MEK4350, fall 2016
Oblig I — To be handed in

Exercise 1 — Interpolation and differentiation with Fourier transform

We shall consider the function $\sin x$ and its derivative $\cos x$ on the interval $0 \leq x < 2\pi$.

Define the vectors `a = [0 1 0 -1]` and `x = 2*pi*(0:3)/4`. We see that the vector `a` consists of four samples of the function $\sin x$ for $x = x_j = \frac{2\pi j}{4}$ for $j = 0, 1, 2, 3$.

We shall make an interpolation with 100 points by means of truncated Fourier series, based on the four sample points, but without knowing that the “answer” should be a sine. We start with the four points as input to a DFT, the DFT shall be reinterpreted as having 100 points, and then the inverse DFT should give us the desired interpolation. Thus we want to insert 24 new points after each of the four already given in the vector `a`. The interpolated vector with 100 elements will be called `aa`. We also need a new vector `xx` with 100 elements uniformly distributed on the interval $0 \leq x < 2\pi$.

Show how we by means of one `fft` and one `ifft` can do this interpolation.

The following commands will show if the interpolation is correct:

```
plot (xx,aa)
hold on
plot (x,a,'*')
hold off
```

Check that the stars fall exactly on the smooth curve that represents one period of $\sin x$.

Then we shall compute the derivative with respect to x as follows: Compute the Fourier transform `aft`, multiply `aft` according to problem 3 in Exercises IV to obtain the Fourier transform `daft`, and compute the inverse transform to get the derivative `da`.

Now we shall do exactly the same with the interpolation `aa`. Find the derivative `daa` by means of the Fourier transform `daaft`.

The following commands will show if the differentiations are correct:

```
plot (xx,daa)
hold on
plot (x,da,'o')
hold off
```

Check that the circles fall exactly on top of the curve of $\cos x$.

Exercise 2 — The "New Year Wave"

Read the article of Sverre Haver (2004): "Freak wave event at Draupner jacket, January 1 1995." — downloadable from the semester web page.

If you want to know about recent ongoing research on this event, you may want to read Jean–Raymond Bidlot et al. (2016): "What conditions led to the Draupner freak wave?" *ECMWF Newsletter* **148**, 37–40 — downloadable from the semester web page.

For a nice review of rogue waves research, please read pages 287–291 of the paper Dysthe, Krogstad & Müller (2008): "Oceanic rogue waves." *Annual Review of Fluid Mechanics* **40**, 287–310 — downloadable from the semester web page provided you are on campus.

Then download the measured time series.[1] The time series has constant time intervals over a total length of 20 minutes. What has been measured is the vertical distance between an instrument mounted hight up in the air and the sea surface.

Reproduce the graphs in figures 4 and 5 in the article of Sverre Haver. Use seconds as unit for time along the first axis, meters as unit for surface elevation along the second axis, and make sure the zero level along the second axis is the mean water level.

Exercise 3 — Interpolation of the New Year Wave with Fourier series

In the previous exercise the curves are likely produced by linear interpolation between the discrete sampling points. In that case the largest amplitude plotted is the largest sample measured at the corresponding sample time. Could it have happened that the crest and the troughs around the New Year Wave in reality happened between the discrete measurement times?

In order to answer this question we can try an interpolation by means of Fourier series. First compute the DFT (by means of `fft` or `ifft` on the computer) of the time series

$$\tilde{\eta}_n = \frac{1}{N} \sum_{j=0}^{N-1} \eta_j e^{\mathrm{i}\omega_n t_j}$$

where $\eta_j = \eta(t_j)$ are the discrete samples at the times $t_j = Tj/N$ for $j = 0, 1, 2, \ldots, N-1$. Here $T = 20$ minutes and N is the number of measurements. The angular frequencies are $\omega_n = 2\pi n/T$.

We have chosen positive (+) sign in the exponent since we do the transform with respect to time, opposite of the sign choice we have used for spatial transform. This is respecting the "standard" wave representation $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$.

Carry out an interpolation with a number of points $M \gg N$.

Focus the plot on the extreme wave, plot the measurements by stars and the interpolation with a continuous curve. Find out if such interpolation suggests that the crest was higher, and the neighboring troughs were lower, than the measurements.

[1]We should be grateful to Statoil for making this unique data set available for academic institutions, the release of the data resulted in an avalanche of research on freak waves.

Exercise 4 — How smooth is the sea surface?

The smoothness of the sea surface can be studied by looking at how fast the Fourier coefficients go to zero, i.e. estimating the power α in the asymptotic formula $|\tilde{\eta}_n| \propto |\omega_n|^{-\alpha}$ for large ω_n.

a) Plot $|\tilde{\eta}_n|$ against ω_n in doubly logarithmic coordinate system. Plot some straight lines with various slopes. Can you identify what is a characteristic slope (i.e. the power of ω) for how fast the Fourier coefficients go to zero?

b) Often we do such a consideration for the wave spectrum, which is related to $|\tilde{\eta}_n|^2$ instead of $|\tilde{\eta}_n|$. Explain how the slope of $|\tilde{\eta}_n|^2$ is related to the slope of $|\tilde{\eta}_n|$.

Hint 1: There is no reason why the power α of ω should be an integer.

Hint 2: We can limit to half of the Fourier transform because η is real.

Hint 3: The graph of the Fourier transform is characterized by an interesting domain near the origin that has negative slope, and an uninteresting domain further away from the origin which is "flat" due to limited measurement accuracy in the time series. We are only interested in the domain with negative slope near the origin.