MEK4350, fall 2018
Exercises II

The Dirac delta-function $\delta(x)$ is a "generalized" function with the property

$$\int_{-\infty}^{\infty} f(x)\delta(x)\,\mathrm{d}x = f(0)$$

where $f(x)$ is a test-function. No ordinary function can have this property. We describe the generalized function by its action on the test-function under the integral.

We require that the test-functions $f(x)$ should be infinitely smooth (can be differentiated infinitely many times and all of their derivatives are also smooth) and vanish outside som finite interval. In particular the test functions are zero as $x \to \pm\infty$.

The Heaviside step function $H(x)$ is given by

$$H(x) = \begin{cases} 0 & \text{for} \quad x < 0 \\ 1 & \text{for} \quad x > 0 \\ \text{either 0 or } \frac{1}{2} \text{ or 1} & \text{for} \quad x = 0 \end{cases}$$

For continuous x it usually does not matter which finite value we select for $H(0)$.

Exercise 1

Let the function $h(x)$ be given by

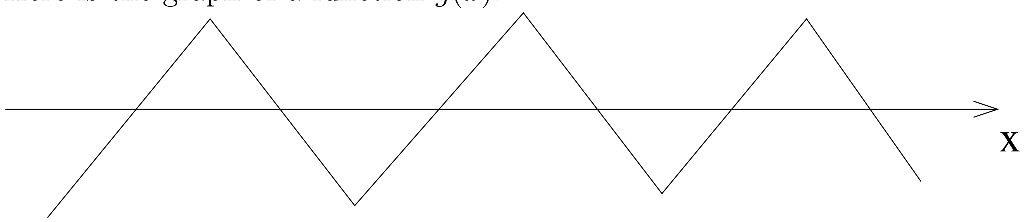
$$h(x) = \begin{cases} h_1 & \text{for} \quad x < a \\ h_2 & \text{for} \quad x \geq a \end{cases}$$

for arbitrary constants h_1, h_2 and a. Compute the derivative $h'(x)$.

Hint: Compute $h'(x)$ for $x \neq a$, and compute $\int_{-\infty}^{\infty} f(x)h'(x)\,\mathrm{d}x$ for a test-function $f(x)$. Use integration by parts.

Exercise 2

Here is the graph of a function $g(x)$:



Sketch the graphs of the first derivative $g'(x)$ and the second derivative $g''(x)$.

Exercise 3

Compute $\delta''(x)$.

Hint: Integration by parts twice.

These two exercises are nice to review, but not necessary to do:

Exercise 4 — The sinc function $\operatorname{sinc} x = \frac{\sin x}{x}$

The sinc function is also known as the cardinal sine function. Our definition is the unnormalized sinc function adopted by Mathematica. The alternative definition, the normalized sinc function, $\operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}$ is adopted by Python and Matlab and Octave and by DLMF. Please consult with Wikipedia which recognizes both definitions.

Interestingly, the Spherical Bessel function of the first kind is $j_0(x) = \frac{\sin x}{x}$, see DLMF and MathWorld.

a) Show using l'Hôpital's rule that $\operatorname{sinc} 0 = 1$.

b) Show that $\int_{-\infty}^{\infty} \operatorname{sinc} x \, \mathrm{d}x = \pi$.

This can be done in several ways, one way is as follows:

1. Observe that the integrand is even, integrate only from 0 to ∞.
2. Rewrite as a double integral using $\int_0^\infty e^{-xt} \, \mathrm{d}t = \frac{1}{x}$.
3. Reverse the order of integration.
4. Perform two integrations by parts.
5. Recognize the integral that defines $\arctan \infty = \pi/2$.

Exercise 5 — Justification that $\int_{-\infty}^{\infty} e^{ikx} \, \mathrm{d}x = 2\pi\delta(k)$

Define $I(k,a) = \int_{-a}^{a} e^{ikx} \, \mathrm{d}x$. We shall investigate the action of $I(k,a)$ on a test-function $f(k)$ under the integral $\int_{-\infty}^{\infty} f(k) I(k,a) \, \mathrm{d}k$ for arbitrarily large a.

a) Show that $I(k,a) = 2a \operatorname{sinc}(ka)$. Plot $I(k,a)$ for small a, moderate a and large a.

b) According to the method of stationary phase (taught in MEK4320) we argue that the extremely fast oscillations of $I(k,a)$ for large a will produce cancellations under the integral everywhere except near $k = 0$ such that we can limit the integration domain to a small neighborhood of the origin $-\epsilon < k < \epsilon$, thus for large a

$$
\begin{aligned}
\int_{-\infty}^{\infty} f(k) \int_{-\infty}^{\infty} e^{ikx} \, \mathrm{d}x \, \mathrm{d}k &\approx \int_{-\infty}^{\infty} f(k) I(k,a) \, \mathrm{d}k \\
&\approx \int_{-\epsilon}^{\epsilon} f(k) I(k,a) \, \mathrm{d}k \\
&\approx f(0) \int_{-\epsilon}^{\epsilon} I(k,a) \, \mathrm{d}k \\
&\approx f(0) \int_{-\infty}^{\infty} I(k,a) \, \mathrm{d}k \; = \; 2\pi f(0)
\end{aligned}
$$

In the limit $a \to \infty$ this result becomes exact.

Note: This exercise shows that the essential behavior $\int_{-\infty}^{\infty} f(x)\delta(x) \, \mathrm{d}x = f(0)$ does not depend on an additional assumption that $\delta(x) = 0$ for $x \neq 0$. Indeed in this exercise such an additional assumption is not satisfied!