- Ocean (gravity) waves are wind-generated waves.
- That occur on the free surface of oceans, seas, lakes, etc.
- When directly being generated and affected by the local winds, a wind wave system is called wind sea.
- When these waves propagate to other locations wind waves are called swell.
- A wave field can be the result of a superposition of a wind sea and several swell systems.

Wind-Generated Waves

- Wind waves are not tsunamis, nor tidal waves, nor internal waves,....

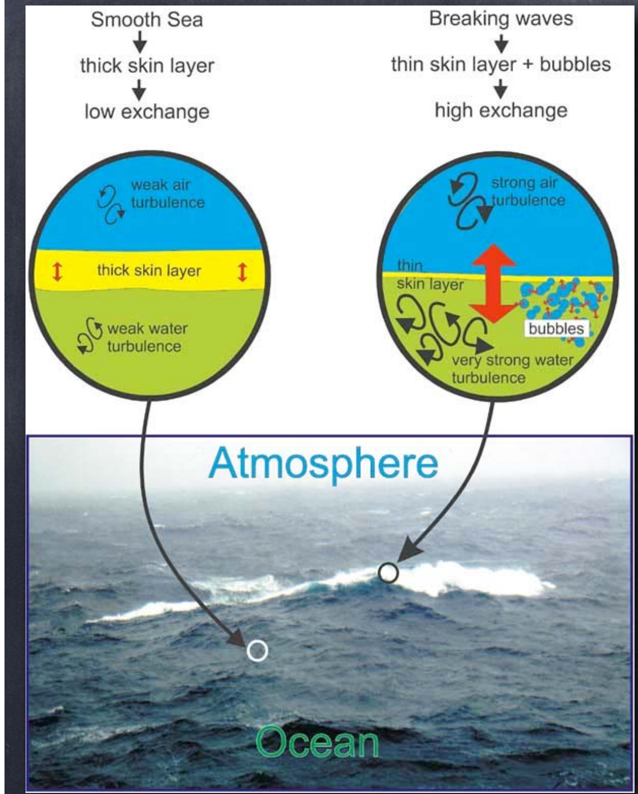


Adapted from Pond and Pickard, 1983

Wind-Generated Waves

- Wind waves play an important role in the energy transference mechanisms between the atmosphere and the ocean.
- Not all the hydrodynamical mechanisms of wind waves are well understood.
- They can be extremely nonlinear.
- To improve the wave forecast.
- To improve the design of vessels, off and onshore structures
  - Oil platforms.
  - Breakwaters.
  - Coastal protection.

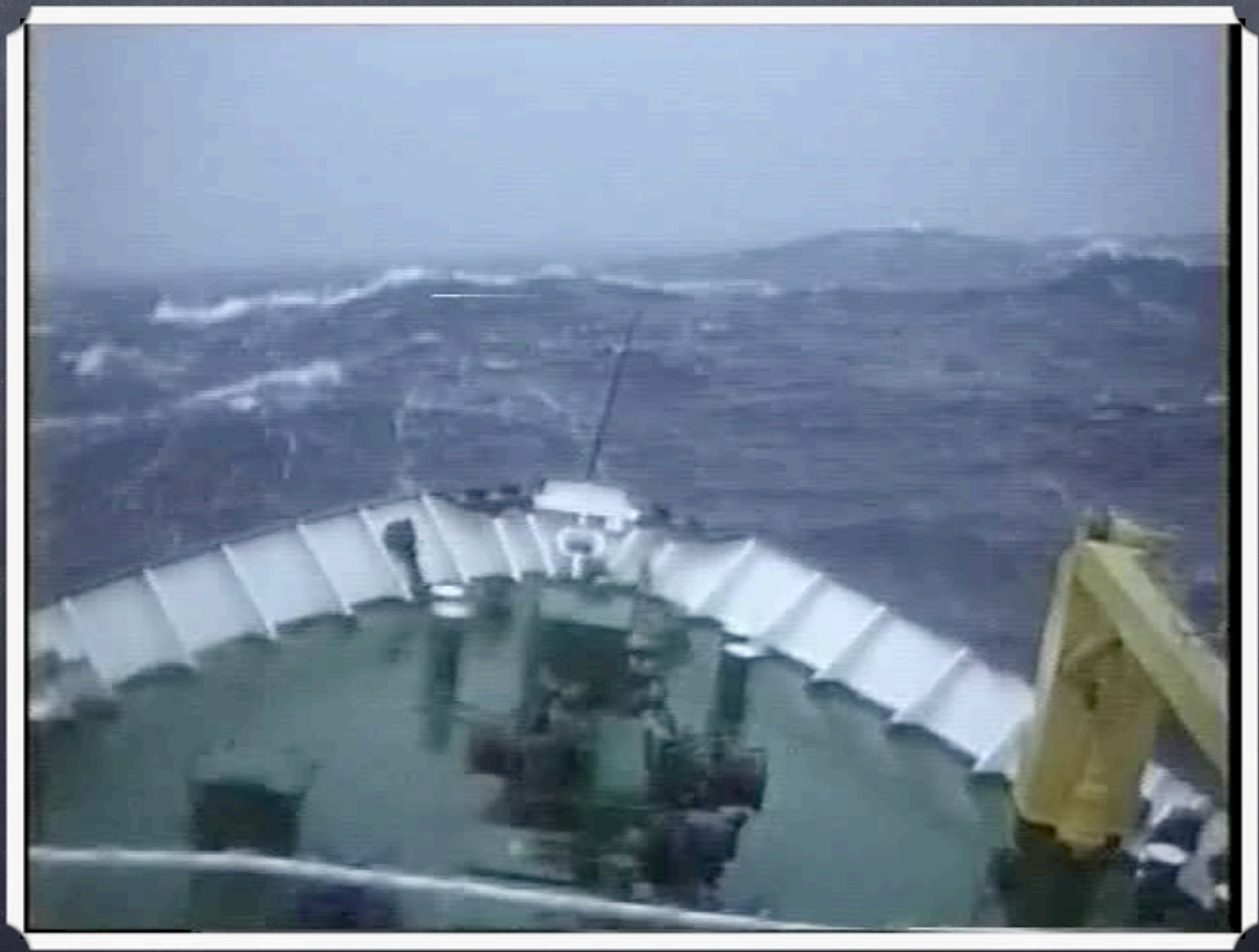
There are still many open questions on the study of wave generation and propagation!!



How wind waves look like?

- Oceanographic cruise of the German vessel **FSS Gauss** in 1992.
- Part of the mission was to carry out field experiments to measure wind waves with marine radars during severe storm conditions.

Filmed by Dr. Friedwart Ziemer
(Helmholtz-Zentrum Geesthacht, Germany)



Spectral Description of Wave Fields

Mathematical Description of Ocean Waves

- Ocean surface waves are caused by the wind.
  - Interaction of two fluids: atmosphere-ocean
- Ocean Waves are described by the spatio-temporal evolution of the vertical elevation of the free sea surface over the sea level

$$\eta(\mathbf{r}, t)$$

Horizontal sea surface coordinates: $\mathbf{r} = (x, y)$

Time: t

Vertical coordinate of the sea surface: $z = \eta(\mathbf{r}, t)$ ← This is the information we try to find or to measure!!

Linear solutions for the wind-generated waves

- Linearizing the hydrodynamic equations that describe the elevation of the sea surface:
  - The sea surface can be described as linear superpositions of several monochromatic waves
  - Those monochromatic waves (e.g. wave spectral components) take the form of
    - sinusoidal waves
    - cosinusoidal waves
    - complex exponential waves

Equivalent descriptions

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

Monochromatic wave in time

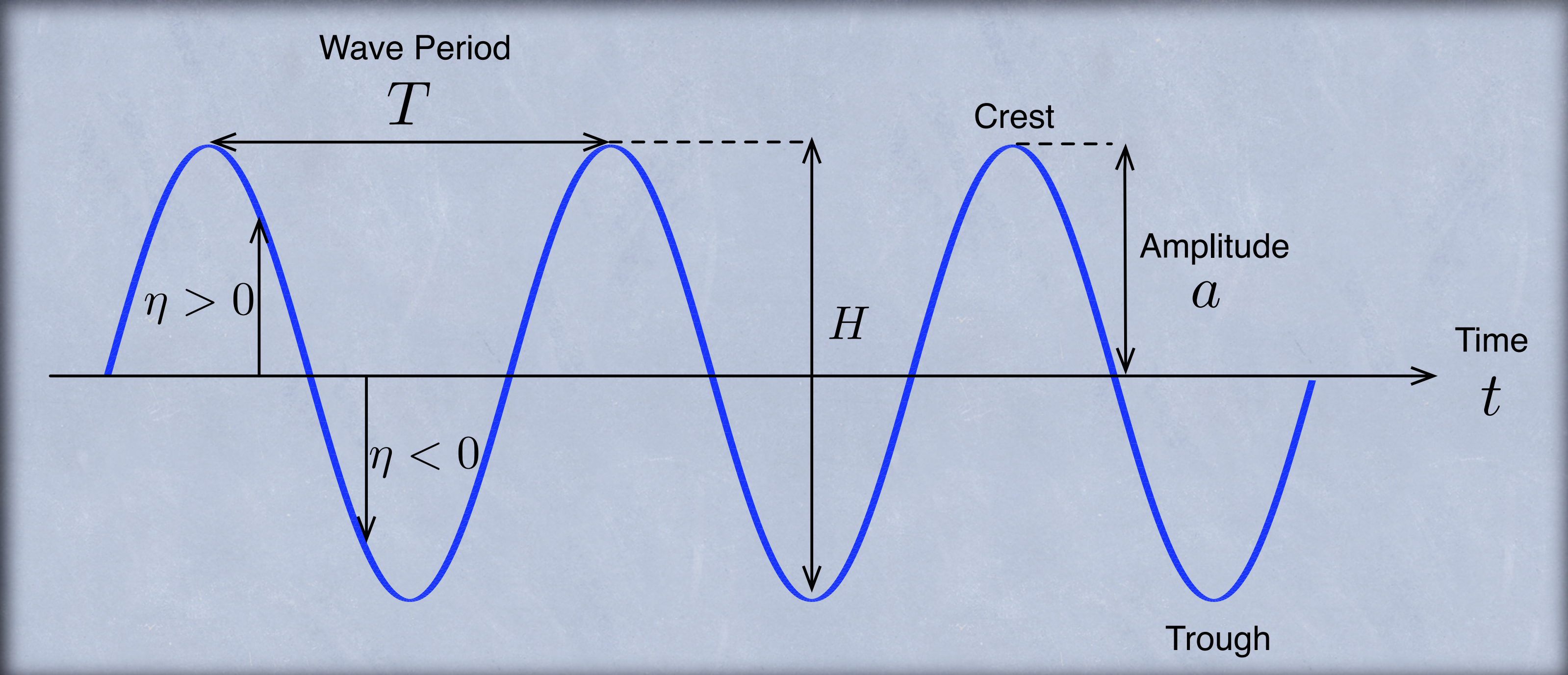
$$\eta(t) = a\cos(\omega t + \varphi)$$

Amplitude [m]: a

Angular frequency [rad/s]: $\omega = 2\pi f$

Wave period [s]: $T = \frac{1}{f}$

Phase [rad]: φ



Monochromatic wave in space (1D)

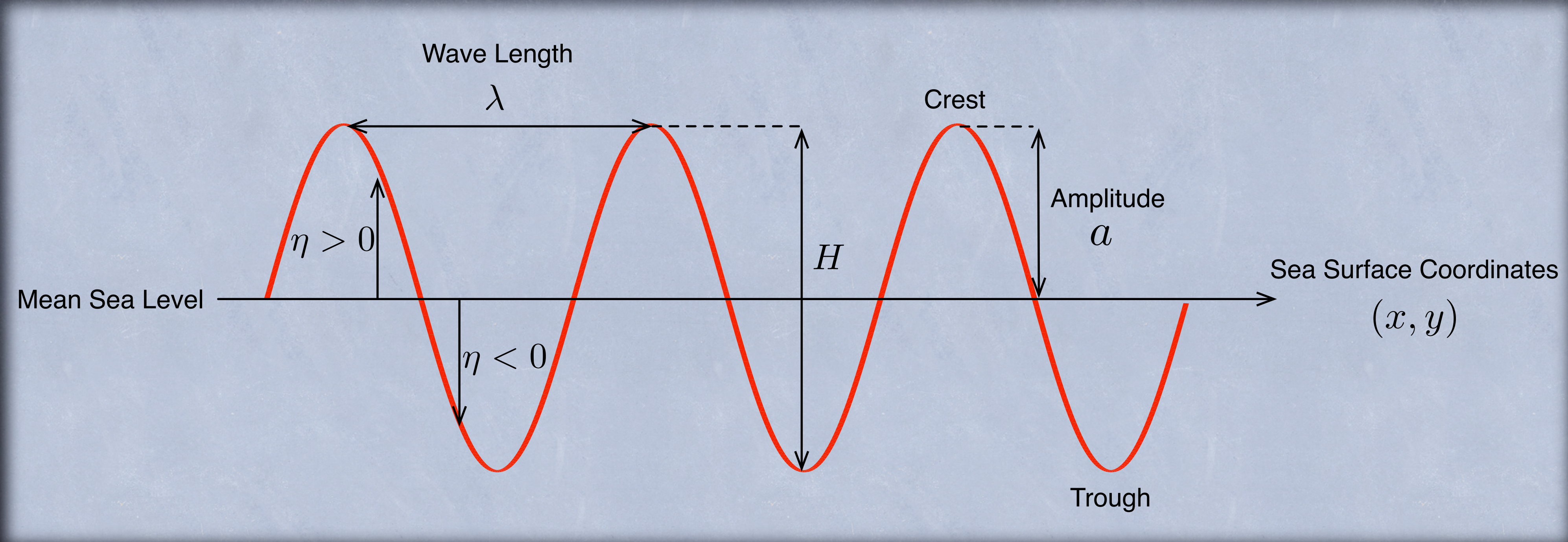
$$\eta(x) = a\cos(kx + \varphi)$$

Amplitude [m]: a

Phase [rad]: φ

Wave number [rad/m]: $k = \frac{2\pi}{\lambda}$

Wave length [m]: λ

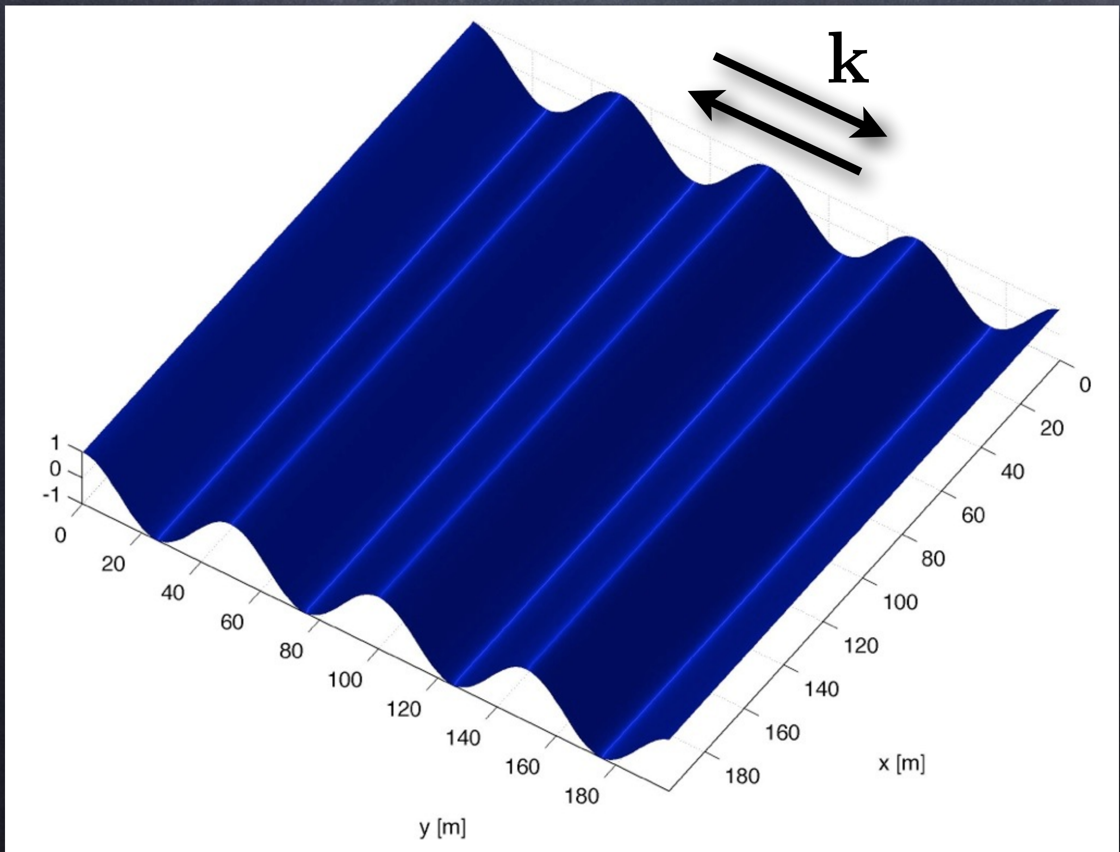


Monochromatic wave in space (2D)

$$\eta(x,y) = a\cos(\mathbf{k}\cdot\mathbf{r}+\varphi) = a\cos(k_x x + k_y y + \varphi)$$

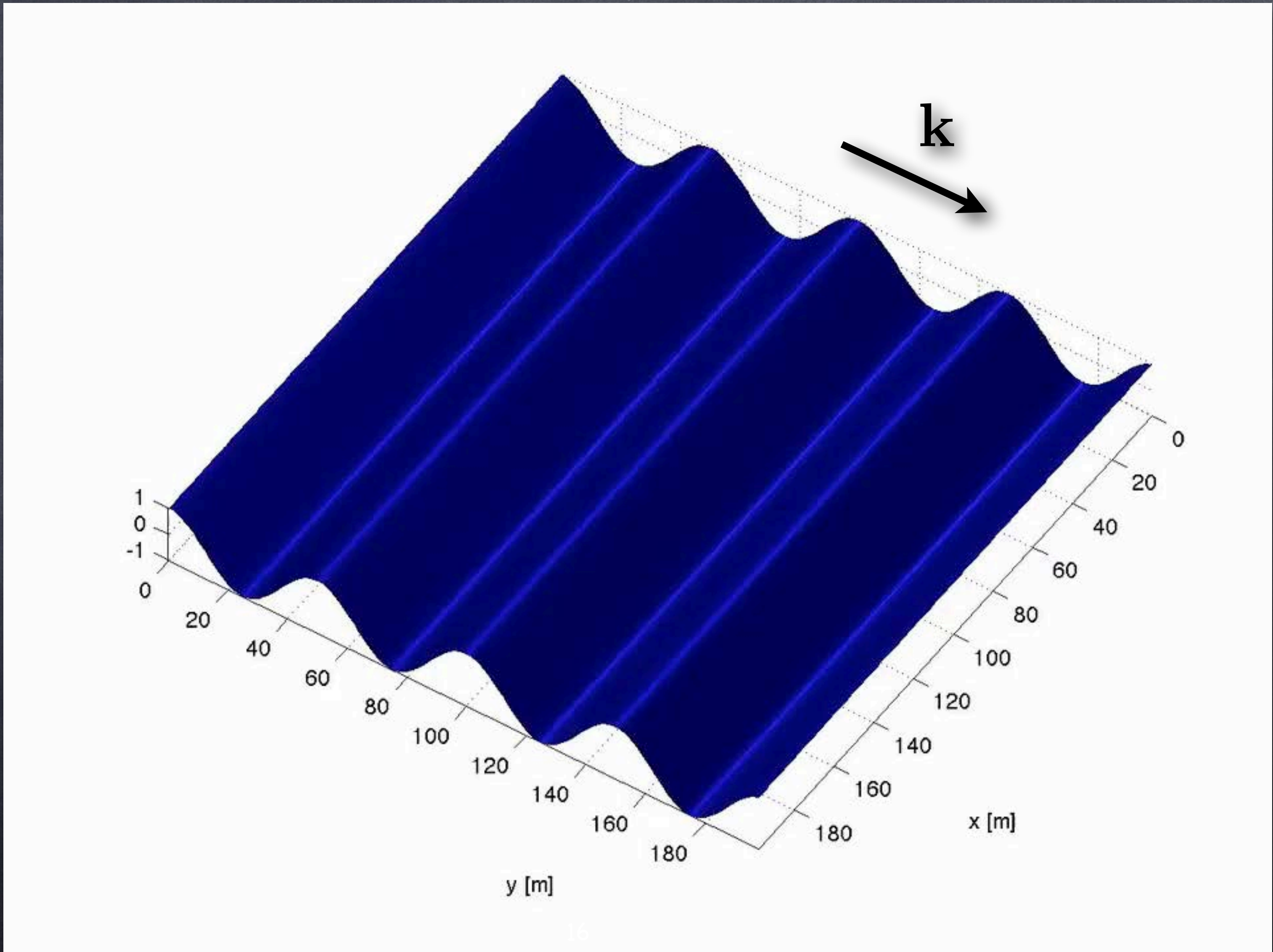
Wave number vector [rad/m]: $\mathbf{k} = (k_x, k_y)$

$$k = |\mathbf{k}| = \sqrt{k_x^2 + k_y^2} = \frac{2\pi}{\lambda}$$



Monochromatic wave in space and time (3D)

$$\eta(x, y, t) = a\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi)$$



Monochromatic wave in space and time (3D) (complex notation)

$$\eta(x,y,t) = a\cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi)$$

$$\cos\alpha = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right)$$

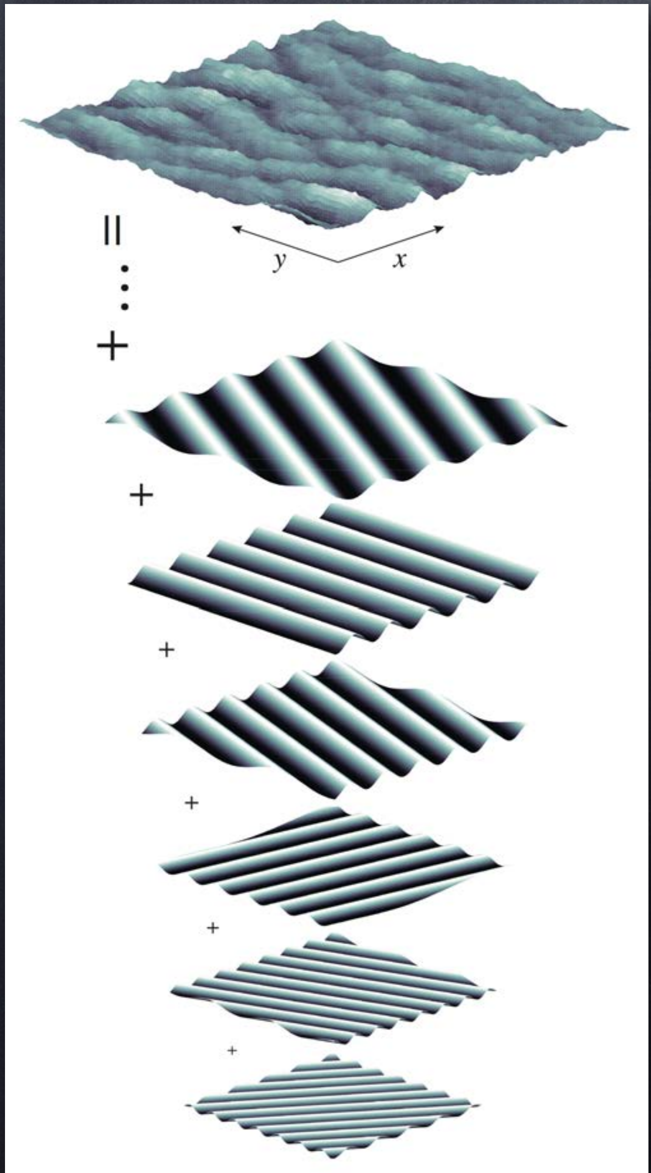
$$\eta(x,y,t) = c\,e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} + \text{c.c.}$$

$$c = \frac{a}{2}\,e^{i\varphi}$$

← Complex amplitude

General solutions of the linear wave theory

- Under the frame of the linear theory, the wave elevation of the free sea surface can be expressed as a liner superposition of different monochromatic waves



- Each wave component is characterized by its:
  - Wave number vector (e.g. wave length and propagation direction)

$$\mathbf{k} = (k_x, k_y) \qquad \lambda = \frac{2\pi}{k}$$

$$\theta = \tan^{-1}\left(\frac{k_y}{k_x}\right)$$

  - Frequency ω
  - Amplitude a
  - Phase φ

> **Comment**: to avoid ambiguity of 180 degrees, the wave propagation direction should be computed numerically using `atan2`-type functions, e.g. `atan2(ky, kx)`.

General solutions of the linear wave theory

- Notations (I):

$$\eta(\mathbf{r}, t) = \sum_n a_n \cos(\mathbf{k}_n \cdot \mathbf{r} - \omega_n t + \varphi_n)$$

$$\eta(\mathbf{r}, t) = \sum_n c_n e^{i(\mathbf{k}_n \cdot \mathbf{r} - \omega_n t)} + \mathrm{c.c.}$$

General solutions of the linear wave theory

- Notations (II):

$$\eta(\mathbf{r},t) = \sum_{k_x}\sum_{k_y}\sum_{\omega} a(k_x,k_y,\omega)\cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi(k_x,k_y,\omega)\right]$$

$$\eta(\mathbf{r},t) = \sum_{k_x}\sum_{k_y}\sum_{\omega} c(k_x,k_y,\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + \mathrm{c\ .c.}$$

$\begin{pmatrix} a(k_x,k_y,\omega) \\ \varphi(k_x,k_y,\omega) \end{pmatrix}$ and $c(k_x,k_y,\omega)$ are estimated from the Fourier Transform

General solutions of the linear wave theory

- Notations (III): continuous notation

This expression normally includes the complex conjugates

$$\eta(\mathbf{r}, t) = \int_{\Omega_{\mathbf{k},\omega}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} dZ(\mathbf{k}, \omega)$$

Complex amplitude: $dZ(\mathbf{k}, \omega)$

- Spectral domain where ocean waves are defined: $\Omega_{\mathbf{k},\omega} = \Omega_{\mathbf{k}} \times \Omega_{\omega}$
- In practice the spectral domain is limited by the resolution of the sensor in space and time

$$\Omega_{\mathbf{k}} = [-k_{x_c}, k_{x_c}) \times [-k_{y_c}, k_{y_c})$$

$$\Omega_{\omega} = [-\omega_c, \omega_c)$$

Dispersion Relation

- Ocean waves are dispersive.
- The dispersion relation is given by

$$\omega = \underbrace{\sqrt{gk \tanh(kh)}}_{\text{Intrinsic Frequency}} + \underbrace{\mathbf{k} \cdot \mathbf{U}}_{\text{Doppler shift}}$$

Intrinsic Frequency

Doppler shift

Current of encounter: $\mathbf{U} = (U_x, U_y)$

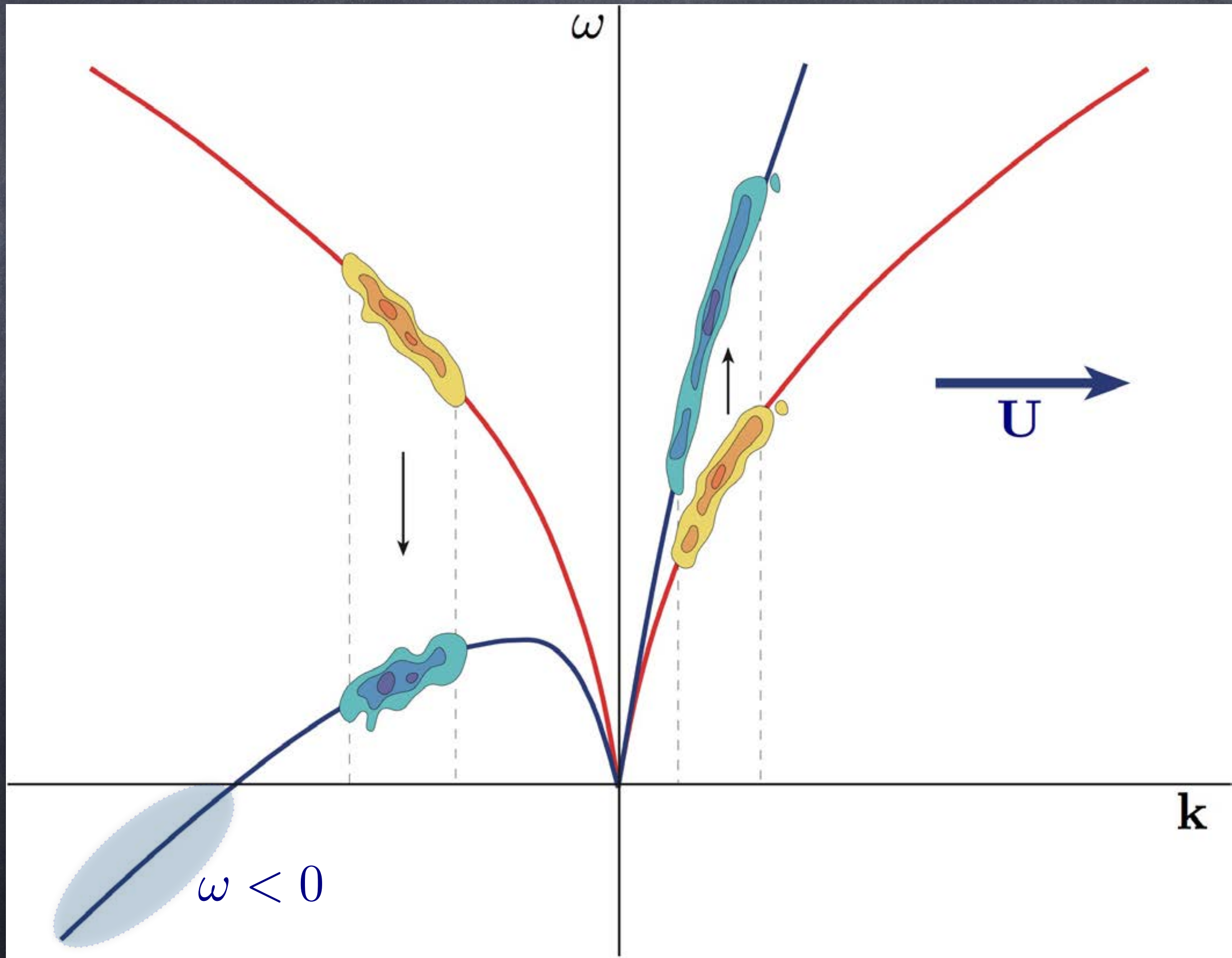
Water depth: h

The "current of encounter" is the combination of different effects:

- Relative motion between the observer and the wave field.
- Geophysical current: wind-, wave-induced current, Stokes drift, geostrophic flow, tides, etc.

Dispersion Relation

$$\omega = \sqrt{gk\tanh(kh)} + \mathbf{k}\cdot\mathbf{U}$$



Dispersion Relation

- General Case:
  - Phase velocity: $v_p = \dfrac{\omega}{k}$
  - Group velocity: $v_g = \dfrac{d\omega}{dk}$

$v_p \neq v_g$ (dispersive)

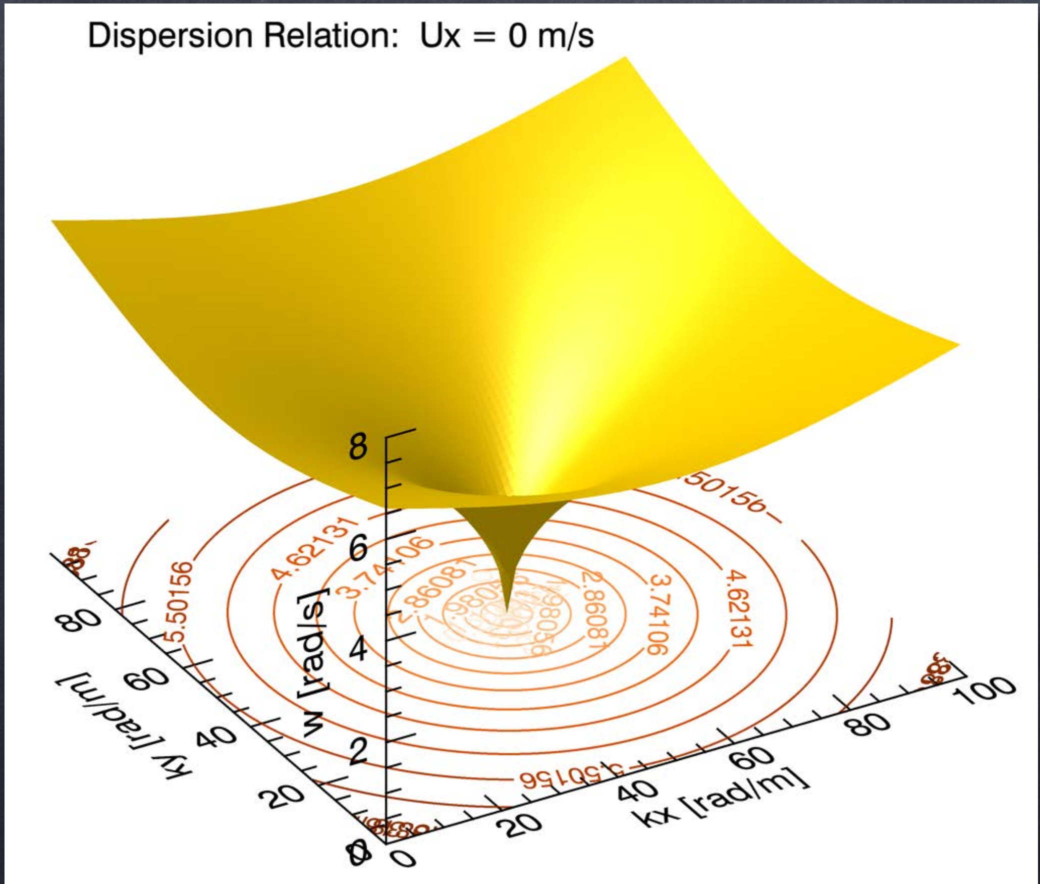
- Approximations (without current of encounter):
  - Deep water: $kh \gg 1 \Longrightarrow \tanh(kh) \approx 1 \Longrightarrow \omega \approx \sqrt{gk}$

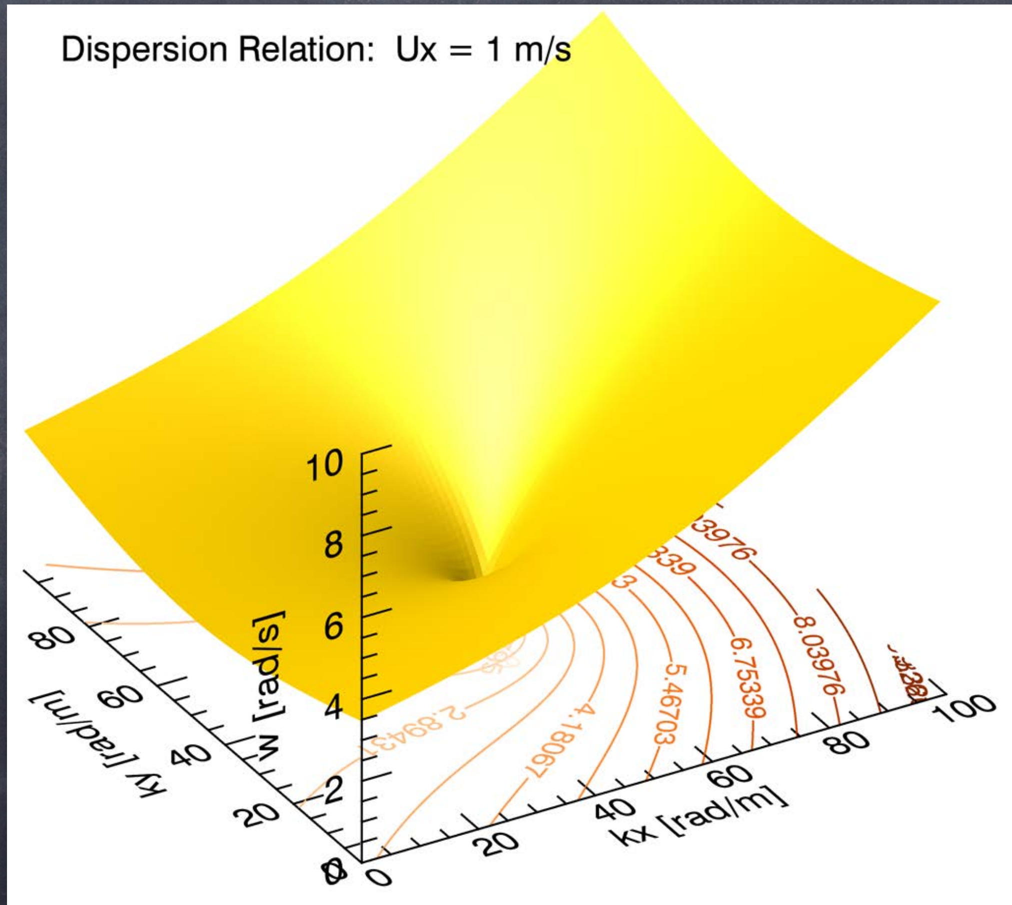
    (dispersive)

  - Shallow water: $kh \ll 1 \Longrightarrow \tanh(kh) \approx kh \Longrightarrow \omega \approx k\sqrt{gh}$

    (non dispersive)

Dispersion Relation





Dispersion Relation

- Considering the dispersion relation the linear wave field can be expressed as

$$\eta(\mathbf{r},t) = \sum_{k_x}\sum_{k_y}\sum_{\omega} a(k_x,k_y,\omega)\cos\left[\mathbf{k}\cdot\mathbf{r} - \omega t + \varphi(k_x,k_y,\omega)\right]$$

$$\Big\downarrow \; \omega(\mathbf{k})$$

$$\eta(\mathbf{r},t) = \sum_{k_x}\sum_{k_y} a(k_x,k_y)\cos\left[\mathbf{k}\cdot\mathbf{r} - \omega(\mathbf{k})t + \varphi(k_x,k_y)\right]$$

Sea State (I)

- The movement of the ocean free surface is complicated, even assuming the linear wave theory.
- A way to improve the the sea surface description given by the linear wave theory is to assume that the wave elevation presents a stochastic behavior.
  - The parameters if the linear wave solutions are considered as random variables
  - The statistical properties of those parameters depend on the meteorological and geophysical conditions
- Under these considerations, the concept of sea state is defined from:
  - Temporal domain where the wave field is statistically stationary.
  - Area of the ocean where the wave field is statistically homogeneous.

Sea State (II)

- Random parameters for different linear wave theory notations:

$$\eta(\mathbf{r},t) = \sum_n a_n \cos(\mathbf{k}_n \cdot \mathbf{r} - \omega_n t + \varphi_n)$$

$$\eta(\mathbf{r},t) = \sum_n c_n e^{i(\mathbf{k}_n \cdot \mathbf{r} - \omega_n t)} + \mathrm{c.c.}$$

$$\eta(\mathbf{r},t) = \int_{\Omega_{\mathbf{k},\omega}} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} dZ(\mathbf{k},\omega)$$

η is a stochastic process

Gaussian Sea States

- The Gaussian sea is the simplest stochastic model to describe sea surface variability
  - It considers different components that are statistically independent (uncorrelated).
  - Statistical symmetry between crests and troughs.
  - η is a zero-mean Gaussian stochastic process: $\mathsf{E}[\eta] = 0$
  - Variance: $\mathsf{Var}[\eta] = \sigma^2$

a_n is Rayleigh distributed

φ_n is uniformly distributed in $[-\pi, \pi)$

c_n is complex-Gaussian distributed

$dZ(\mathbf{k}, \omega)$ is complex-Gaussian distributed

Gaussian Sea States

$\mathsf{E}[\eta] = 0 \qquad \mathsf{Var}[\eta] = \sigma^2$

- Considering that the spectral components are statistically independent:

$$\sigma^2 = \mathsf{E}\left[\eta^2\right] = \frac{1}{2}\sum_n \mathsf{E}\left[a_n^2\right]$$

$$\sigma^2 = \mathsf{E}\left[\eta^2\right] = 2\sum_n \mathsf{E}\left[|c_n|^2\right]$$

due to c.c.

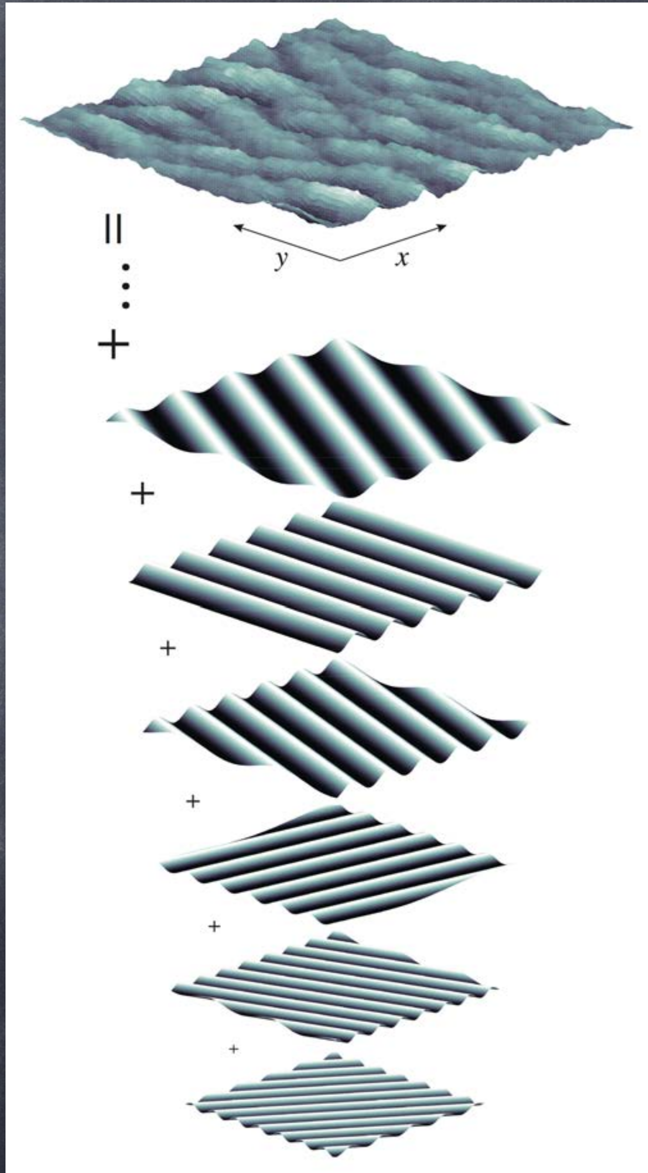
$$\sigma^2 = \mathsf{E}\left[\eta^2\right] = \int_{\Omega_{\mathbf{k},\omega}} \mathsf{E}\left[|dZ(\mathbf{k},\omega)|^2\right]$$

Spectral Representation of Sea States

- Using the continuos representation of sea states:
  (it could be done with the discrete notations as well)

$$\eta(\mathbf{r},t) = \int_{\Omega_{\mathbf{k},\omega}} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\, dZ(\mathbf{k},\omega)$$

Spectral random measure



Spectral Representation of Sea States

- This spectral representation corresponds to the **Eulerian** description of the sea surface.

- Different individual wave components are uncorrelated (statistically independent).

$$\mathsf{E}\left[dZ(\mathbf{k},\omega)dZ^*(\mathbf{k}',\omega')\right] = 0$$

$$\forall\omega \neq \omega' \quad \forall\mathbf{k} \neq \mathbf{k}'$$

$$\mathsf{E}\left[dZ(\mathbf{k},\omega)\right] = 0$$

$$\mathsf{E}\left[\eta\right] = 0$$

Zero-mean Gaussian process
(for the Eulerian description)

Three-dimensional Wave Spectrum

$$F^{(3)}(\mathbf{k},\omega)d^2kd\omega = \mathsf{E}\left[\left|dZ(\mathbf{k},\omega)\right|^2\right]$$

$$F^{(3)}(\mathbf{k},\omega) = F^{(3)}(-\mathbf{k},-\omega)$$

Variance of the sea surface

Zeroth-order moment

$$\sigma^2 = \mathsf{Var}\left[\eta\right] = \int_{\Omega_{\mathbf{k},\omega}} F^{(3)}(\mathbf{k},\omega)\, dk_x dk_y d\omega = m_0$$

Location of the wave spectral components

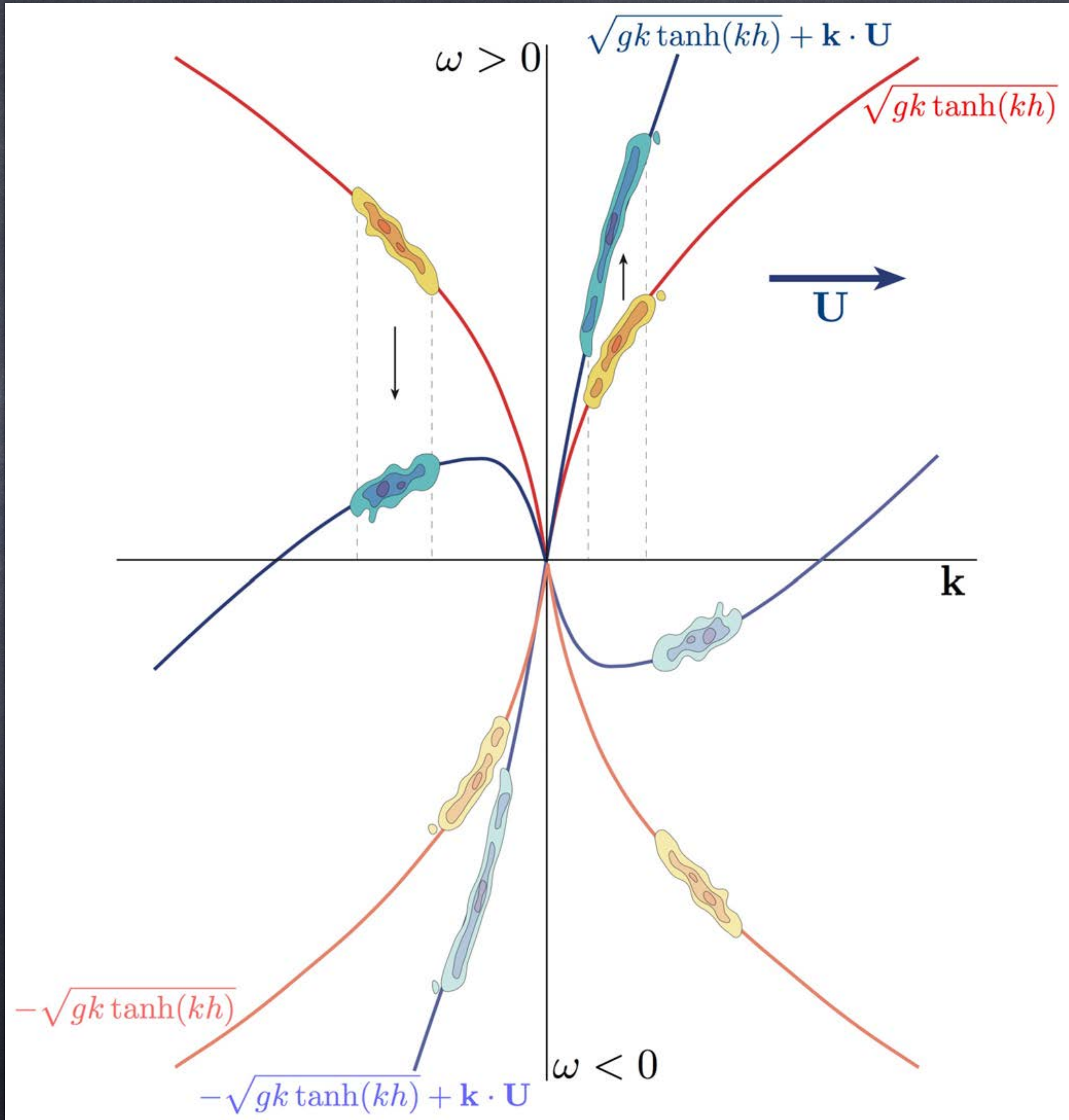
- For real-valued wave elevation field the spectrum is an even function

$$F^{(3)}(\mathbf{k},\omega) = F^{(3)}(-\mathbf{k},-\omega)$$

- For linear wave theory, the observed spectrum (and similarly the spectral amplitude) will be supported on a modified dispersion shell

$$\Omega = \left\{ (\mathbf{k},\omega) \,\middle|\, \omega = \pm\sqrt{gk\tanh(kh)} + \mathbf{k}\cdot\mathbf{U} \right\}$$

Location of the wave spectral components



Alternative Wave Spectral Descriptions

- Different spectral density functions can be derived by integrating the 3D wave spectrum over different subsets of the spectral domain.
- All these spectral density functions must preserve the total energy (e.g. the variance of the wave elevation process).
- The transformations of the spectral density functions assume:
  - The dispersion relation.
  - The wave field is statistically homogeneous in space.
  - The wave field is statistically stationary in time.
    - The sea surface elevation is assumed to be an Ergodic process.

Alternative Wave Spectral Descriptions: Frequency Spectrum

- This spectral density is obtained integrating over all the wave number domain.
- It represents the spectrum obtained from a point measurement.
  - E.g. a record of a buoy moored at a fixed ocean location.

$$S(\omega) = \int_{\Omega_{\mathbf{k}}} F^{(3)}(\mathbf{k}, \omega)\, dk_x dk_y$$

$$S(\omega) = S(-\omega)$$

$$S(\omega) \longmapsto 2 \cdot S(\omega) \quad , \quad \forall \omega > 0$$

Alternative Wave Spectral Descriptions: Directional Wave Number Spectrum

- Integrating over all the frequency domain:

$$F^{(2)}(\mathbf{k}) = \int_{\Omega_\omega} F^{(3)}(\mathbf{k}, \omega) d\omega$$

- This spectrum presents symmetric dependence on the wave propagation direction

$$F^{(2)}(\mathbf{k}) = F^{(2)}(-\mathbf{k})$$

- There is an ambiguity of 180 degrees.

Alternative Wave Spectral Descriptions: Unambiguous Directional Wave Number Spectrum

- Integrating over the positive frequency domain:

$$F_{+}^{(2)}(\mathbf{k}) = 2\int_{0}^{\omega_c} F^{(3)}(\mathbf{k},\omega)d\omega$$

- This spectrum resolves the directional ambiguity

$$F_{+}^{(2)}(\mathbf{k}) \neq F_{+}^{(2)}(-\mathbf{k})$$

Alternative Wave Spectral Descriptions: 3D spectrum from 2D spectrum

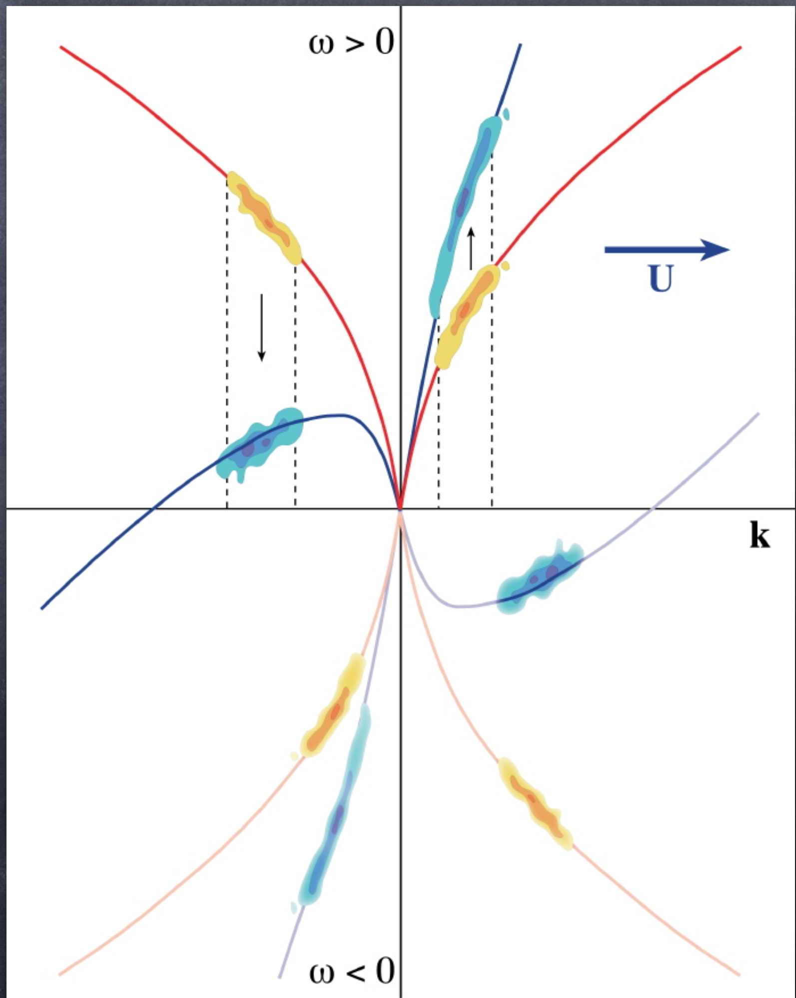
- Assuming the dispersion relation the three-dimensional wave spectrum can be obtained from the unambiguous wave number spectrum as

$$F^{(3)}(\mathbf{k},\omega) = \frac{1}{2}\left\{F_{+}^{(2)}(\mathbf{k})\delta\left[\omega - \varpi(\mathbf{k})\right] + F_{+}^{(2)}(-\mathbf{k})\delta\left[\omega + \varpi(-\mathbf{k})\right]\right\}$$

$$F^{(2)}(\mathbf{k}) = \frac{1}{2}\left[F_{+}^{(2)}(\mathbf{k}) + F_{+}^{(2)}(-\mathbf{k})\right]$$

Alternative Wave Spectral Descriptions: 3D spectrum from 2D spectrum

$$F^{(3)}(\mathbf{k},\omega) = \frac{1}{2}\left\{F_{+}^{(2)}(\mathbf{k})\delta\left[\omega - \varpi(\mathbf{k})\right] + F_{+}^{(2)}(-\mathbf{k})\delta\left[\omega + \varpi(-\mathbf{k})\right]\right\}$$



Alternative Wave Spectral Descriptions: Directional Spectrum (I)

- Transforming the coordinate system from Cartesian to polar coordinates

$$(k_x, k_y) \longmapsto (k, \theta)$$

$$k = \sqrt{k_x^2 + k_y^2} \qquad \theta = \tan^{-1}\left(\frac{k_y}{k_x}\right)$$

Wave propagation direction

$$\tilde{F}^{(2)}(k, \theta) = F_+^{(2)}[\mathbf{k}(k, \theta)] \cdot k$$

Jacobian from Cartesian to polar coordinates

Comment: to avoid ambiguity of 180 degrees, the wave propagation direction should be computed numerically using `atan2`-type functions, e.g. `atan2(ky, kx)`.

Alternative Wave Spectral Descriptions: Directional Spectrum (II)

- Transforming from wave number to frequency:
- The dispersion relation is assumed

$$(k,\theta) \longmapsto (\omega,\theta)$$

$$E(\omega,\theta) = \tilde{F}^{(2)}[k(\omega),\theta] \cdot \frac{dk}{d\omega}$$

Jacobian:
Group Velocity^{-1}

$$v_g = \frac{d\omega}{dk}$$

Alternative Wave Spectral Descriptions: Directional Spectrum (III)

- The frequency-direction spectrum is factorized as

$$E(\omega, \theta) = S(\omega) D(\omega, \theta)$$

Directional spreading function: $D(\omega, \theta)$

$$D(\omega, \theta) \geq 0 \ , \quad \forall (\omega, \theta)$$

$$\int_{-\pi}^{\pi} D(\omega, \theta) d\theta = 1 \ , \quad \forall \omega$$

directional probability density distribution

$$S(\omega) = \int_{-\pi}^{\pi} E(\omega, \theta) d\theta$$

Frequency Spectrum

Alternative Wave Spectral Descriptions

$$F^{(3)}(\mathbf{k},\omega)$$

$$S(\omega) = \int_{\Omega_{\mathbf{k}}} F^{(3)}(\mathbf{k},\omega)\, dk_x dk_y$$

$$F_+^{(2)}(\mathbf{k}) = 2\int_0^{\omega_c} F^{(3)}(\mathbf{k},\omega) d\omega$$

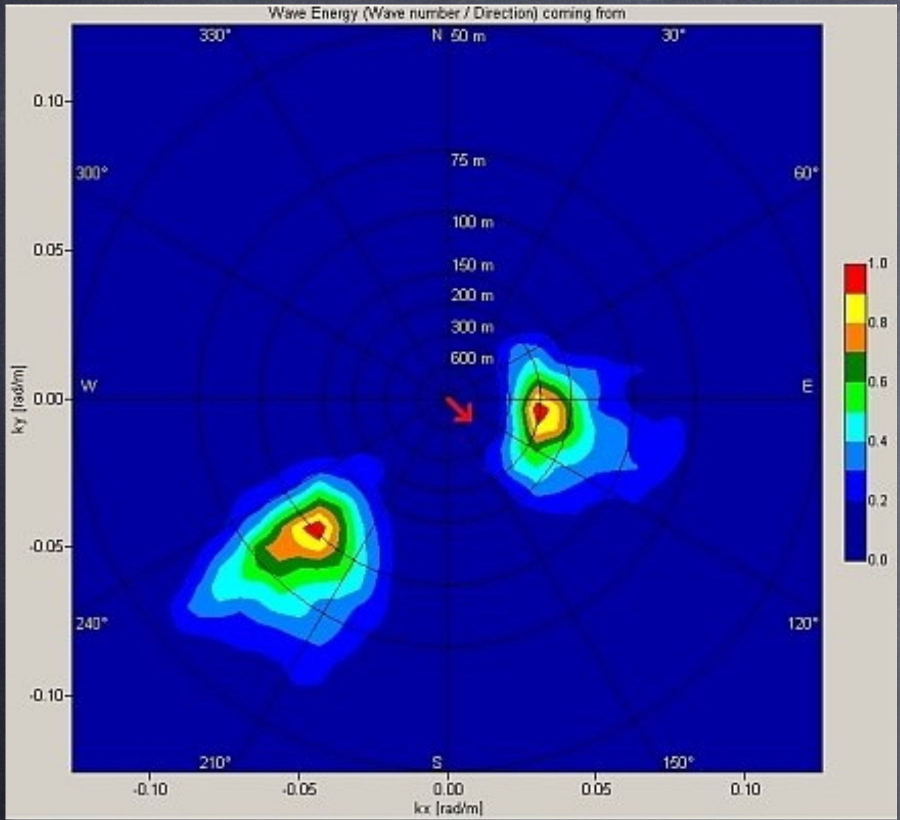
$$\tilde{F}^{(2)}(k,\theta) = F_+^{(2)}[\mathbf{k}(k,\theta)] \cdot k$$

$$E(\omega,\theta) = \tilde{F}^{(2)}[k(\omega),\theta] \cdot \frac{dk}{d\omega}$$

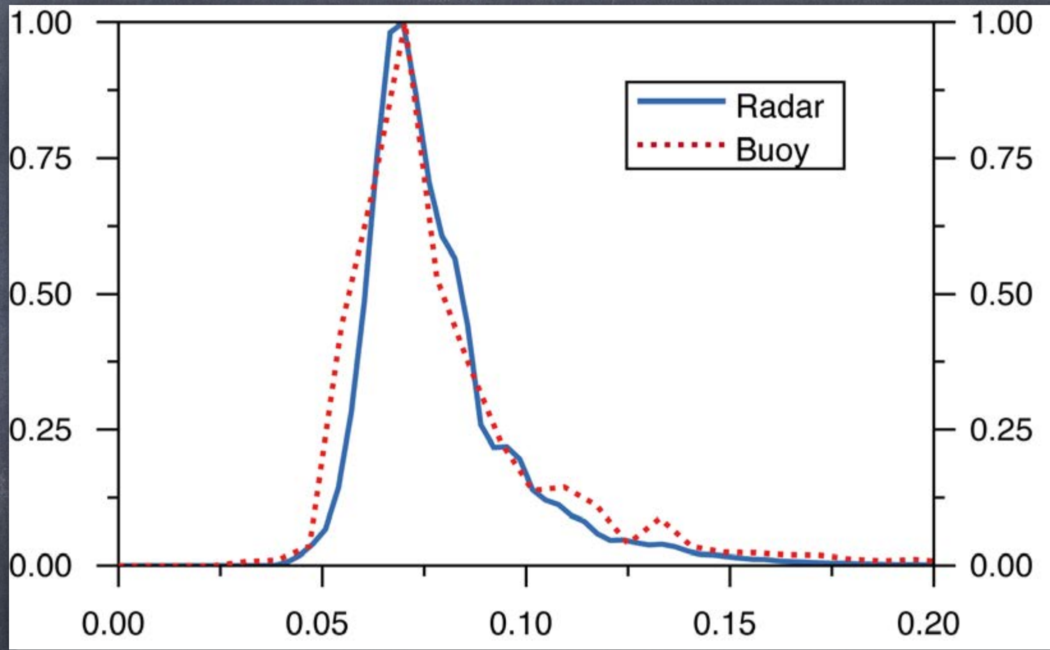
$$S(\omega) = \int_{-\pi}^{\pi} E(\omega,\theta) d\theta$$

Examples of Wave Spectra

Wave number spectrum $F_+^{(2)}(\mathbf{k})$



Frequency spectrum $S(f)$



wave frequency [Hz]

Statistical and spectral estimation of sea state parameters

- Assuming the concept of sea state different parameters related to the wave elevation of the free sea surface can be retrieved.
- Those parameters can be derived from…
  - Statistical analysis of the data in the temporal, spatial, or spatio-temporal domains
  - Spectral analysis of the data: frequencies and/or wave number domains.
  - Under the Gaussian wave field assumptions, both approaches are equivalent.
  - With real data sets, the two different approaches give different and complementary descriptions.

Statistical estimation of sea state parameters

Domain	Example	Dimension	Variables
Temporal	Buoy Record (punctual measurement)	1D	t
Spatial	Measurements along a wave channel	1D	x
Spatial	Video or radar image	2D	(x, y)
Spatio-temporal	Temporal sequences of video or radar images	3D	(x, y, t)

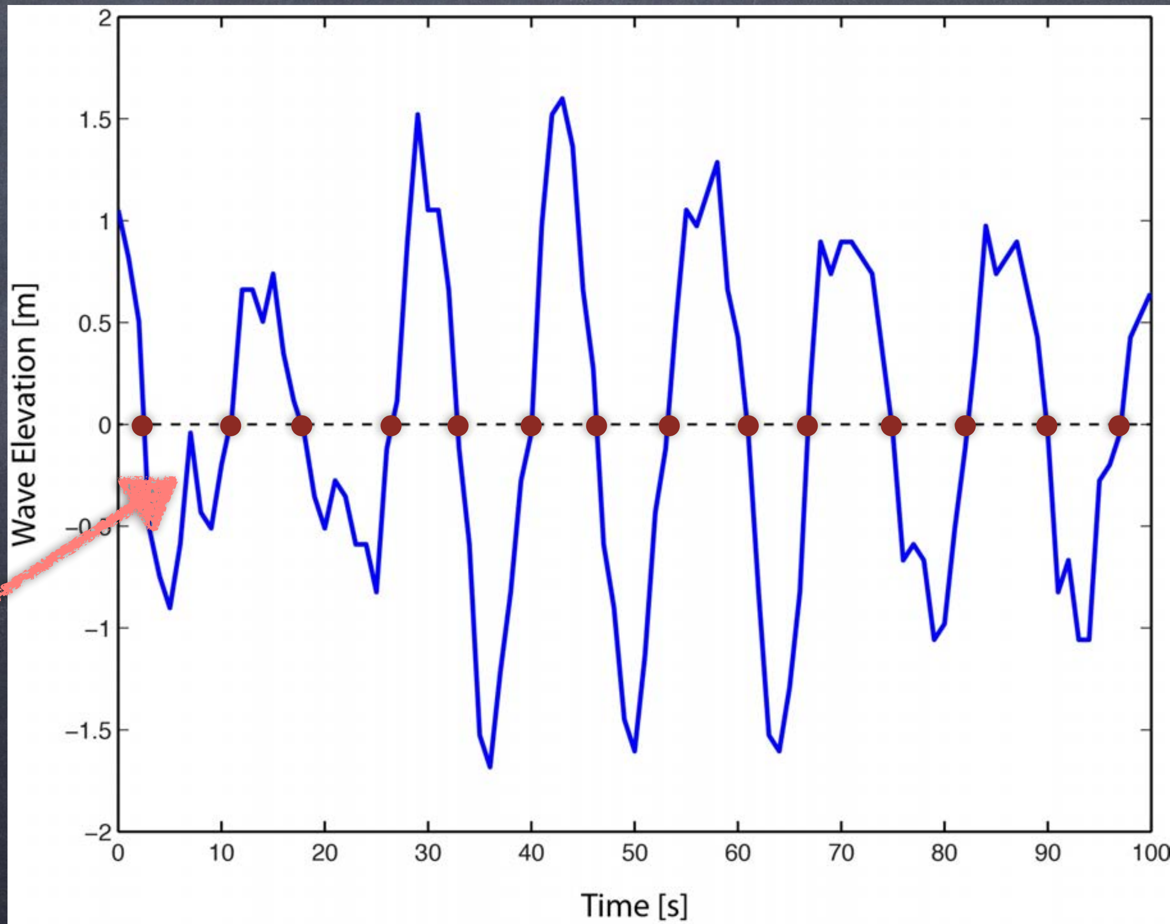
Spectral estimation of sea state parameters

Domain	Example	Dimension	Variables
Frequency	Buoy Record (punctual measurement)	1D	ω
Wave number	Measurements along a wave channel	1D	k_x
Wave number	Video or radar image	2D	(k_x, k_y)
Wave number and frequency	Temporal sequences of video or radar images	3D	(k_x, k_y, ω)

Statistical estimation of sea state parameters (1D)

- This analysis considers the definition of a wave taking into account when the elevation of the sea surface cross the mean sea level

Part of a buoy measurement record



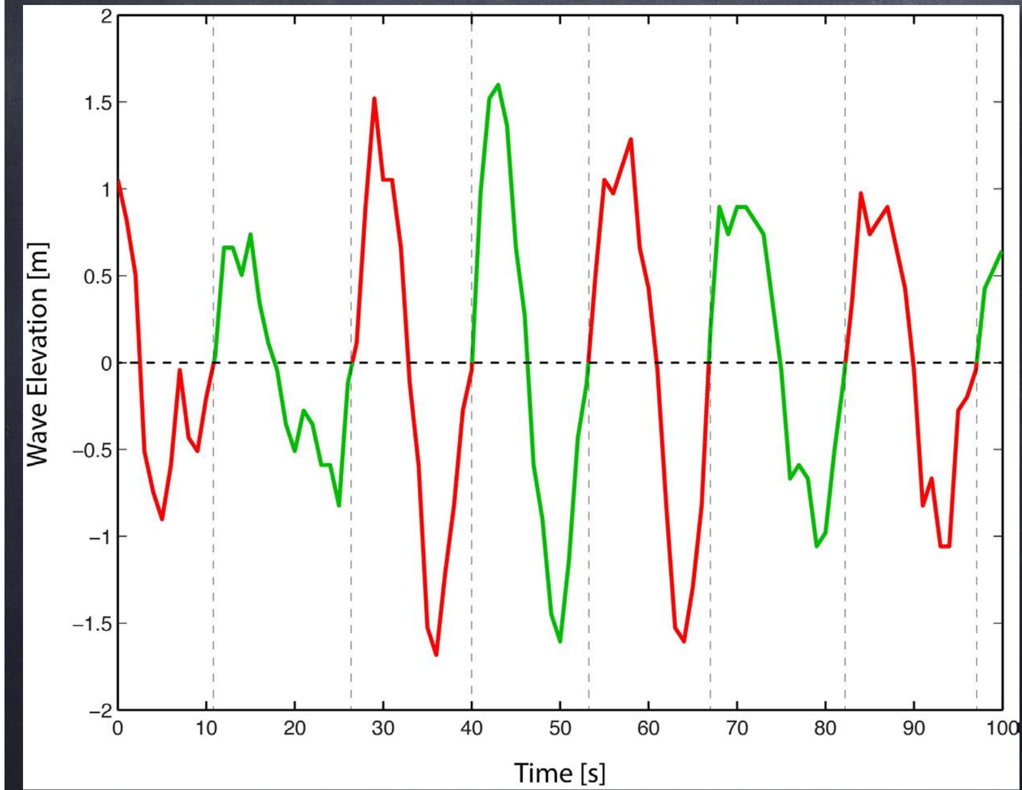
Comment: "practical" indication that the wave record is not exactly a narrow-banded process

Statistical estimation of sea state parameters (1D)

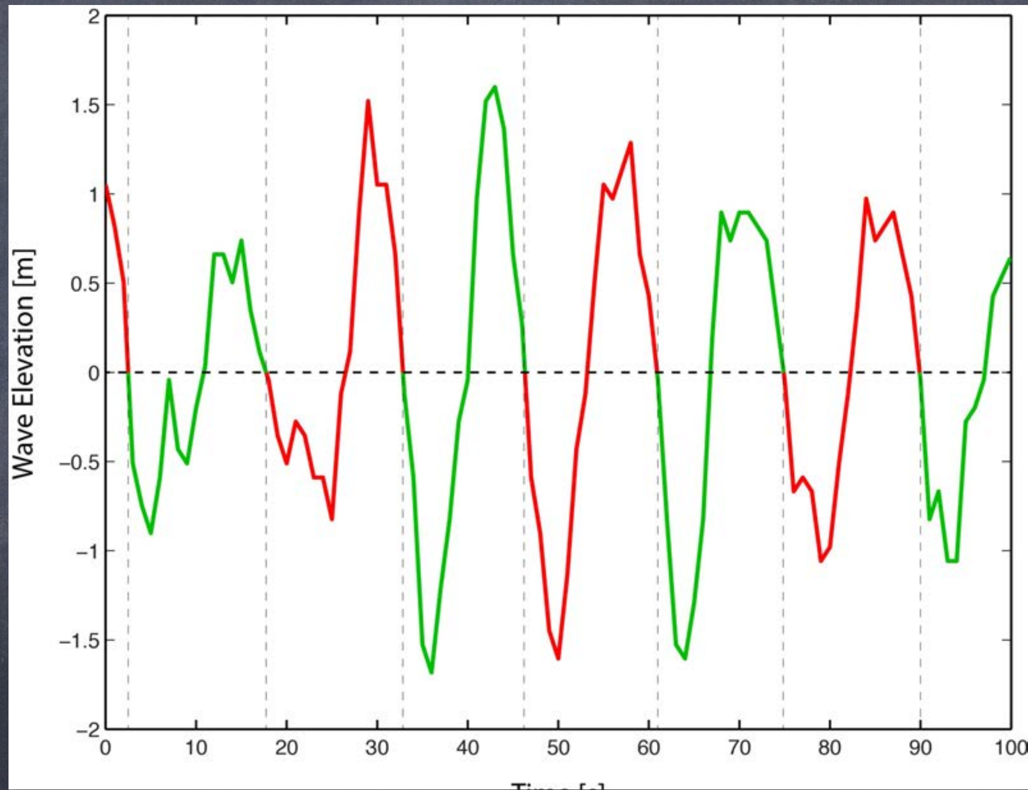
- Two criteria to define waves can be considered:
  - Zero-up crossing.
  - Zero-down crossing.
- For Gaussian stochastic processes both criteria are equivalent.
- In practice the sea state parameters obtained from the different criteria are slightly different.
- The zero-up crossing criterium has been chosen as the standard for practical and engineering purposes.

Statistical estimation of sea state parameters (1D)

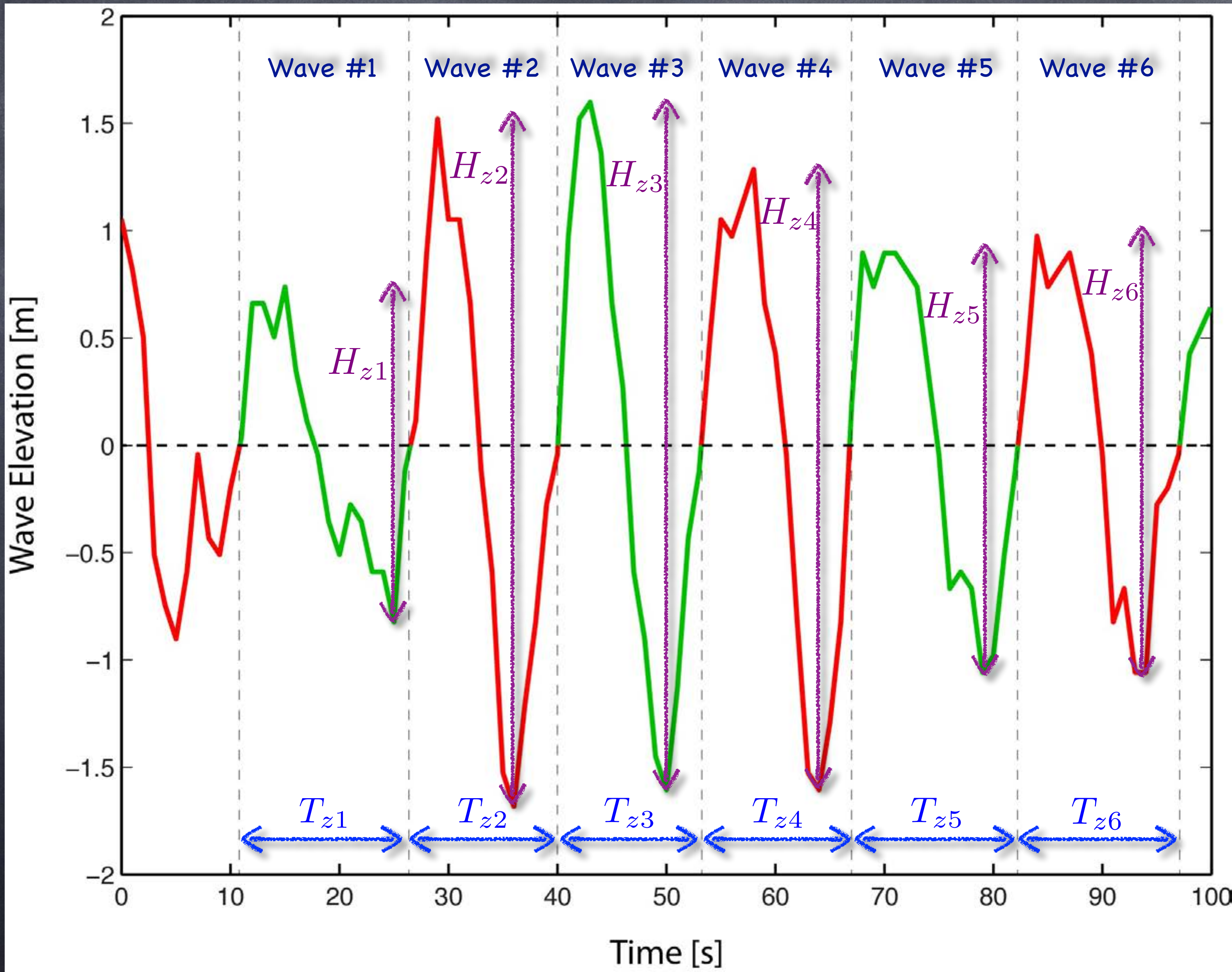
Zero-up crossing



Zero-down crossing



Statistical estimation of sea state parameters (1D)

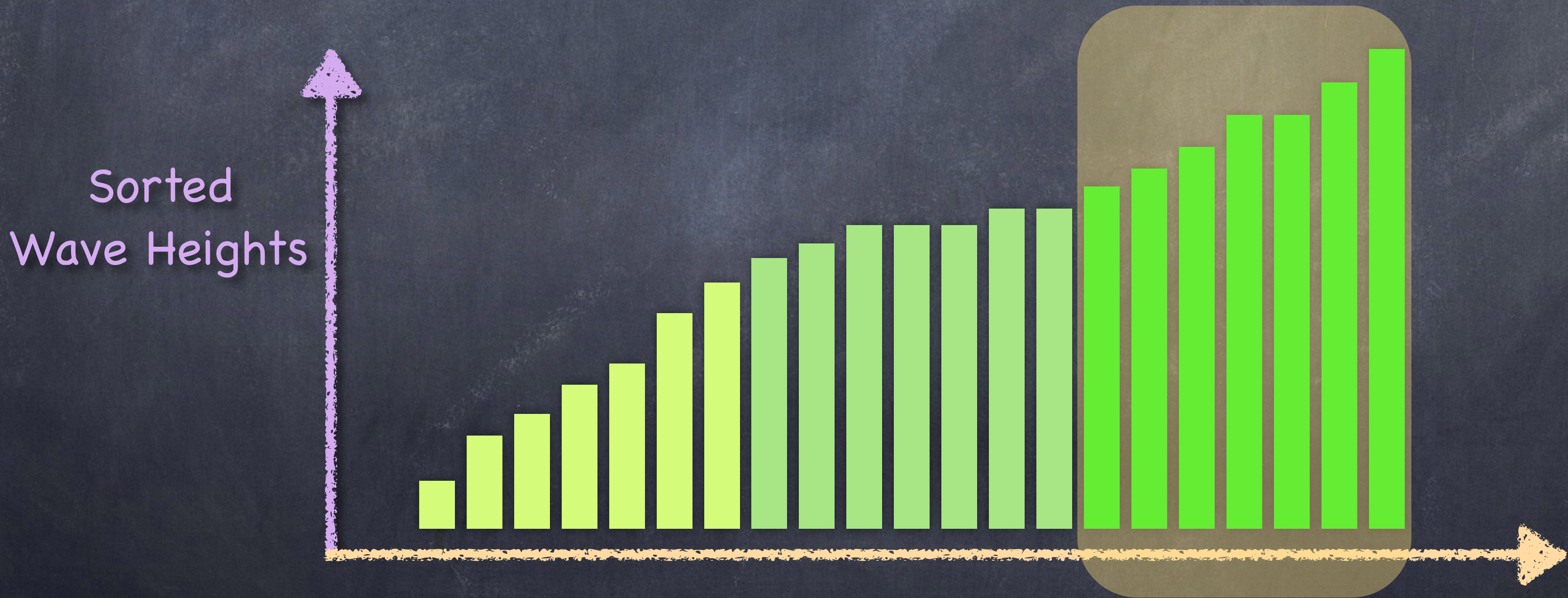


Statistical estimation of sea state parameters (1D)

- For each individual wave a wave height and a wave period is identified
  - For wave channel measurements wavelengths are measured.
- From those data different parameters can be extracted:
  - Mean wave height
  - Mean wave period $T_z = \frac{1}{N} \sum_{i=1}^{N} T_{zi}$, N: number of individual waves
  - Mean wave steepness
  - PDFs distributions of wave heights or/and periods
  - ...

Statistical estimation of sea state parameters (1D)

- One important parameter is the so-called significant wave height
- This parameter is the mean of the one-third of the highest waves in the record
    - Each individual wave height within the record is sorted from the lowest to the highest



$$H_s \equiv H_{1/3}$$

Statistical estimation of sea state parameters (1D)

- For a Gaussian sea state the mean of the one-third of the highest waves is related to the standard deviation of the wave elevation as

$$H_{1/3} \approx 4\sigma$$

- Or, as a function of the variance of the wave elevation,

$$H_{1/3} \approx 4\sqrt{\sigma^2}$$

Sea state parameters derived from the wave spectra

- Spectral moments: $\omega = 2\pi f$

$$m_j = \int f^j S(f) df \qquad (j = \ldots,\ -1,\ 0,\ 1,\ 2,\ \ldots)$$

- Significant wave height $H_s = 4\sqrt{m_0}$; $m_0 = \sigma^2$

- Mean period estimations:

$$T_e = \frac{m_{-1}}{m_0}$$

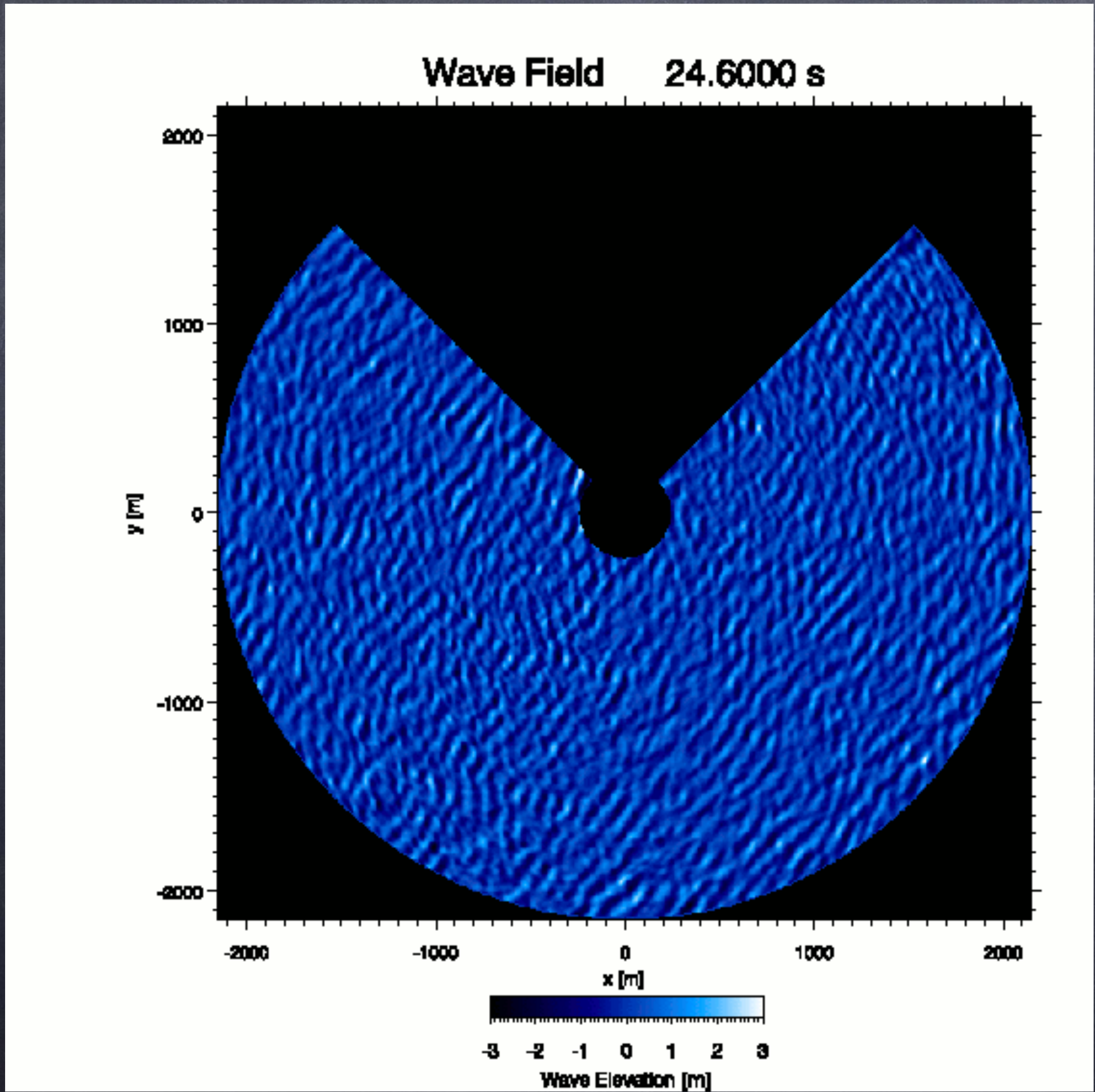
← Used for designs of wave energy extraction systems

$$T_{m01} = \frac{m_0}{m_1}$$

$$T_{m02} = \sqrt{\frac{m_0}{m_2}}$$

← Used as spectral estimation of the zero-up crossing mean period

Estimation of sea state parameters (2D and 3D)



Wave elevation field retrieved from a temporal sequence of marine radar images

$$\eta(x, y, t)$$

Estimation of sea state parameters (2D and 3D)

- There is not a clear extension of the zero-up crossing method for 2D and 3D sea surface data sets.
- In fact, the definition of wave height is not yet well understood for these cases.
  - Discussion about the necessity of taking into account proper 2D and 3D wave descriptions:
    - Paul C. Liu et al., "From single point gauge to spatio-temporal measurement of ocean waves - prospects and perspectives". OMAE 2014.
- However, it is easy to estimate sea state parameters from the different spectral density functions.

Spectral sea state parameters (2D and 3D)

- Mean wave propagation direction depending on the frequency

$$\bar{\theta}(\omega) = \tan^{-1}\left[\frac{\int_0^{2\pi} D(\omega,\theta)\sin\theta d\theta}{\int_0^{2\pi} D(\omega,\theta)\cos\theta d\theta}\right]$$

- Mean wave direction over all frequencies

$$MDIR = \tan^{-1}\left[\frac{\int_\omega S(\omega)\sin\bar{\theta}(\omega)d\omega}{\int_\omega S(\omega)\cos\bar{\theta}(\omega)d\omega}\right]$$

Spectral sea state parameters (2D and 3D)

- Mean wave lengths and wave crest
  - They are computed from the wave number spectrum $F_+^{(2)}(k_x, k_y)$
  - A way is to use the covariance matrix

$$\Lambda = \begin{pmatrix} \overline{k_x^2} & \overline{k_x k_y} \\ \overline{k_x k_y} & \overline{k_y^2} \end{pmatrix}$$

$$\overline{k_i k_j} = \frac{\int_{\Omega_{\mathbf{k}}} k_i k_j \, F_+^{(2)}(k_1, k_2) \, dk_1 dk_2}{m_0}$$

$$i, j = 1, 2$$

$$k_1 \equiv k_x$$

$$k_2 \equiv k_y$$

Spectral sea state parameters (2D and 3D)

- Mean wave lengths and wave crest
  - The two eigenvalues of the covariance matrix are computed

$$(\xi_1^2, \xi_2^2)$$

$$\Lambda = \begin{pmatrix} \overline{k_x^2} & \overline{k_x k_y} \\ \overline{k_x k_y} & \overline{k_y^2} \end{pmatrix} \xrightarrow{\text{Matrix diagonalization}} \Lambda_d = \begin{pmatrix} \xi_1^2 & 0 \\ 0 & \xi_2^2 \end{pmatrix}$$

Spectral sea state parameters (2D and 3D)

- Mean wave lengths and wave crest
  - These values are derived from the two eigenvalues (ξ_1^2, ξ_2^2)
  - Mean wavelength $\bar{\lambda} = \dfrac{2\pi}{\sqrt{\max(\xi_1^2, \xi_2^2)}}$
  - Mean wave crest $\bar{\lambda}_c = \dfrac{2\pi}{\sqrt{\min(\xi_1^2, \xi_2^2)}}$
  - Mean wave size $WS = \bar{\lambda} \cdot \bar{\lambda}_c$

Sea state parameters derived from the wave spectra

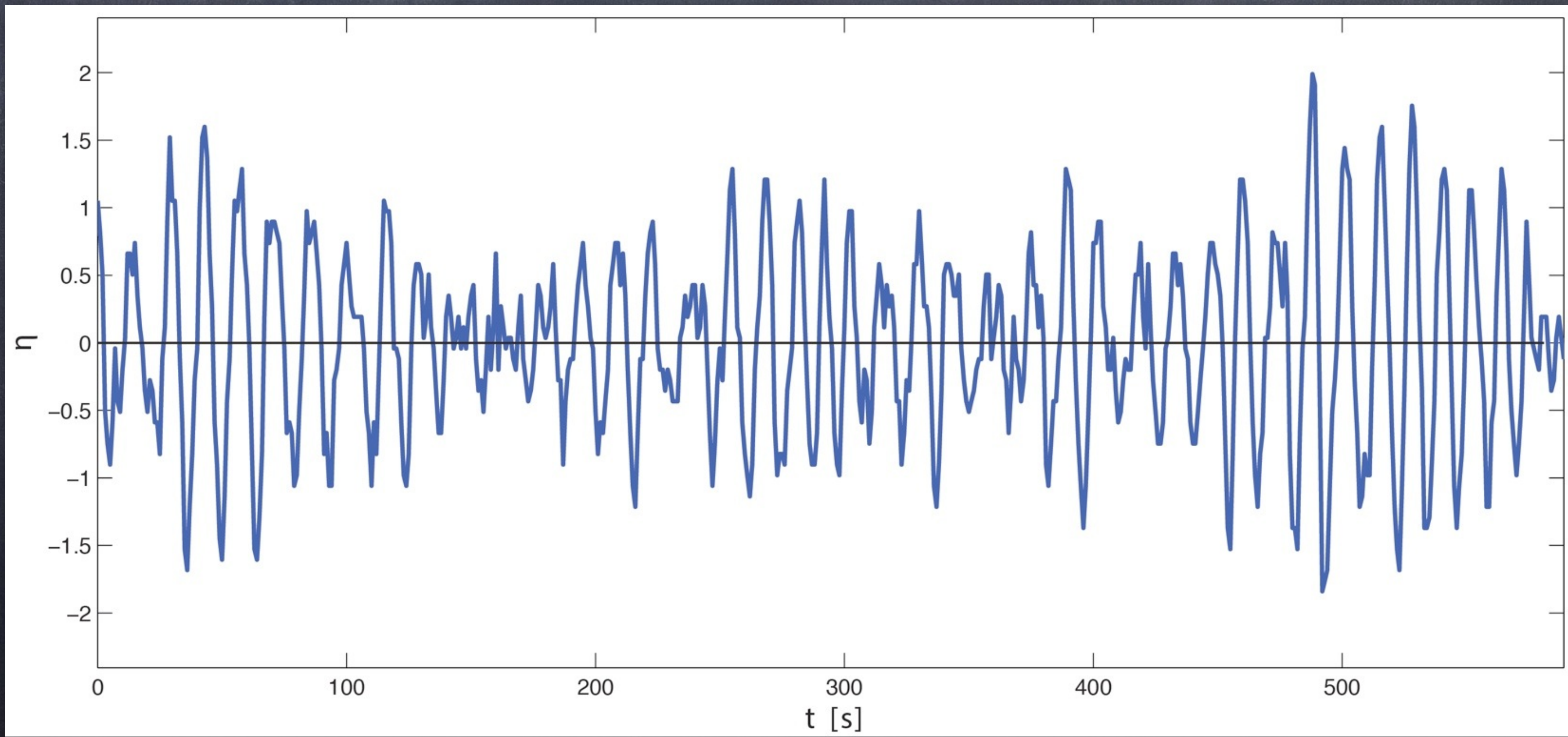
- Other parameters can be derived as
  - Peak period,
  - Significant steepness,
  - Stability parameters,
  - Wave grouping parameters,
  - etc.
- From the de directional spectra (wave number, etc.) additional parameters are:
  - Mean, peak wave length,
  - Mean, peak wave propagation direction,
  - etc.

Wave grouping

- Waves propagate in the ocean in form of wave packages of consecutive high waves
- These packages of waves are known as wave grouping or wave groups
- Wave groups are dangerous for marine systems (e.g. ships, platforms, etc.) not only because there are high waves within each group, but those waves present similar periods and can cause problems of resonance in the marine system.
- Although the existence of a power spectrum predicts wave groups, experiences in laboratories, as well as measurements indicate that this phenomenon is still not well understood.
  - Wave groups are responsible of the wave energy propagation.
  - Affected by wave breaking effects during the wave propagation.

Wave grouping

- Traditionally wave groups have been analyzed in the temporal domain (1D).
- New results have been developed to analyze wave groups in 2D and 3D.
- Example of wave grouping measured by a buoy:



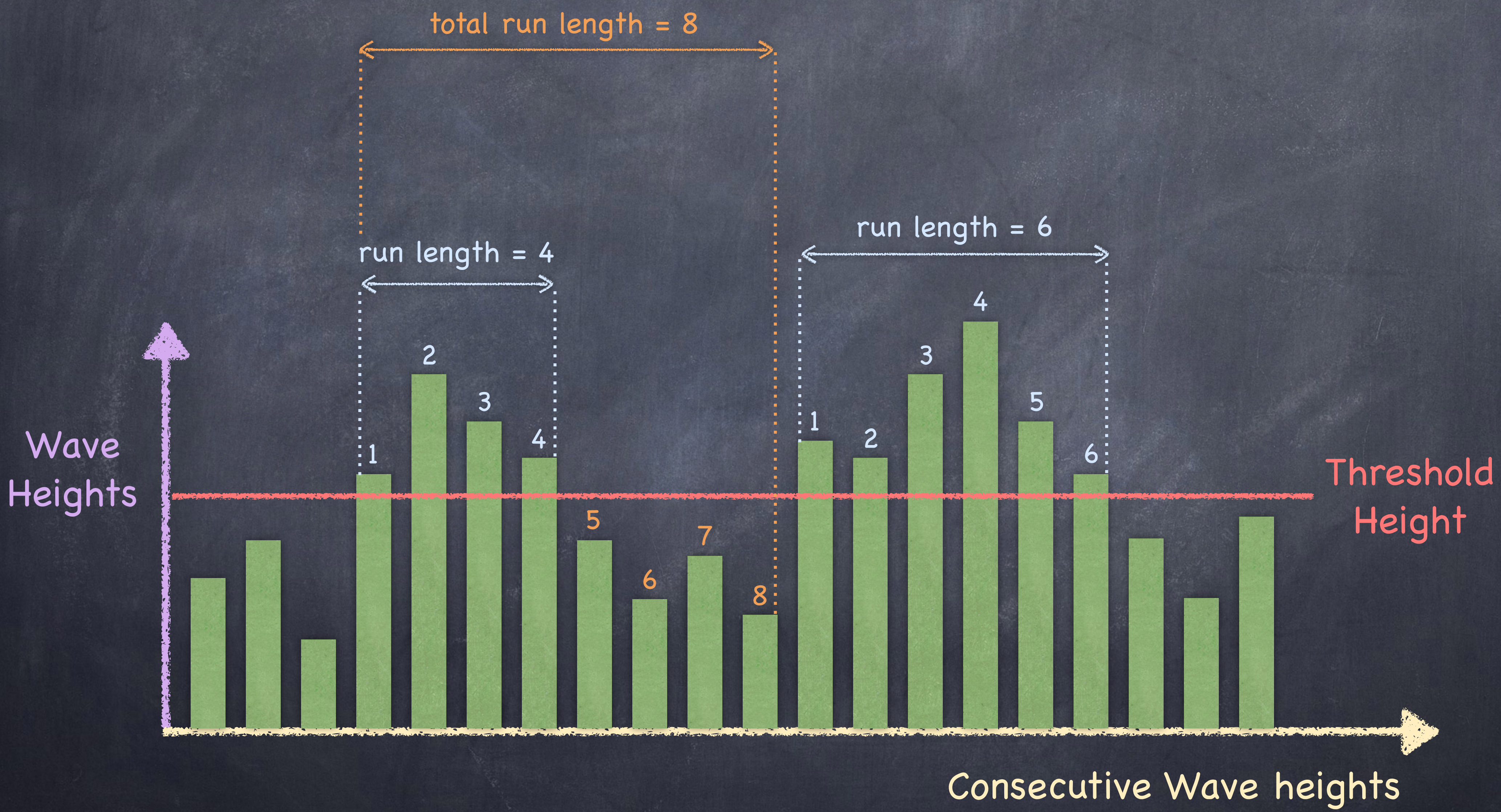
Analysis of wave groups in time

- There are different approaches to analyze the number of waves within a group
  - Correlation between consecutive wave heights
    - Statistical behavior of the groups
    - Mean number of waves within a group
  - Study of the envelope properties
    - Statistical behavior of the groups
    - Hydrodynamic information of the wave grouping
      - Wave energy propagation

Correlation between consecutive wave heights

- This method (Kimura, 1980; Longuet-Higgins, 1984) considers that wave heights evolve in time as a memory stochastic process.
  - It derives the statistical properties thought a Markov chain approach.
- Concept of run: number of consecutive waves that are higher than a given threshold height.
  - Typical threshold heights:
    - Significant wave height
    - Mean wave height
    - Median wave height
  - Parameters used for this approach
    - Run length: number of waves within a run.
    - Total run length: number of waves between two wave groups.

Definition of runs



Correlation between consecutive wave heights

- Parameters derived from this approach
  - Mean run length.
  - Mean total run length.
  - Probability of number of waves during two wave groups
  - Correlation between two wave heights

$$\gamma = \frac{\mathcal{E}(\kappa) - (1-\kappa^2)\mathcal{K}(\kappa)/2 - \pi/4}{1-\pi/4}$$

$\mathcal{E}(\kappa)$: complete elliptic integral of 1st kind

$\mathcal{K}(\kappa)$: complete elliptic integral of 2nd kind

κ : Kimura parameter

Correlation between consecutive wave heights

- The Kimura parameter is the important parameter in this wave grouping description
- This parameter can be derived from the spectrum using the Markov chain approach (Rice, 1944; Battjes and Vledder, 1984; Longuet-Higgins, 1984)

$$\kappa = \left| \frac{1}{m_0} \int S(\omega) e^{i\omega\tau} d\omega \right|$$

τ is a characteristic time of the wave field, e.g. T_{m01}

Analysis of wave groups from the envelope

- Definition of the envelope
  - The envelope as a stochastic process
- Numerical computation of the envelope
  - Hilbert transform

Definition of the envelope (I)

- Gaussian wave elevation at a fixed position: $\eta(t) = \sum_n a_n \cos(\omega_n t + \varphi_n)$
- Consider a characteristic frequency where most of the energy is concentrated: $\overline{\omega}$
  - Examples:
    - Peak frequency: $\overline{\omega} \equiv \omega_p$
    - Mean frequency: $\overline{\omega} \equiv 2\pi / T_{m01}$
- the wave elevation field is factorized as: $\eta(t) = \eta_c(t)\cos(\overline{\omega} t) - \eta_s(t)\sin(\overline{\omega} t)$

$$\eta_c(t) \equiv \sum_n a_n \cos\left[(\omega_n - \overline{\omega})t + \varphi_n\right]$$

$$\eta_s(t) \equiv \sum_n a_n \sin\left[(\omega_n - \overline{\omega})t + \varphi_n\right]$$

← Low frequency oscillations $\omega_n - \overline{\omega}$

Definition of the envelope (II)

$$\eta_c(t) \equiv \sum_n a_n \cos\left[(\omega_n - \overline{\omega})t + \varphi_n\right] \qquad \eta_s(t) \equiv \sum_n a_n \sin\left[(\omega_n - \overline{\omega})t + \varphi_n\right]$$

$$A(t) \equiv \sqrt{\eta_c^2(t) + \eta_s^2(t)} \qquad \phi(t) \equiv \tan^{-1}\left[\frac{\eta_s(t)}{\eta_c(t)}\right]$$

$$\eta_c(t) = A(t)\cos\phi(t) \qquad \eta_s(t) = A(t)\sin\phi(t)$$

$$\eta(t) = A(t)\cos\left[\overline{\omega}t + \phi(t)\right]$$

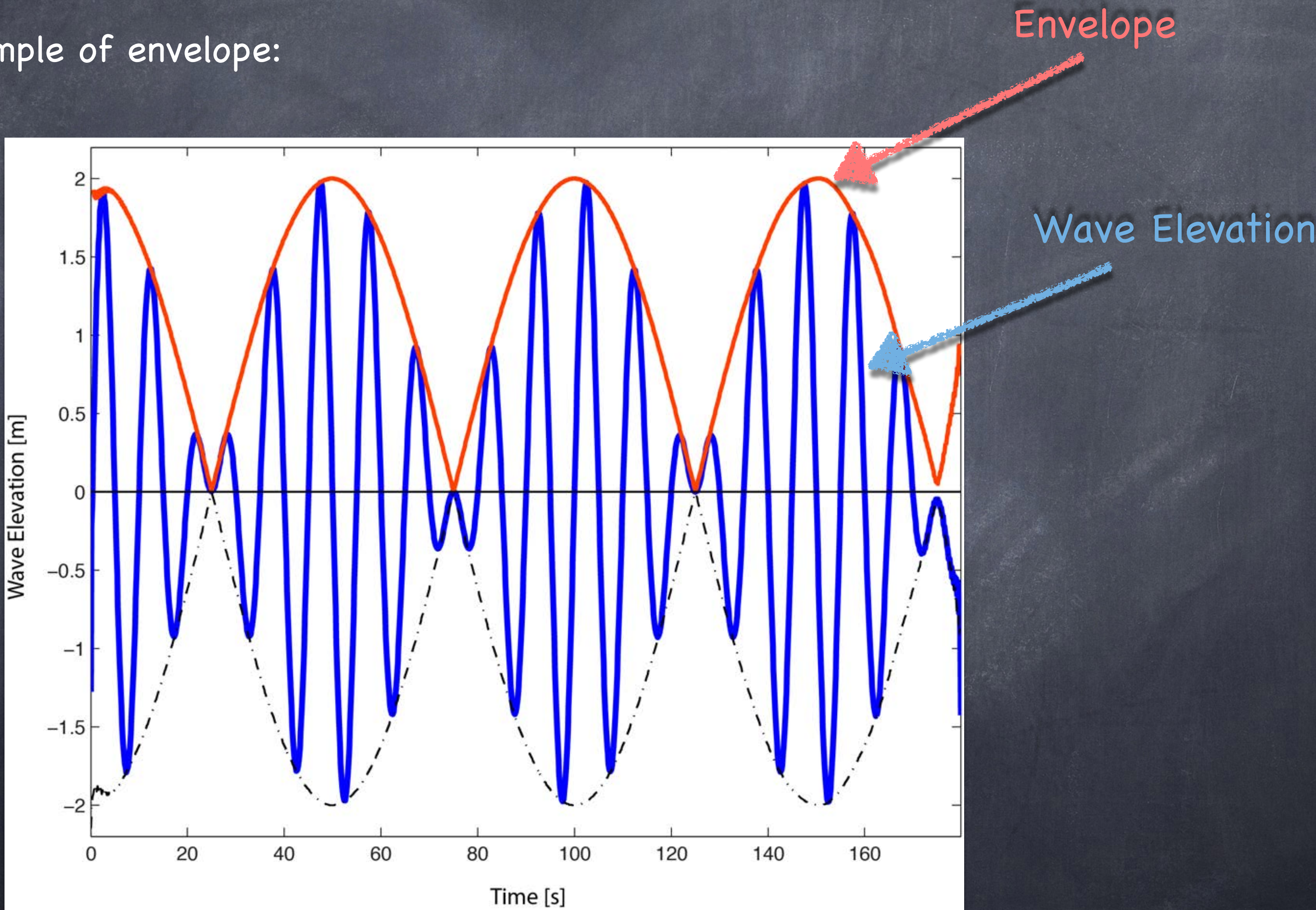
Definition of the envelope (III)

$$\eta(t) = A(t)\cos\left[\overline{\omega}t + \phi(t)\right] = A(t)\cos\Phi(t)$$

- Instantaneous amplitude or envelope: $A(t)$
- Instantaneous phase: $\Phi(t) \equiv \overline{\omega}t + \phi(t)$
- If the wave elevation field is a stochastic process, the envelope is a stochastic process too.

Definition of the envelope (IV)

- Example of envelope:



Envelope as a stochastic process (I)

$\eta(t)$ Gaussian zero-mean process $\longrightarrow \left\{ \begin{array}{c} \eta_c(t) \\ \\ \eta_s(t) \end{array} \right\}$ Gaussian zero-mean processes

$$\mathsf{E}[\eta] = 0 \qquad \mathsf{Var}[\eta] = \mathsf{E}[\eta^2] = \sigma^2$$

$$\mathsf{E}[\eta_c] = 0 \qquad \mathsf{Var}[\eta_c] = \mathsf{E}[\eta_c^2] = \sigma^2$$

$$\mathsf{E}[\eta_s] = 0 \qquad \mathsf{Var}[\eta_s] = \mathsf{E}[\eta_s^2] = \sigma^2$$

$\mathsf{E}[\eta_c\,\eta_s] = 0 \longrightarrow \left\{ \begin{array}{c} \eta_c \\ \\ \eta_s \end{array} \right\}$ Independent Gaussian processes

Envelope as a stochastic process (II)

- Joint probability density function of two independent Gaussian processes with same variance:

$$p(\eta_c,\, \eta_s) = \frac{1}{2\pi\sigma^2} \exp\left[-\frac{\eta_c^2 + \eta_s^2}{2\sigma^2}\right]$$

- Change to polar coordinates: $(\eta_c,\, \eta_s) \mapsto (A,\, \phi)$ ⟶ $d\eta_c d\eta_s \mapsto A dA d\phi$

$$p(A, \phi) = \frac{A}{2\pi\sigma^2} \exp\left[-\frac{A^2}{2\sigma^2}\right]$$

- Integrating over all the angles ϕ

$$p(A) = \frac{A}{\sigma^2} \exp\left[-\frac{A^2}{2\sigma^2}\right]$$

Rayleigh distribution

Hilbert Transform (I)

$$\zeta(t) = \frac{1}{\pi}\mathcal{P}\int_{\mathbb{R}} \frac{\eta(\tau)}{t-\tau}d\tau = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\left[\int_{-\infty}^{t-\varepsilon}\frac{\eta(\tau)}{t-\tau}d\tau + \int_{t+\varepsilon}^{\infty}\frac{\eta(\tau)}{t-\tau}d\tau\right]$$

$\mathcal{P}$: Cauchy's principal value

From the Hilbert transform the so-called analytic signal is defined:

$$\xi(t) = \eta(t) + i\,\zeta(t) = A(t)e^{i\Phi(t)}$$

$$A(t) = \sqrt{\eta^2(t) + \zeta^2(t)} \qquad \Phi(t) = \tan^{-1}\left[\frac{\zeta(t)}{\eta(t)}\right]$$

$$\eta(t) = \mathsf{Re}[\xi(t)] = A(t)\cos\Phi(t)$$

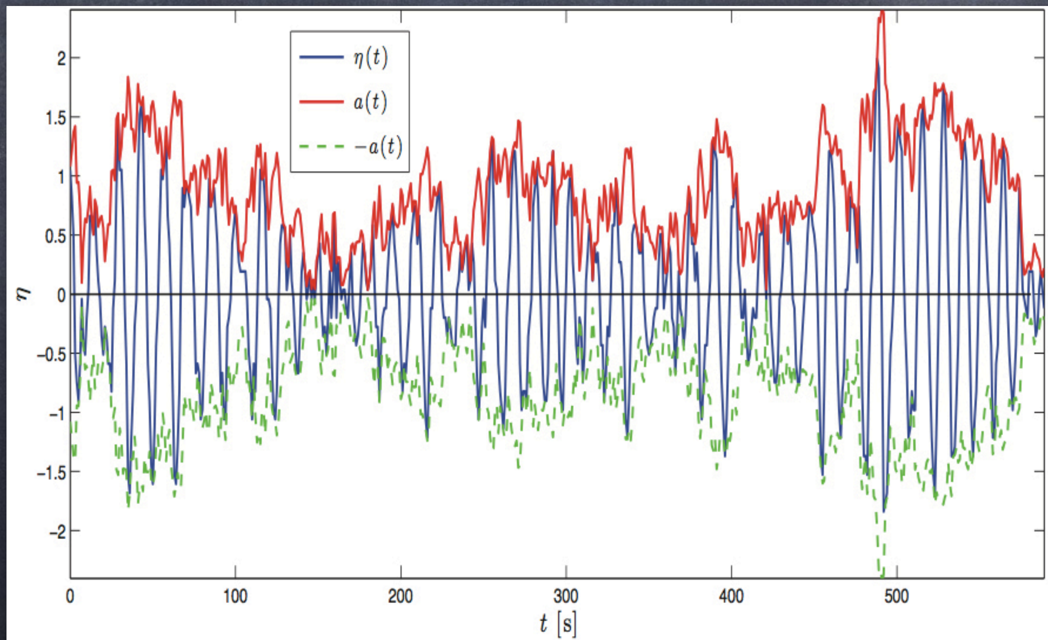
Hilbert Transform (II)

For practical applications the Hilbert transform is computed using the Fourier transform

$\eta(t)$ —HT→ $\zeta(t)$

FT ↓ FT^{-1} ↑

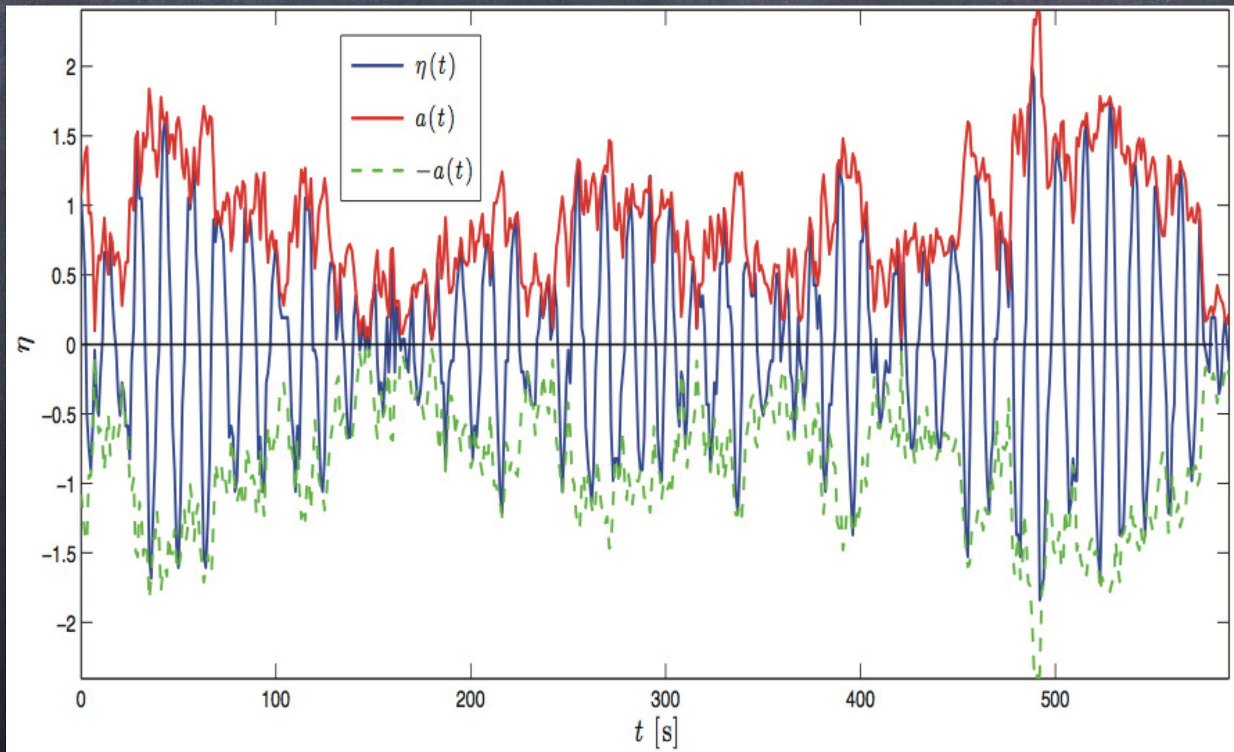
$H(\omega)$ —Transfer Function→ $Z(\omega) = -i\,\mathrm{sgn}(\omega) H(\omega)$



Hilbert Transform (III)

- The Hilbert transform is the analytical solution of the Lagrangian horizontal displacement of a wave under the frame of the linear theory (Krogstad and Trulsen, 2010).

- The envelope can be understood as the modulus of the wave elevation and the horizontal wave displacement during a wave cycle.



Analysis of wave groups from the envelope (I)

- From the estimated envelope different studies of wave grouping can be achieved
  - Correlation time of the envelope
  - Mean persistence time of the envelope over a given threshold.
    - Mean number of waves within groups.
  - Spectral density of the envelope
    - It is related to the spectral density of the wave field.
- The envelope is directly related to the propagation of wave energy features:
  - Wave energy per unit of area
  - Energy flux

Analysis of wave groups from the envelope

- In general two envelopes are defined:
  - Upper envelope: $A^+(t)$
  - Lower envelope: $A^-(t)$
  - For linear narrow-banded wave fields: $A^+(t) = -A^-(t) \equiv A(t)$
    - Under these conditions the wave height is regarded as twice the envelope

$$H \sim 2A$$

  - For other cases:
    - Non linear waves: crests higher above mean sea level than troughs deep below

$$A^+(t) \neq -A^-(t)$$

    - Linear but non narrow banded waves: neighboring crests and troughs are not statistically symmetrical around the mean sea level

$$A^+(t) \not\approx -A^-(t + T/2)$$

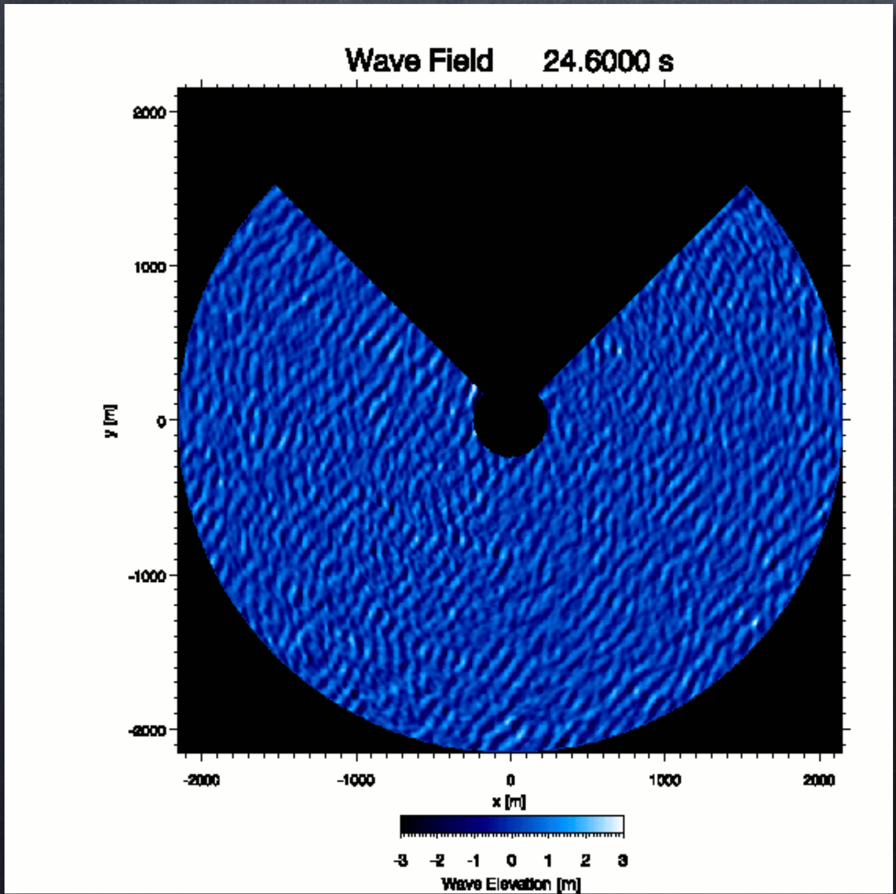
Analysis of wave groups in more dimensions (I)

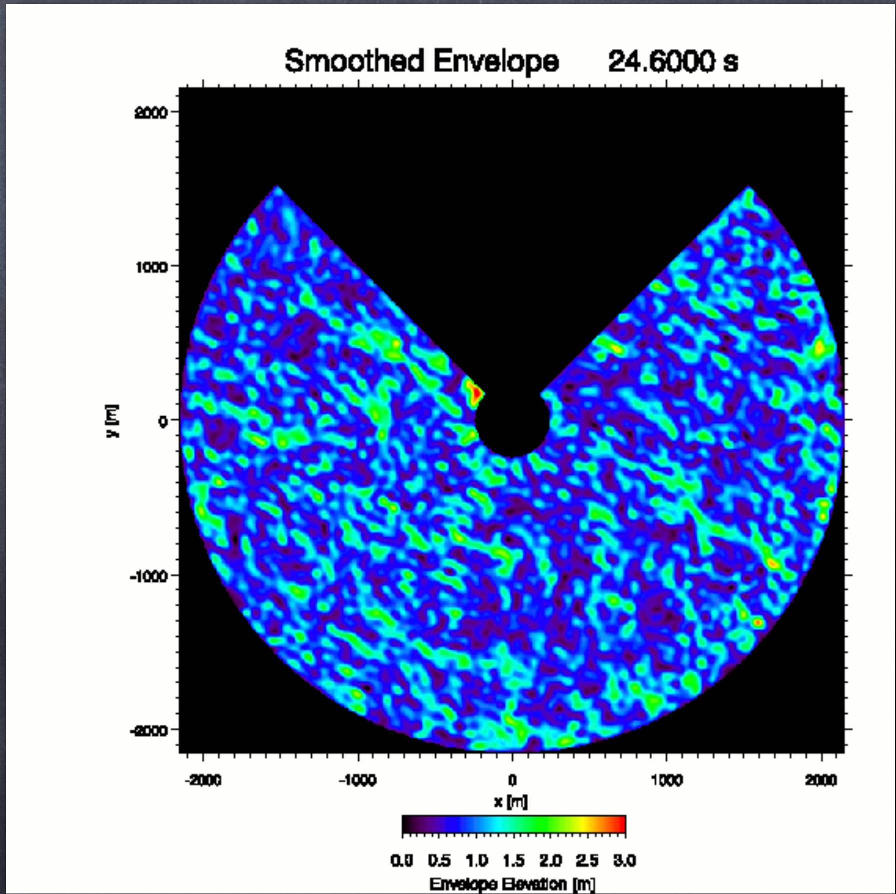
- There is not a unique generalization of the Hilbert Transform for higher dimensions.
  - Any multidimensional generalization corresponds to the Hilbert transform in 1D
- For linear ocean waves the proper generalization is the Riesz Transform.
  - In the same way than the Hilbert transform for 1D, the Riesz transform corresponds to the Lagrangian horizontal wave displacements under the frame of the linear wave theory (Nieto Borge et al., 2013).
- The spatio-temporal evolution of groups, energy, etc. can be achieved.

Analysis of wave groups in more dimensions (II)

- Example of spatio-temporal envelope derived from the wave elevation estimation by using temporal sequences of X-band radar images (Nieto Borge et al., 2013).

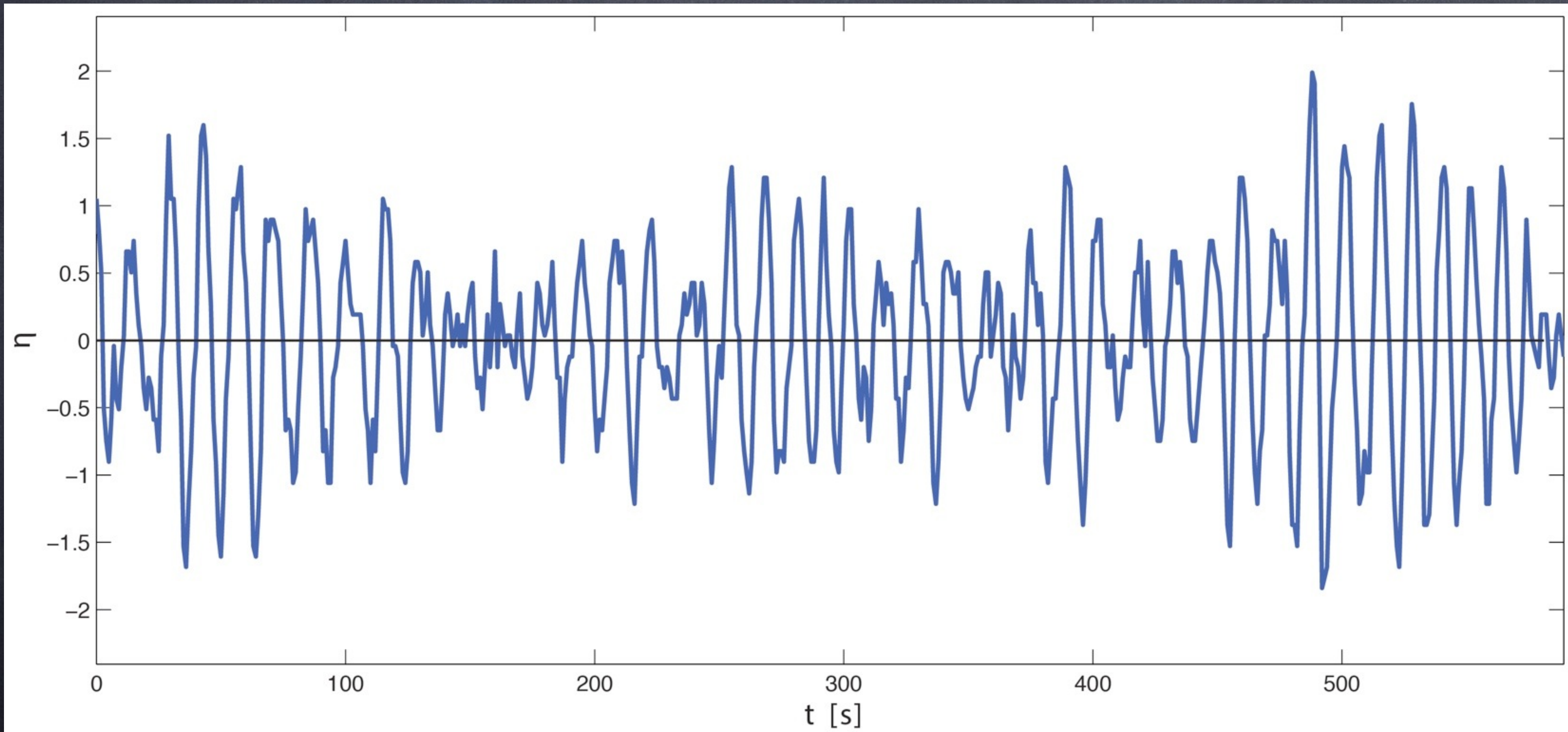
$\eta(x, y, t)$ $A(x, y, t)$



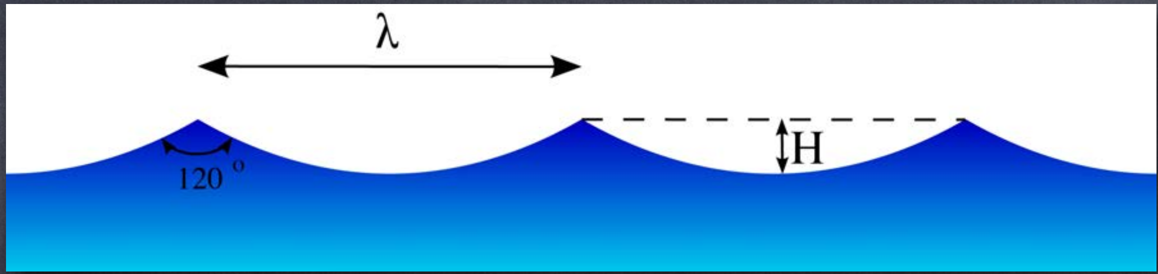


Examples of Gaussian waves

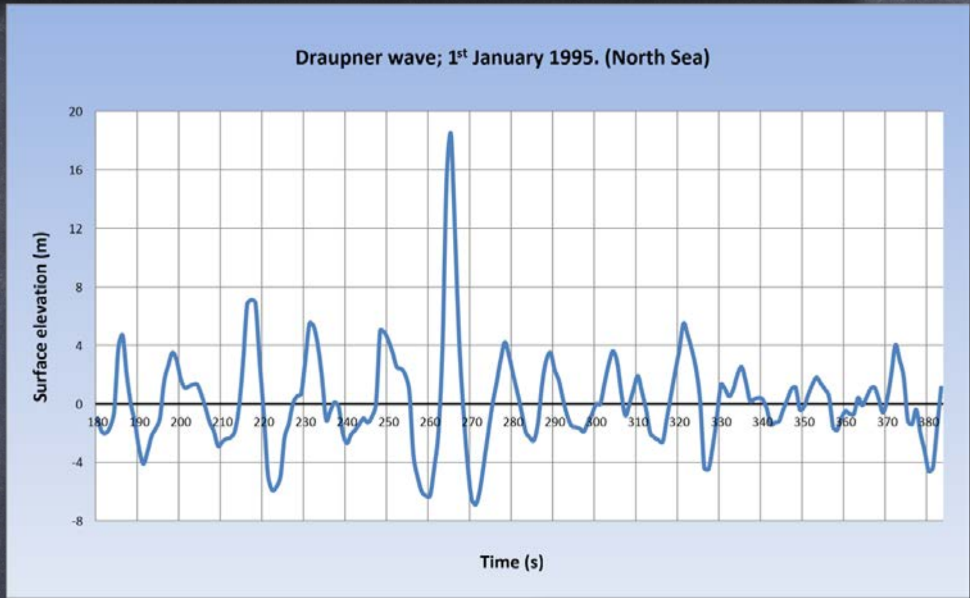
- Example of wave record measured by a buoy deployed in the Northern coast of Spain (Bay of Biscay)



Examples of non Gaussian waves



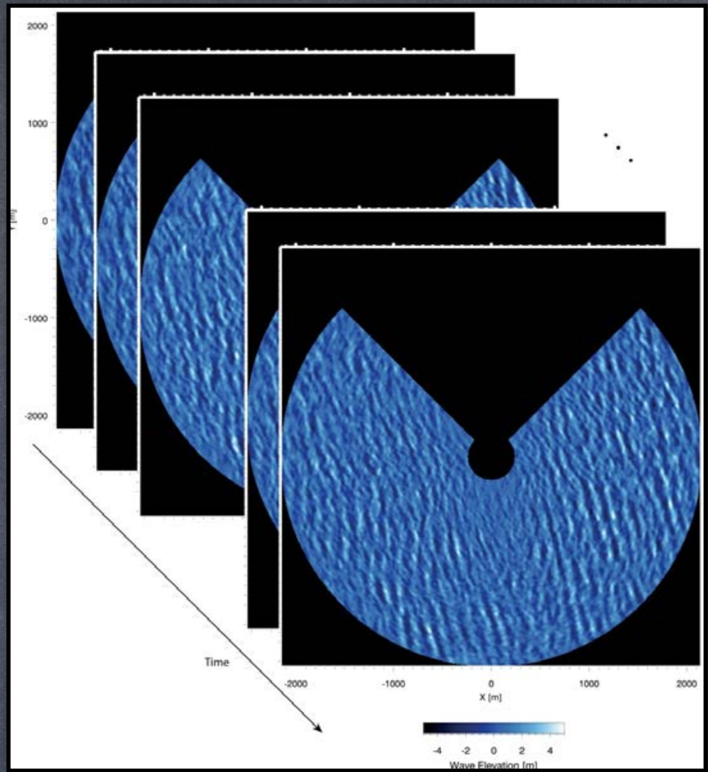
Stokes waves



Freak wave (New Year Wave)

Breaking wave

DFT Estimation of the Three-Dimensional Spectrum



Spectral Analysis Techniques Applied to Image Time Series

Sampling of an Image Time Series

- Consider time series of images sampled in space and time

$$\xi_{mnl} = \xi(x_m, y_n, t_l)$$

$$x_m = m \cdot \Delta x \quad ; \quad m = 0, \ldots, N_x - 1$$

$$y_n = n \cdot \Delta y \quad ; \quad n = 0, \ldots, N_y - 1$$

$$t_l = l \cdot \Delta t \quad ; \quad l = 0, \ldots, N_t - 1$$

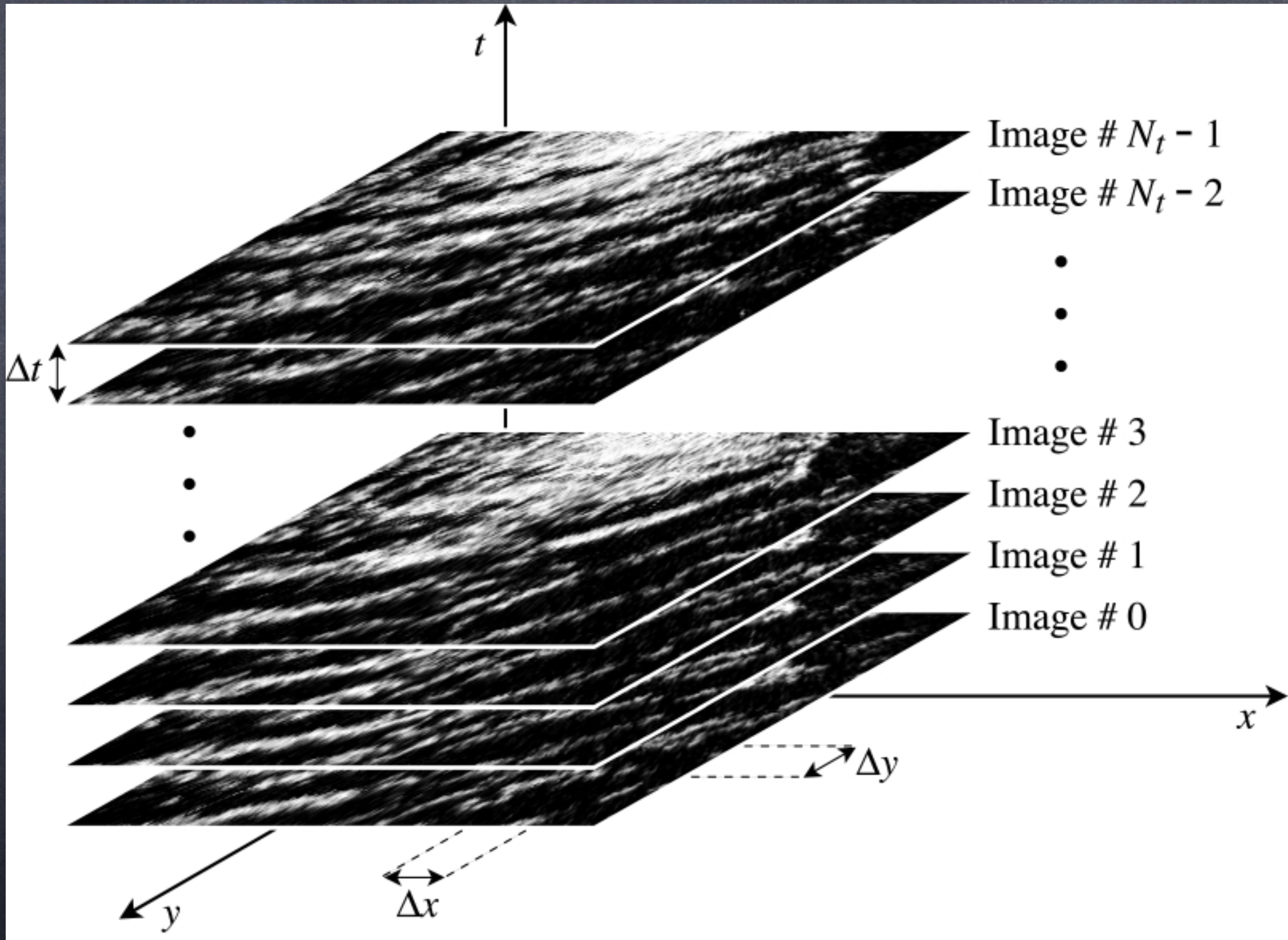
Sampling time: Δt

Spatial resolution along X-axis: Δx

Spatial resolution along Y-axis: Δy

Sampling of an Image Time Series

- Example: temporal sequence of X-band marine radar images



3D Discrete Fourier Transform (3D-DFT)

- The three-dimensional Fourier coefficients of $\xi_{mnl} = \xi(x_m, y_n, t_l)$

$$\Xi_{m'n'l'} = \frac{1}{N_x N_y N_t} \sum_{m=0}^{N_x-1} \sum_{n=0}^{N_y-1} \sum_{l=0}^{N_t-1} \xi_{mnl} \, e^{-i2\pi(mm'+nn'+ll')/(N_x N_y N_t)}$$

$$m' = 0, \ldots, N_x - 1$$

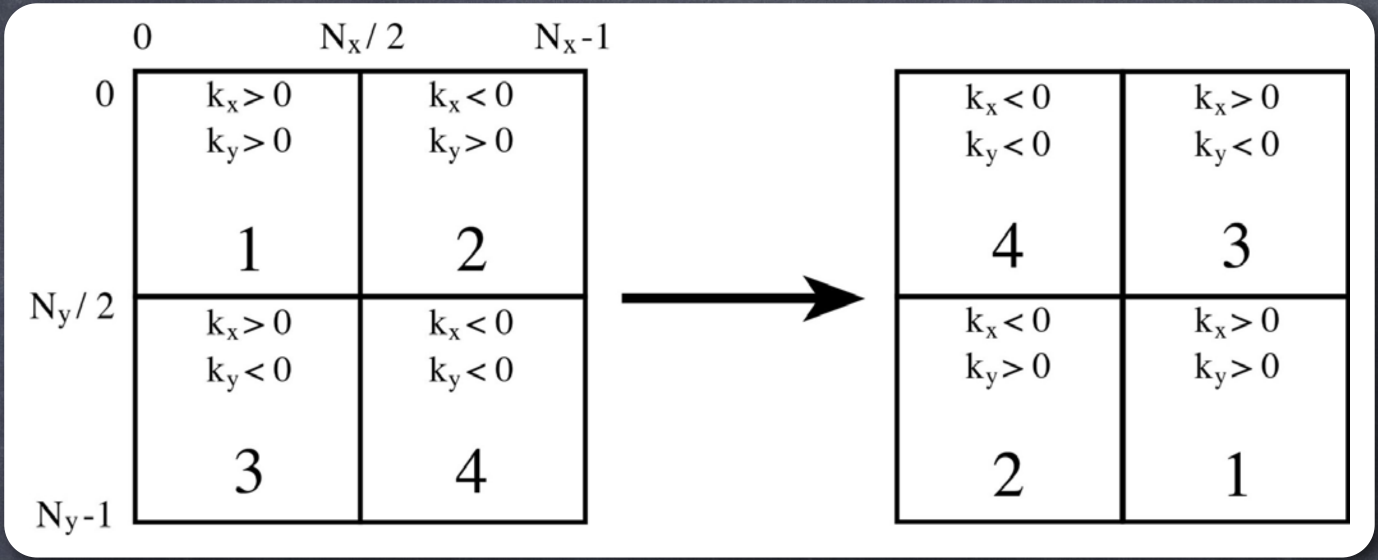
$$\Xi_{m'n'l'} = \Xi(k_{x_{m'}}, k_{y_{n'}}, \omega_{l'}) \qquad n' = 0, \ldots, N_y - 1$$

$$l' = 0, \ldots, N_t - 1$$

- In practice, the DFT is computed using the Fast Fourier Transform (FFT) algorithm.
- The output of a FFT function has to be reordered to have the negative branch of each spectral variable before the positive branch.

3D Discrete Fourier Transform (3D-DFT)

- Two-dimensional example of data reordering (half-period shift)



- Most of the data analysis softwares include a specific function for that:
- Examples:
  - IDL: **SHIFT** function.
  - Matlab: **fftshift** function.

3D Discrete Fourier Transform (3D-DFT)

- Scheme of the application of the 3D FFT:

$$\xi(x_m, y_n, t_l)$$

3D FFT ↓

$$\Xi(k_{x_{m'}}, k_{y_{n'}}, \omega_{l'})$$

half-period shift ↓

$$\Xi(k_{x_{m'}}, k_{y_{n'}}, \omega_{l'})$$

3D Discrete Fourier Transform (3D-DFT) Spectral Variables

- Once the half-period shift is carried out, the sampled spectral variables are

$$k_{x_{m'}} = -k_{x_c} + m' \cdot \Delta k_x \qquad ; \qquad m' = 0, \ldots, N_x - 1$$

$$k_{y_{n'}} = -k_{y_c} + n' \cdot \Delta k_y \qquad ; \qquad n' = 0, \ldots, N_y - 1$$

$$\omega_{l'} = -\omega_c + l' \cdot \Delta \omega \qquad ; \qquad l' = 0, \ldots, N_t - 1$$

$$k_{x_c} = \frac{\pi}{\Delta x} \qquad \Delta k_x = \frac{2\pi}{N_x \Delta x}$$

$$k_{y_c} = \frac{\pi}{\Delta y} \qquad \Delta k_y = \frac{2\pi}{N_y \Delta y}$$

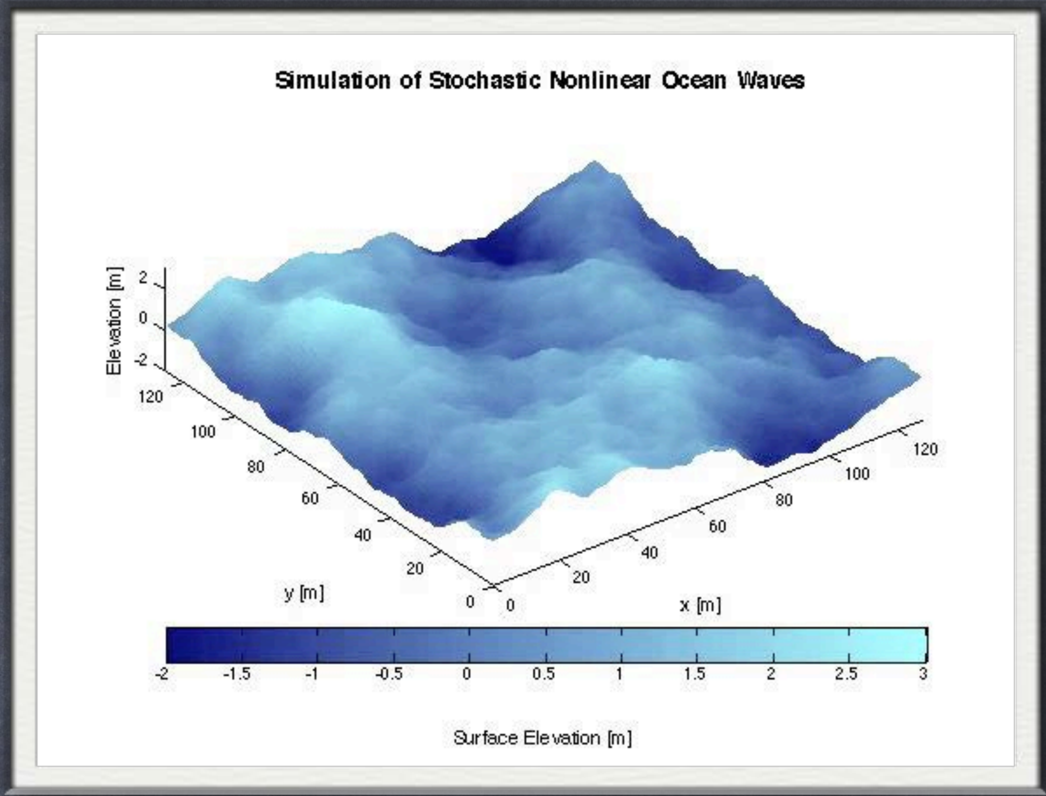
$$\omega_c = \frac{\pi}{\Delta t} \qquad \Delta \omega = \frac{2\pi}{N_t \Delta t}$$

3D Discrete Fourier Transform (3D-DFT) Spectral Estimation

- The spectral estimation using the DFT is called **periodogram**.

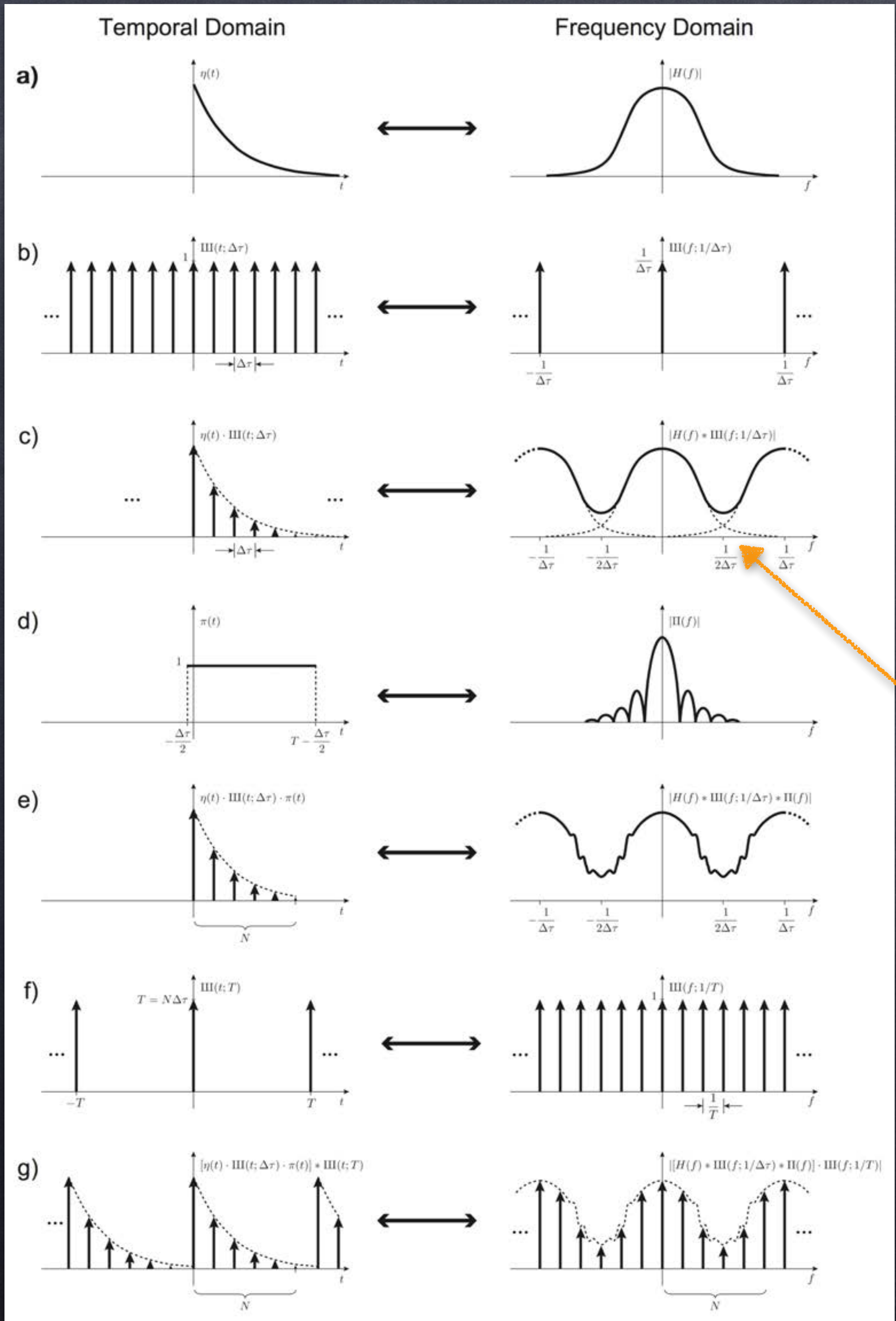
$$F^{(3)}(m', n', l') = \frac{1}{\Delta k_x \Delta k_y \Delta \omega} \left| \Xi(m', n', l') \right|^2$$

$$F^{(3)}(m', n', l') \equiv F^{(3)}(k_{x_{m'}}, k_{y_{n'}}, \omega_{l'})$$



3D Discrete Fourier Transform (DFT) for wave analysis

1D DFT: Overview of the needed mathematical operations



a) Original (continuous) functions

b-c) Sampling in time

d-c) Time truncation

f-g) Sampling in frequency

Aliasing

3D Discrete Fourier Transform (DFT) for wave analysis

- The DFT permits to estimate the complex Fourier coefficients

$$\eta(x, y, t) \longrightarrow \boxed{\text{3D DFT}} \longrightarrow c(k_x, k_y, \omega)$$

- Or the amplitude and phase of the spectral components:

$$a(k_x, k_y, \omega) \qquad \varphi(k_x, k_y, \omega)$$

- The computation of the DFT from its definition needs extremely large CPU time
  - Fast Fourier Transform (FFT) algorithms permit to estimate the DFT in short CPU time

Examples

- Three different examples to illustrate ow the FFT works in 1D, 2D and 3D:
  - 1D (t): Time series of wave elevations measured by a buoy at a fixed location.
  - 2D (x, y): Sea surface elevations to estimate the 2D wave number spectra:
    - Three cases: wind sea, swell, bimodal sea state.
  - 3D (x, y, t): Time series of sea surface elevations to estimate:
    - 3D wave number-frequency spectrum
    - 2D wave number spectrum
    - Three cases: wind sea, swell, bimodal sea state.

Example 1: Wave elevation time series

- Read the data file WaveTimeSeries.dat
- Estimate the variance of the time series.
- Estimate the spectral density using the function fft.
- Plot the spectral density for each frequency.
  - What do you see in the plot?
- Compute the variance from the spectral estimation.
- Proposed exercise:
  - Compute the autocorrelation function from the spectral density.

Example 2: 2D Sea surface elevations (I)

- Read the data files for each case (Matlab data format):
  - Wind sea, files: WindSea2D_1.dat, WindSea2D_2.dat and WindSea2D_3.dat
  - Swell, files: Swell2D_1.dat, Swell2D_2.dat and Swell2D_3.dat
  - Bimodal sea state, files: Bimodal2D_1.dat, Bimodal2D_2.dat and Bimodal2D_3.dat
- Estimate the variance of the surfaces
- Plot the spectra
  - Compare the different obtained the spectra for each case.
    - Are they equal? Why?
  - Do they present even-symmetry on the wave numbers? Why?
- Estimate the variance from the spectral estimation.

Example 2: 2D Sea surface elevations (II)

- Way to proceed:
  - Load the data in the Matlab workspace
  - Compute the 2D FFT by using the Matlab functions fft2 or fftn.
  - shift a half-period in the wave number-frequency domain using the function fftshift
  - compute the 32D spectral estimation (wave number space)

Example 3: spatio-temporal evolution of a wave field (I)

- Purpose: analyze a spatio-temporal wave field evolution by using a 3D FFT.
- Data:
  - Matlab data files:
    - Wind sea: WindSea.mat
    - Swell: Swell.mat
    - Bimodal sea state: Bimodal.mat

Example 3: spatio-temporal evolution of a wave field (II)

- Way to proceed (I):
  - Load the data in the Matlab workspace
  - Estimate the variance from the wave elevation field
  - Compute the 3D spectral estimation (wave number and frequency space)
    - Compute the 3D FFT using the function fftn
    - Shift a half-period in the wave number-frequency domain using the function fftshift
    - Estimate the 3D spectrum.

Example 3: spatio-temporal evolution of a wave field (II)

- Way to proceed (II):
  - Identify the dispersion relation by applying different transects on the wave number-frequency domain.
  - Estimate the variance of the wave field from the 3D spectrum
  - compute the 2D wave number spectrum, by integrating over all the positive frequencies.
    - Is this wave number spectrum symmetric? Why?
  - Estimate the variance of the wave field from the 2D spectrum

Thanks