MEK4350
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2 Nonlinear theory

2.1 Free and forced oscillations

This is a review of several techniques of applied mathematics that are likely to be taught in various other courses. The main point here is to illustrate how nonlinearity can act in two different ways.

2.1.1 Free oscillations

Consider the linear harmonic oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = 0 \tag{1}$$

where t is time and ω is the angular frequency. This equation is homogeneous, or unforced. It can be solved by assuming $x = \mathrm{e}^{\lambda t}$, giving the characteristic polynomial

$$\lambda^2 + \omega^2 = 0, \tag{2}$$

which has two distinct roots $\lambda = \mathrm{i}\omega$ and $\lambda = -\mathrm{i}\omega$ both with multiplicity one. If we want a real solution we can write

$$x = a\cos\omega t + b\sin\omega t = A\mathrm{e}^{\mathrm{i}\omega t} + A^*\mathrm{e}^{-\mathrm{i}\omega t} = A\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} \tag{3}$$

where a and b are real, A is complex, a raised * signifies complex conjugation, and c.c. denotes the complex conjugate of the previous terms. This oscillation will be denoted "free" since it does not depend on being forced.

2.1.2 Forced oscillations

Consider the forced linear harmonic oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \omega^2 x = f\cos\mu t \tag{4}$$

where the forcing is given on the right-hand side as an oscillation with amplitude f and angular frequency μ. The equation is inhomogeneous or forced due to the expression on the right-hand side. The solution can be written

$$x = x_h + x_p \tag{5}$$

where x_h is the solution of the homogeneous problem previously found in (3) and x_p is a particular (or inhomogeneous or forced) solution proportional to the forcing amplitude f. We may anticipate $x_p = c\cos\mu t$, which after substitution into (4) yields

$$c = \frac{f}{\omega^2 - \mu^2}. \tag{6}$$

This solution breaks down when the forcing frequency μ coincides with the natural frequency ω of the linear oscillator, in which case we have resonance.

In order to find a solution for the resonant case $\mu = \omega$ we may resort to the method of undetermined coefficients. We then need to identify an operator, called an annihilator, which has the property that the right-hand side of (4) becomes zero. The desired operator is

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \mu^2. \tag{7}$$

Applying this operator to both sides of (4) we get

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \mu^2\right)\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} - \omega^2\right) x = 0. \tag{8}$$

For the resonant case $\mu = \omega$ we get the characteristic polynomial

$$(\lambda^2 - \omega^2)^2 = 0 \tag{9}$$

which has two distinct roots, each with multiplicity two, $\lambda \in \{\mathrm{i}\omega, \mathrm{i}\omega, -\mathrm{i}\omega, -\mathrm{i}\omega\}$. The general real solution can now be written as

$$\begin{aligned} x &= a\cos\omega t + b\sin\omega t + ct\cos\omega t + dt\sin\omega t \\ &= A\mathrm{e}^{\mathrm{i}\omega t} + Ct\mathrm{e}^{\mathrm{i}\omega t} + \mathrm{c.c.} \end{aligned} \tag{10}$$

To complete the method of undetermined coefficients, this solution must be substituted into the original equation (4) which yields

$$c = 0, \qquad d = \frac{f}{2\omega}, \qquad C = -\frac{\mathrm{i}f}{4\omega} \tag{11}$$

while a, b and A are free. The resonant solution is seen to grow linearly in time, without bounds, see figure 1.

2.1.3 Nonlinearly forced oscillations

Consider the nonlinear oscillator

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = \epsilon\alpha x^3 \tag{12}$$

where $0 < \epsilon \ll 1$ is a small (non-dimensional) ordering parameter and α is a physical constant of order 1. If the solution x is bounded, then we may expect that the nonlinear term is small compared to the terms on the left-hand side, so to a leading order we have the same linear harmonic oscillator as we studied above. At a higher order the linear harmonic oscillator is forced by its own oscillation at a previous order. The analysis is conveniently done by means of a regular perturbation expansion.

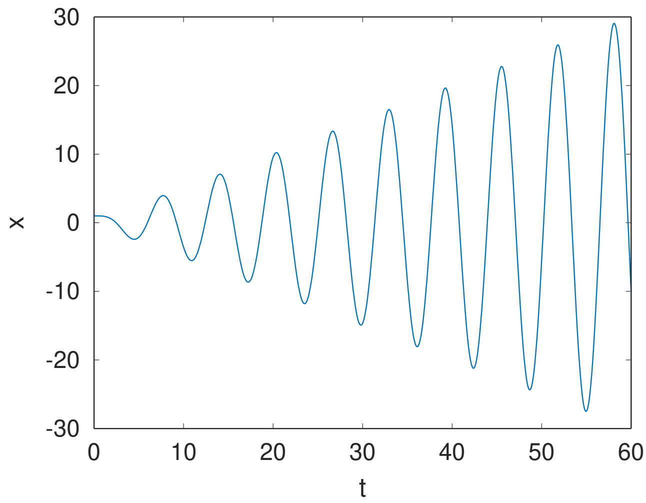

Figure 1: Resonant solution with unbounded growth, $\omega = \mu = a = f = 1$ and $b = 0$.

2.1.4 Regular perturbation expansion

Let us assume the regular perturbation expansion

$$x = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \dots \tag{13}$$

When this is substituted into equation (12) we get

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + \epsilon \frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \dots + \omega^2 x_1 + \epsilon\omega^2 x_2 + \dots = \epsilon\alpha x_1^3 + \dots \tag{14}$$

From here we extract equations of the first two orders

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + \omega^2 x_1 = 0 \tag{15}$$

and

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \omega^2 x_2 = \alpha x_1^3 \tag{16}$$

The solution of (15) is known to be $x_1 = A\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.}$, with complex amplitude A. Substituting this into (16) we get

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \omega^2 x_2 = \alpha A^3 \mathrm{e}^{3\mathrm{i}\omega t} + 3\alpha |A|^2 A \mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.} \tag{17}$$

It is convenient to express the solution of this equation as the superposition

$$x_2 = x_{2,h} + x_{2,1} + x_{2,3} \tag{18}$$

where $x_{2,h} = B\mathrm{e}^{\mathrm{i}\omega t} + \text{c.c.}$ is the homogeneous solution, $x_{2,1}$ is a particular solution in response to the forcing terms proportional to $\mathrm{e}^{\pm\mathrm{i}\omega t}$, and $x_{2,3}$ is a particular solution in response to the forcing terms proportional to $\mathrm{e}^{\pm 3\mathrm{i}\omega t}$.

We know from equations (10) and (11) that the solution for $x_{2,1}$ is unbounded as time increases. In fact, we will show that equation (12) has unbounded solutions.

However, if it had been our goal to study an unbounded solution we should not have employed the small parameter ϵ to characterize the smallness of the solution as it appears in the perturbation expansion (13). On the other hand, equation (12) also has solutions that remain bounded. Let us first verify that both bounded and unbounded solutions exist, and then modify the perturbation approach in order to study the bounded solutions.

2.1.5 Global solution illustrated by phase portrait

Let us express the second-order equation (12) as a system of two first-order equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y \tag{19}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \epsilon\alpha x^3 - \omega^2 x \tag{20}$$

We can interpret (x, y) to be a two-dimensional position, and $(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t})$ to be a two-dimensional velocity field. We can check that the velocity field is divergence free. Therefore we know that a stream function exists

$$\psi = \frac{\alpha\epsilon}{4}x^4 - \frac{\omega^2}{2}x^2 - \frac{1}{2}y^2 \tag{21}$$

such that $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{\partial\psi}{\partial y}$ and $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial\psi}{\partial x}$. The phase portrait, i.e. the streamlines, is given by the contours of constant ψ. A special case is shown by the solid curves in figure 2.

For positive α there are three fixed points of the system (19)–(20), they are all on the x-axis with $x = 0$ or $x^2 = \frac{\omega^2}{\alpha\epsilon}$. Through the latter two fixed points we can compute the separatrices $y = \pm\sqrt{\frac{\epsilon\alpha}{2}}\left(x^2 - \frac{\omega^2}{\epsilon\alpha}\right)$ which are two parabolas. Clearly, a solution is bounded for all time provided it is inside the "eye" delimited by the separatrices near the origin, a solution is unbounded if it is outside the separatrices.

For negative α there is only one fixed point at the origin.

2.1.6 Multiple scales perturbation expansion

Looking carefully at figure 1 we may anticipate that the amplitude of the oscillation should vary slowly in comparison with the fundamental oscillation. Therefore we now try a multiple scales approach to see if that can arrest the unbounded growth in the case that the solution should remain bounded.

Now we assume that while the rapid time for oscillation is $t_0 = t$, the relevant time for modulation is $t_1 = \epsilon t$ or even slower scales $t_n = \epsilon^n t$. The variable x will now be assumed to be a function of both t_0 and t_1 and all the slower scales. We develop the time derivative by means of the chain rule

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}t_0}{\mathrm{d}t}\frac{\partial}{\partial t_0} + \frac{\mathrm{d}t_1}{\mathrm{d}t}\frac{\partial}{\partial t_1} + \ldots = \frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1} + \ldots \tag{22}$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \left(\frac{\partial}{\partial t_0} + \epsilon\frac{\partial}{\partial t_1}\ldots\right)^2 = \frac{\partial^2}{\partial t_0^2} + 2\epsilon\frac{\partial^2}{\partial t_0 \partial t_1} + \ldots. \tag{23}$$

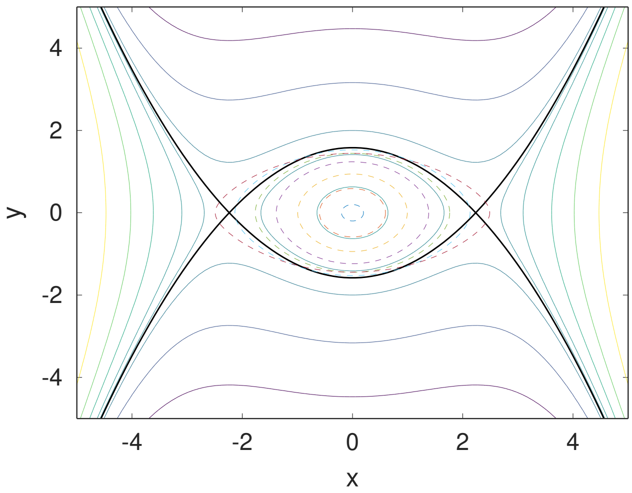

Figure 2: Solid lines: Phase portrait of the nonlinear oscillator, for $\omega = \alpha = 1$ and $\epsilon = 0.2$. Dashed lines: Approximation found by the multiple scales perturbation expansion.

Equation (12) now becomes

$$\frac{\partial^2 x}{\partial t_0^2} + 2\epsilon \frac{\partial^2 x}{\partial t_0 \partial t_1} + \ldots + \omega^2 x = \epsilon \alpha x^3 \tag{24}$$

which is conveniently solved by assuming the regular perturbation expansion

$$x = x_1 + \epsilon x_2 + \epsilon^2 x_3 + \ldots \tag{25}$$

When this is substituted into the above equation we get

$$\frac{\partial^2 x_1}{\partial t_0^2} + 2\epsilon \frac{\partial^2 x_1}{\partial t_0 \partial t_1} + \ldots + \epsilon \frac{\partial^2 x_2}{\partial t_0^2} + \ldots + \omega^2 x_1 + \epsilon \omega^2 x_2 + \ldots = \epsilon \alpha x_1^3 + \ldots \tag{26}$$

From here we extract equations for the first two orders

$$\frac{\partial^2 x_1}{\partial t_0^2} + \omega^2 x_1 = 0 \tag{27}$$

and

$$\frac{\partial^2 x_2}{\partial t_0^2} + \omega^2 x_2 = \alpha x_1^3 - 2\frac{\partial^2 x_1}{\partial t_0 \partial t_1}. \tag{28}$$

The solution of (27) is known to be $x_1 = A(t_1)\mathrm{e}^{\mathrm{i}\omega t_0} + \text{c.c.}$, where the complex amplitude $A(t_1)$ is now a function of the slow modulation time. Substituting this into (28) we get

$$\frac{\partial^2 x_2}{\partial t_0^2} + \omega^2 x_2 = \alpha A^3 \mathrm{e}^{3\mathrm{i}\omega t_0} + 3\alpha |A|^2 A \mathrm{e}^{\mathrm{i}\omega t_0} - 2\mathrm{i}\omega \frac{\partial A}{\partial t_1} \mathrm{e}^{\mathrm{i}\omega t_0} + \text{c.c.} \tag{29}$$

It is convenient to express the solution of this equation as the superposition

$$x_2 = x_{2,h} + x_{2,1} + x_{2,3} \tag{30}$$

where $x_{2,h} = B\mathrm{e}^{\mathrm{i}\omega t_0} + \text{c.c.}$ is the homogeneous solution, $x_{2,1}$ is a particular solution due to the forcing terms proportional to $\mathrm{e}^{\pm\mathrm{i}\omega t_0}$, and $x_{2,3}$ is a particular solution due to the forcing terms proportional to $\mathrm{e}^{\pm 3\mathrm{i}\omega t_0}$.

In order to avoid unbounded growth of $x_{2,1}$ we need to make sure that there is no forcing on the right-hand side proportional to $\mathrm{e}^{\mathrm{i}\omega t_0}$. This can now be achieved by requiring such terms to cancel out, giving a governing equation for A

$$\frac{\partial A}{\partial t_1} + \frac{3\alpha}{2\omega}\mathrm{i}|A|^2 A = 0 \tag{31}$$

which is readily solved as

$$A = a\mathrm{e}^{-\frac{3\alpha}{2\omega}\mathrm{i}a^2 t_1 + \mathrm{i}\theta} \tag{32}$$

where a and θ are two real constants.

We may now set both $x_{2,h}$ and $x_{2,1}$ to zero since they to not contribute anything new.

The only inhomogeneous solution that needs to be considered is

$$x_{2,3} = -\frac{\alpha}{8\omega^2}A^3\mathrm{e}^{3\mathrm{i}\omega t_0} + \text{c.c.}. \tag{33}$$

Thus we have obtained the solution accurate to second order

$$x = 2a\cos(\omega' t + \theta) - \frac{\epsilon\alpha}{4\omega^2}a^3\cos 3(\omega' t + \theta) \tag{34}$$

where $\omega' = \omega - \frac{3\epsilon\alpha}{2\omega}a^2$.

The solution in equation (34) is shown as dashed curves in figure 2. The shape of the streamlines is well captured near the origin and well inside the “eye”, but is not so well captured near the separatrices.

The second-order modification causes the oscillations to be slower for $\alpha > 0$ and faster for $\alpha < 0$. This is not captured by figure 2, but is suggested in figure 3. Orbits along the separatrices take infinite time, but this is not captured by the second-order approximation.

2.1.7 Solution in the style of WKB

With hindsight, we could have anticipated that the bounded solution had to be a superposition of oscillations proportional to $\mathrm{e}^{\pm\mathrm{i}\omega t}$ and $\mathrm{e}^{\pm 3\mathrm{i}\omega t}$, with appropriately slowly varying amplitudes. Using such hindsight, it is possible to focus on the slow modulation timescale only, $t_1 = \epsilon t$, and express the equation in terms of this slow coordinate only

$$\epsilon^2\frac{\mathrm{d}^2 x}{\mathrm{d}t_1^2} + \omega^2 x = \epsilon\alpha x^3 \tag{35}$$

and assume a solution that is a superposition of first and third harmonic oscillations of the form

$$x = \mathcal{A}_1\mathrm{e}^{\epsilon^{-1}\mathrm{i}\omega t_1} + \epsilon\mathcal{A}_3\mathrm{e}^{3\epsilon^{-1}\mathrm{i}\omega t_1} + \mathcal{O}(\epsilon^2) + \text{c.c.} \tag{36}$$

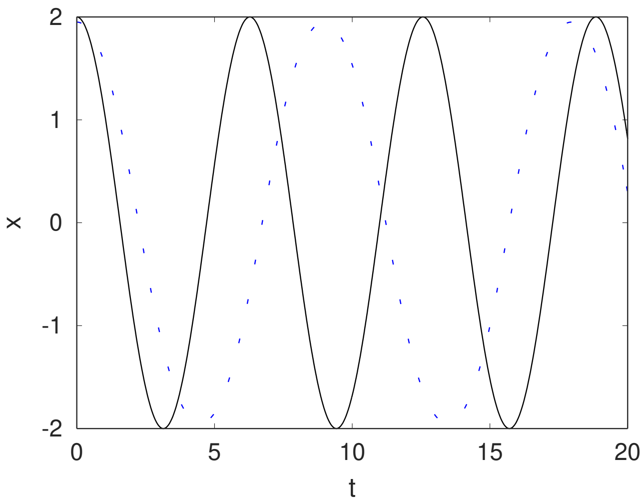

Figure 3: Leading order solution $2a\cos(\omega t+\theta)$ shown in solid black line, second order solution (34) shown in dotted blue line, for $\omega = a = \alpha = 1$, $\theta = 0$ and $\epsilon = 0.2$.

The symbol "c.c." indicates the complex conjugate of the previous terms, as usual. Substituting this into (35) yields

$$\epsilon^2\left\{(-\epsilon^{-2}\omega^2\mathcal{A}_1 + 2\epsilon^{-1}\mathrm{i}\omega\frac{\partial\mathcal{A}_1}{\partial t_1})\mathrm{e}^{\epsilon^{-1}\mathrm{i}\omega t_1} + (-9\epsilon^{-1}\omega^2\mathcal{A}_3)\mathrm{e}^{3\epsilon^{-1}\mathrm{i}\omega t_1} + \ldots + \text{c.c.}\right\}$$
$$+\,\omega^2\left\{\mathcal{A}_1\mathrm{e}^{\epsilon^{-1}\mathrm{i}\omega t_1} + \epsilon\mathcal{A}_3\mathrm{e}^{3\epsilon^{-1}\mathrm{i}\omega t_1} + \ldots + \text{c.c.}\right\}$$
$$= \epsilon\alpha\left\{3\mathcal{A}_1^3\mathrm{e}^{3\epsilon^{-1}\mathrm{i}\omega t_1} + 3|\mathcal{A}_1|^2\mathcal{A}_1\mathrm{e}^{\epsilon^{-1}\mathrm{i}\omega t_1} + \ldots + \text{c.c.}\right\} \quad (37)$$

Extracting the first and third harmonics we get

$$-\omega^2\mathcal{A}_1 + 2\epsilon\mathrm{i}\omega\frac{\partial\mathcal{A}_1}{\partial t_1} + \omega^2\mathcal{A}_1 = 3\alpha\epsilon|\mathcal{A}_1|^2\mathcal{A}_1 + \mathcal{O}(\epsilon^2) \quad (38)$$

and

$$-9\epsilon\omega^2\mathcal{A}_3 + \epsilon\omega^2\mathcal{A}_3 = \epsilon\alpha 3\mathcal{A}_1^3 + \mathcal{O}(\epsilon^2) \quad (39)$$

which are seen to yield exactly the same result as previously achieved by the multiple scales approach, however, now with much less effort.

Please note that in a typical applied mathematics course the WKB method is likely taught in a more compact and efficient way, though less intuitive for the present application.

2.1.8 Summary of nonlinear effects in this section

We have found that, for a description of bounded solutions of an essentially linear oscillator with a small nonlinear modification, it is useful to distinguish two essentially different types of nonlinear response: "static" and "dynamic". The "static" nonlinear response is due to non-resonant forcing (the symbols $x_{2,3}$ and $\mathcal{A}_3$ above). The "dynamic" nonlinear response is governed by an "evolution equation" imposed to avoid resonant forcing leading to unbounded growth (the differential equations for A and $\mathcal{A}_1$ above).

On the other hand, for a description of the solutions that actually blow up, the distinction between "static" and "dynamic" response is not so useful.

2.1.9 Exercises

1. **Nonlinear oscillator with quadratic nonlinear term.**

   Consider the nonlinear oscillator

   $$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega^2 x = \epsilon \alpha x^2$$

   where $0 < \epsilon \ll 1$ is a small (non-dimensional) ordering parameter and α is a physical constant of order 1.

   (a) In the case that we attempt a solution by a regular perturbation expansion (13), with no consideration of slow time modulation, discuss why this equation will not suffer from unbounded resonant growth in the solution to second-order accuracy, but will suffer from unbounded resonant growth in the solution to third-order accuracy.

   (b) Discuss the global solution in terms of a phase portrait, and try to find evidence that a solution that remains bounded for all time can exist.

   (c) Use either a multiple-scales or a WKB approach to solve this equation to the third order, without allowing unbounded resonant growth.

   Hint: The relevant slow scale is likely $t_2 = \epsilon^2 t$. The solution is probably

   $$x = 2a\cos(\omega' t + \theta) + \epsilon \frac{2\alpha}{\omega^2} a^2 - \epsilon \frac{2\alpha}{3\omega^2} a^2 \cos 2(\omega' t + \theta) + \frac{\epsilon^2 \alpha^2}{6\omega^4} a^3 \cos 3(\omega' t + \theta) + \ldots$$

   where $\omega' = \omega - \frac{5\epsilon^2\alpha^2}{3\omega^3} a^2$ and where a and θ are real constants. In figure 4 the solution has been compared to the leading order, the second order and the third order. The second-order modification causes the oscillations to be slightly uplifted, while the third-order modification causes the oscillations to be slower.

2. **Physical pendulum.**

   The differential equation which represents the motion of a simple pendulum is

   $$\frac{\mathrm{d}^2 \theta}{\mathrm{d}t^2} + \frac{g}{l} \sin\theta = 0$$

   where g is the acceleration of gravity, l is the length of the pendulum, and θ is the angular displacement.

   For small displacements this can be approximated by a simple-harmonic oscillator. For slightly larger displacements the solution can be approximated by the technique of multiple-scale expansion illustrated above. For arbitrary displacements it is possible to draw the phase portrait distinguishing motion back and forth or motion always rotating around in the same direction.

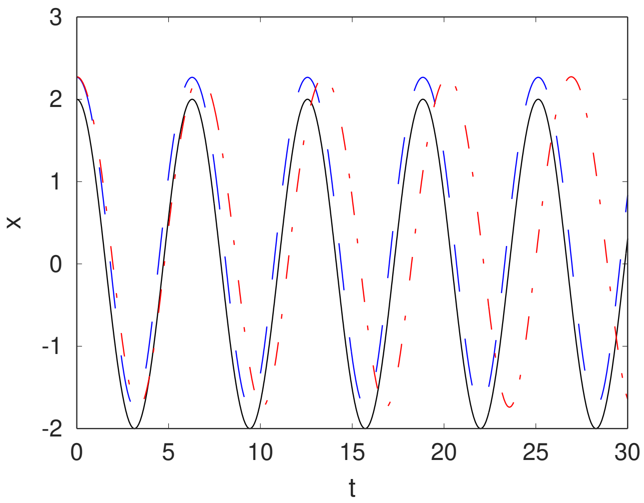

Figure 4: Solutions of problem 1. Leading order solution $2a\cos(\omega t + \theta)$ shown in solid black line, second order solution $2a\cos(\omega t + \theta) + \epsilon\frac{2\alpha}{\omega^2}a^2 - \epsilon\frac{2\alpha}{3\omega^2}a^2\cos 2(\omega t + \theta)$ shown in dashed blue line, third order solution shown in dash-dotted red line, for $\omega = a = \alpha = 1$, $\theta = 0$ and $\epsilon = 0.2$.

2.2 Nonlinear water surface waves: Governing equations, potential theory and non-resonant waves

We consider a liquid such as water, with density $\rho(\boldsymbol{r}, t)$, pressure $p(\boldsymbol{r}, t)$, and velocity $\boldsymbol{v}(\boldsymbol{r}, t)$. The vertical component of the velocity will be denoted w. The liquid is bounded below by the bottom at $z = -h(\boldsymbol{x}, t)$ and above by the free surface $z = \eta(\boldsymbol{x}, t)$. Above the free surface we have air with density $\rho_a(\boldsymbol{r}, t)$ and pressure $p_a(\boldsymbol{r}, t)$, however since the density of air is small in comparison with water we will not consider the movement of the air. We denote time by t, the three-dimensional position vector by $\boldsymbol{r} = x\boldsymbol{i}_x + y\boldsymbol{i}_y + z\boldsymbol{i}_z$, and the two-dimensional horizontal position vector by $\boldsymbol{x} = x\boldsymbol{i}_x + y\boldsymbol{i}_y$. We let $\boldsymbol{i}_x, \boldsymbol{i}_y, \boldsymbol{i}_z$ be unit vectors in the x, y, z-directions. The positive z-direction is oriented up. The acceleration of gravity is $\boldsymbol{g} = -g\boldsymbol{i}_z$. The free surface is subject to surface tension, and the coefficient of surface tension is γ.

2.2.1 The basic equations

The continuity equation expresses the conservation of mass

$$\frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho + \rho \nabla \cdot \boldsymbol{v} = 0. \tag{40}$$

In the case of an incompressible fluid,

$$\frac{\mathrm{D}\rho}{\mathrm{d}t} \equiv \frac{\partial \rho}{\partial t} + \boldsymbol{v} \cdot \nabla \rho = 0, \tag{41}$$

including the special case of constant density, the continuity equation reduces to

$$\nabla \cdot \boldsymbol{v} = 0. \tag{42}$$

Euler's equation expresses the conservation of momentum for an inviscid fluid

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\frac{1}{\rho}\nabla p + \boldsymbol{g}. \tag{43}$$

The kinematic bottom condition is the requirement that liquid cannot flow through the bottom

$$\frac{\partial h}{\partial t} + \boldsymbol{v} \cdot \nabla h = -w \qquad \text{at } z = -h \tag{44}$$

where $w = \boldsymbol{i}_z \cdot \boldsymbol{v}$ is the vertical component of the velocity. In the case of a stationary and horizontal bottom, we have

$$w = 0 \qquad \text{at } z = -h. \tag{45}$$

The kinematic free surface condition is the requirement that liquid cannot flow through the free surface, or alternatively that a fluid particle at the free surface must stay at the free surface

$$\frac{\partial \eta}{\partial t} + \boldsymbol{v} \cdot \nabla \eta = w \qquad \text{at } z = \eta. \tag{46}$$

The dynamic free surface condition is the requirement that the sum of all forces on an infinitesimal surface element is zero. Let S be a surface element of the interface between water and air. On S there will be a pressure force $\boldsymbol{F}_w$ from the water below, a pressure force $\boldsymbol{F}_a$ from the air above, and surface tension force $\boldsymbol{F}_\gamma$ on the lateral side ∂S. Here ∂S is a closed curve on the interface between water and air, delimiting the surface element S from the rest of the interface. The pressure forces are

$$\boldsymbol{F}_w = \int_S p\boldsymbol{n}\, \mathrm{d}\sigma \tag{47}$$

and

$$\boldsymbol{F}_a = -\int_S p_a\boldsymbol{n}\, \mathrm{d}\sigma \tag{48}$$

where p is the pressure in the water below, p_a is the pressure in the air above, $\boldsymbol{n}$ is a unit normal vector to the free surface pointing out from water into air, and $\mathrm{d}\sigma$ is an infinitesimal surface element. The surface tension force is

$$\boldsymbol{F}_\gamma = -\int_{\partial S} \gamma\boldsymbol{n} \times \mathrm{d}\boldsymbol{r} = -\int_S \gamma\boldsymbol{n}\nabla \cdot \boldsymbol{n}\, \mathrm{d}\sigma \tag{49}$$

where the transition from a curve integral to a surface integral has been achieved by application of Stokes theorem. If we can assume the surface element S has a "thickness", within which a mass m is contained, then the acceleration $\boldsymbol{a}$ of the surface element is, according to Newton's second law, $\boldsymbol{F}_w + \boldsymbol{F}_a + \boldsymbol{F}_\gamma = m\boldsymbol{a}$. Now letting S shrink to zero thickness and extent, the mass shrinks to zero as well. In order that the acceleration does not become infinite, we must insist that the sum of the forces vanishes, thus

$$p - p_a - \gamma\nabla \cdot \boldsymbol{n} = 0 \qquad \text{at } z = \eta. \tag{50}$$

As an exercise one can show that

$$\nabla \cdot \boldsymbol{n} = -\frac{\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} + \frac{\partial^2\eta}{\partial x^2}\left(\frac{\partial\eta}{\partial y}\right)^2 + \frac{\partial^2\eta}{\partial y^2}\left(\frac{\partial\eta}{\partial x}\right)^2 - 2\frac{\partial\eta}{\partial x}\frac{\partial\eta}{\partial y}\frac{\partial^2\eta}{\partial x\partial y}}{\left(1 + \left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2\right)^{\frac{3}{2}}}. \tag{51}$$

It is common to see this expressed compactly in terms of the principal radii of curvature of the surface, however we prefer to use the above expression for subsequent analysis.

2.2.2 Potential flow

It is often a good approximation to consider the flow to be irrotational, $\nabla \times \boldsymbol{v} = 0$. Then the velocity can be derived from a velocity potential $\phi(\boldsymbol{r}, t)$

$$\boldsymbol{v} = \nabla\phi. \tag{52}$$

The continuity equation for an incompressible fluid now becomes the Laplace equation

$$\nabla^2\phi = 0. \tag{53}$$

The Euler equation becomes, in the case of constant density,

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 = -\frac{p - p_0}{\rho} - gz + f(t) \tag{54}$$

where p_0 is a constant reference pressure and $f(t)$ is a constant of integration (constant with respect to space). We can get rid of this constant of integration by a redefinition of the velocity potential $\phi \to \phi + \int^t f(\xi)\,\mathrm{d}\xi$. This redefinition does not affect the definition of velocity in (52), therefore we can simply set $f(t) \equiv 0$ in (54). A slight rewriting provides the Euler pressure equation[1]

$$p = p_0 - \rho\left(\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + gz\right). \tag{55}$$

The kinematic bottom condition becomes, in the case of a stationary and horizontal bottom,

$$\frac{\partial\phi}{\partial z} = 0 \qquad \text{at } z = -h. \tag{56}$$

The kinematic free surface condition becomes

$$\frac{\partial\eta}{\partial t} + \nabla\phi\cdot\nabla\eta = \frac{\partial\phi}{\partial z} \qquad \text{at } z = \eta. \tag{57}$$

The dynamic free surface condition can now be rewritten with the help of the Euler pressure equation (55). Upon selecting the reference pressure p_0 equal to the constant air pressure $p_0 = p_a$ we get

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + g\eta + \frac{\gamma}{\rho}\nabla\cdot\boldsymbol{n} = 0 \qquad \text{at } z = \eta \tag{58}$$

[1]The Euler pressure equation for unsteady potential flow should not be confused with the Bernoulli equation for steady flow stating that $\frac{p}{\rho} + \frac{1}{2}v^2 + gz$ is constant along a streamline with no requirement of irrotationality.

2.2.3 Normalization for weakly nonlinear waves

Let us assume that k_c is a characteristic wave number, ω_c is a characteristic angular frequency, and a_c is a characteristic amplitude for the surface elevation. Let us introduce the (characteristic) steepness $\epsilon = k_c a_c$. We can then perform the following normalizations

$$(x', y', z') = k_c(x, y, z) \quad t' = \omega_c t \quad \eta = a_c \eta' \quad \phi = \frac{\omega_c a_c}{k_c}\phi' \quad g = \frac{\omega_c^2}{k_c} g' \qquad \frac{\gamma}{\rho} = \frac{\omega_c^2}{k_c^3}\frac{\gamma'}{\rho'}$$

such that all the above primed quantities are supposed to be of "order one". We also normalize the depth

$$h' = k_c h$$

which will not be required to be "order one", but can be arbitrarily large. When this is substituted into the equations of the previous subsection we get

$$\frac{\partial \eta'}{\partial t'} + \epsilon \nabla' \phi' \cdot \nabla' \eta' = \frac{\partial \phi'}{\partial z'} \qquad \text{at } z' = \epsilon \eta' \tag{59}$$

$$\frac{\partial \phi'}{\partial t'} + \frac{1}{2}\epsilon \left(\nabla' \phi'\right)^2 + g'\eta' + \frac{\gamma'}{\rho'} \nabla' \cdot \boldsymbol{n} = 0 \qquad \text{at } z' = \epsilon \eta' \tag{60}$$

$$\nabla'^2 \phi' = 0 \qquad \text{for } -h' < z' < \epsilon \eta' \tag{61}$$

$$\frac{\partial \phi'}{\partial z'} = 0 \qquad \text{at } z' = -h' \tag{62}$$

In the above equations we shall drop the primes, but keep the steepness ϵ as an indicator of the magnitude of each term.

The resulting normalized equations look identical to the original equations, except for the presence of the ordering parameter ϵ. This representation is advantageous since it can be interpreted in two different ways: Set $\epsilon = 1$ and we recover the original dimensional and un-normalized equations. Set g and ρ and γ to unity and we get dimensionless and properly normalized equations.

For small steepness, $\epsilon \ll 1$, we can perform a Taylor-expansion around $z = 0$ such that for any function $f(z)$

$$f(\epsilon\eta) = f(0) + \epsilon\eta \frac{\partial f}{\partial z}(0) + \frac{1}{2}\epsilon^2\eta^2 \frac{\partial^2 f}{\partial z^2}(0) + \cdots \tag{63}$$

and we get within the first three orders

$$\frac{\partial \eta}{\partial t} + \epsilon \nabla\phi \cdot \nabla\eta + \epsilon^2 \eta \nabla \frac{\partial \phi}{\partial z} \cdot \nabla \eta - \frac{\partial \phi}{\partial z} - \epsilon\eta \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{2}\epsilon^2\eta^2 \frac{\partial^3 \phi}{\partial z^3} = O(\epsilon^3) \qquad \text{at } z = 0 \tag{64}$$

$$\begin{aligned} \frac{\partial \phi}{\partial t} + \epsilon\eta \frac{\partial^2 \phi}{\partial z \partial t} + \frac{1}{2}\epsilon^2\eta^2 \frac{\partial^3 \phi}{\partial z^2 \partial t} + \frac{1}{2}\epsilon \left(\nabla\phi\right)^2 + \epsilon^2 \eta \nabla\phi \cdot \nabla \frac{\partial \phi}{\partial z} + g\eta - \frac{\gamma}{\rho}\Big\{\nabla^2 \eta \\ - \frac{1}{2}\epsilon^2 \left[\frac{\partial^2 \eta}{\partial x^2}\left(3(\frac{\partial \eta}{\partial x})^2 + (\frac{\partial \eta}{\partial y})^2\right) + \frac{\partial^2 \eta}{\partial y^2}\left(3(\frac{\partial \eta}{\partial y})^2 + (\frac{\partial \eta}{\partial x})^2\right) + 4\frac{\partial^2 \eta}{\partial x \partial y}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\right]\Big\} \\ = O(\epsilon^3) \qquad \text{at } z = 0 \end{aligned} \tag{65}$$

$$\nabla^2\phi = 0 \qquad \text{for } -h < z < 0 \tag{66}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at } z = -h \tag{67}$$

2.2.4 Regular perturbation expansion

To solve equations (64)–(67) it is tempting to try regular perturbation expansions

$$\eta = \eta_1 + \epsilon\eta_2 + \epsilon^2\eta_3 + \cdots \tag{68}$$

$$\phi = \phi_1 + \epsilon\phi_2 + \epsilon^2\phi_3 + \cdots \tag{69}$$

The leading order problem

After substituting (68)–(69) into (64)–(67), the leading order problem is

$$\frac{\partial \eta_1}{\partial t} - \frac{\partial \phi_1}{\partial z} = 0 \qquad \text{at } z = 0 \tag{70}$$

$$\frac{\partial \phi_1}{\partial t} + g\eta_1 - \frac{\gamma}{\rho}\nabla^2\eta_1 = 0 \qquad \text{at } z = 0 \tag{71}$$

$$\nabla^2\phi_1 = 0 \qquad \text{for } -h < z < 0 \tag{72}$$

$$\frac{\partial \phi_1}{\partial z} = 0 \qquad \text{at } z = -h \tag{73}$$

This system is solved by assuming a monochromatic elementary wave solution

$$\begin{pmatrix} \eta_1(\boldsymbol{x}, t) \\ \phi_1(\boldsymbol{r}, t) \end{pmatrix} = \begin{pmatrix} \hat{\eta}_1 \\ \hat{\phi}_1(z) \end{pmatrix} \mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)} \tag{74}$$

where $\boldsymbol{k} = k_x\boldsymbol{i}_x + k_y\boldsymbol{i}_y$ is the wave vector, $\boldsymbol{x}$ is the horizontal position vector, $\boldsymbol{r}$ is the three-dimensional position vector, and ω is the angular frequency.

The solution for $\hat{\phi}_1$ is found from (72)–(73) as

$$\hat{\phi}_1(z) = A\frac{\cosh k(z+h)}{\sinh kh} \tag{75}$$

where $k = |\boldsymbol{k}| = \sqrt{k_x^2 + k_y^2}$ is the wavenumber. The ratio of cosh over sinh is preferred in order that for deep water, $h \to \infty$, the limiting behavior is $\hat{\phi}_1(z) = A\mathrm{e}^{kz}$.

The two surface conditions then give the linear system

$$\begin{pmatrix} -\mathrm{i}\omega & -k \\ g + \frac{\gamma}{\rho}k^2 & -\mathrm{i}\omega \coth kh \end{pmatrix} \begin{pmatrix} \hat{\eta}_1 \\ A \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{76}$$

A nontrivial solution requires the determinant of the coefficient matrix to be zero, which provides us the dispersion relation

$$\omega^2 = (gk + \frac{\gamma}{\rho}k^3)\tanh kh. \tag{77}$$

A nontrivial solution for $k > 0$ can then be expressed by

$$\hat{\eta}_1 = \frac{b}{2} \qquad \text{and} \qquad A = -\mathrm{i}\frac{\omega}{k}\frac{b}{2} \tag{78}$$

where b is a complex amplitude.

Corresponding to $k = 0$, the system (70)–(73) has the nontrivial solution $\eta = 0$ and $\phi = Ux + Vy$ corresponding to a horizontal current with a flat water surface.

Since the leading order problem is linear, we can employ the principle of superposition to compose a general solution for an irregular sea as a sum of monochromatic waves. For a discrete superposition of real monochromatic waves we can write

$$\eta_1(\boldsymbol{x}, t) = \frac{1}{2}\sum_j b_j \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_j\cdot\boldsymbol{x}-\omega_j t)} + \text{c.c.} = \sum_j |b_j| \cos(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega_j t + \arg b_j) \tag{79}$$

and

$$\phi_1(\boldsymbol{r}, t) = Ux + Vy + \frac{1}{2}\sum_j \frac{\omega_j \cosh\left(k_j(z+h)\right)}{k_j \sinh(k_j h)} \left(-\mathrm{i}b_j \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_j\cdot\boldsymbol{x}-\omega_j t)} + \text{c.c.}\right) \tag{80}$$

where b_j are complex amplitudes, and where each pair of wave vector $\boldsymbol{k}_j$ and angular frequency ω_j satisfies the dispersion relation (77).

Notice that since we want the surface elevation to be real, the sum over j should include complex conjugates as appropriate.

Higher order problems

The general form of the problem of order $n > 1$ is

$$\frac{\partial \eta_n}{\partial t} - \frac{\partial \phi_n}{\partial z} = F_n \qquad \text{at } z = 0 \tag{81}$$

$$\frac{\partial \phi_n}{\partial t} + g\eta_n - \frac{\gamma}{\rho}\nabla^2\eta_n = G_n \qquad \text{at } z = 0 \tag{82}$$

$$\nabla^2\phi_n = 0 \qquad \text{for } -h < z < 0 \tag{83}$$

$$\frac{\partial \phi_n}{\partial z} = 0 \qquad \text{at } z = -h \tag{84}$$

where we have

$$F_2 = -\nabla\phi_1 \cdot \nabla\eta_1 + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} \tag{85}$$

$$G_2 = -\eta_1 \frac{\partial^2 \phi_1}{\partial z \partial t} - \frac{1}{2}(\nabla\phi_1)^2 \tag{86}$$

$$F_3 = -\nabla\phi_2 \cdot \nabla\eta_1 - \nabla\phi_1 \cdot \nabla\eta_2 - \eta_1 \nabla\frac{\partial \phi_1}{\partial z} \cdot \nabla\eta_1 + \eta_2 \frac{\partial^2 \phi_1}{\partial z^2} + \eta_1 \frac{\partial^2 \phi_2}{\partial z^2} + \frac{1}{2}\eta_1^2 \frac{\partial^3 \phi_1}{\partial z^3} \tag{87}$$

$$G_3 = -\eta_2 \frac{\partial^2 \phi_1}{\partial z \partial t} - \eta_1 \frac{\partial^2 \phi_2}{\partial z \partial t} - \nabla\phi_2 \cdot \nabla\phi_1 - \frac{1}{2}\eta_1^2 \frac{\partial^3 \phi_1}{\partial z^2 \partial t} - \eta_1 \nabla\phi_1 \cdot \nabla \frac{\partial \phi_1}{\partial z}$$
$$- \frac{\gamma}{2\rho} \left\{ \frac{\partial^2 \eta_1}{\partial x^2} \left(3(\frac{\partial \eta_1}{\partial x})^2 + (\frac{\partial \eta_1}{\partial y})^2 \right) + \frac{\partial^2 \eta_1}{\partial y^2} \left(3(\frac{\partial \eta_1}{\partial y})^2 + (\frac{\partial \eta_1}{\partial x})^2 \right) + 4 \frac{\partial^2 \eta_1}{\partial x \partial y} \frac{\partial \eta_1}{\partial x} \frac{\partial \eta_1}{\partial y} \right\} \tag{88}$$

etc.

Second order problem

If the first-order solution is a superposition of two monochromatic waves

$$\eta_1(\boldsymbol{x}, t) = \mathrm{Re}\left\{ b_j \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega_j t)} + b_l \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_l \cdot \boldsymbol{x} - \omega_l t)} \right\} \tag{89}$$

then the second-order forcing will have oscillations with wavenumber vectors $\boldsymbol{K}_{j,l}^{\pm} = \boldsymbol{k}_j \pm \boldsymbol{k}_l$ and angular frequencies $\Omega_{j,l}^{\pm} = \omega_j \pm \omega_l$, and we anticipate the particular solutions

$$\begin{pmatrix} \eta_{2,j,l}^{\pm}(\boldsymbol{x}, t) \\ \phi_{2,j,l}^{\pm}(\boldsymbol{r}, t) \end{pmatrix} = \begin{pmatrix} \hat{\eta}_{2,j,l}^{\pm} \\ \hat{\phi}_{2,j,l}^{\pm}(z) \end{pmatrix} \mathrm{e}^{\mathrm{i}(\boldsymbol{K}_{j,l}^{\pm} \cdot \boldsymbol{x} - \Omega_{j,l}^{\pm} t)} \tag{90}$$

For non-zero $\boldsymbol{K}_{j,l}^{\pm}$ the solution for $\hat{\phi}_{2,j,l}^{\pm}$ is found from (83)–(84) as

$$\hat{\phi}_{2,j,l}^{\pm}(z) = A_{2,j,l}^{\pm} \frac{\cosh K_{j,l}^{\pm}(z+h)}{\sinh K_{j,l}^{\pm} h} \tag{91}$$

where $K_{j,l}^{\pm} = |\boldsymbol{K}_{j,l}^{\pm}|$ and where the ratio of cosh over sinh is chosen in order that for deep water, $h \to \infty$, the limiting behavior is $\hat{\phi}_{2,j,l}^{\pm}(z) = A_{2,j,l}^{\pm} \mathrm{e}^{K_{j,l}^{\pm} z}$.

If we denote the corresponding contributions to F_2 and G_2 by $\hat{F}_{2,j,l}^{\pm} \mathrm{e}^{\mathrm{i}(\boldsymbol{K}_{j,l}^{\pm} \cdot \boldsymbol{x} - \Omega_{j,l}^{\pm} t)}$ and $\hat{G}_{2,j,l}^{\pm} \mathrm{e}^{\mathrm{i}(\boldsymbol{K}_{j,l}^{\pm} \cdot \boldsymbol{x} - \Omega_{j,l}^{\pm} t)}$, then the two surface conditions (81) and (82) with the two right-hand sides (85) and (86) give the linear system

$$\begin{pmatrix} -\mathrm{i}\Omega_{j,l}^{\pm} & -K_{j,l}^{\pm} \\ g + \frac{\gamma}{\rho} K_{j,l}^{\pm\,2} & -\mathrm{i}\Omega_{j,l}^{\pm} \coth K_{j,l}^{\pm} h \end{pmatrix} \begin{pmatrix} \hat{\eta}_{2,j,l}^{\pm} \\ A_{2,j,l}^{\pm} \end{pmatrix} = \begin{pmatrix} \hat{F}_{2,j,l}^{\pm} \\ \hat{G}_{2,j,l}^{\pm} \end{pmatrix} \tag{92}$$

This matrix equation can readily be solved when the pair $\boldsymbol{K}_{j,l}^{\pm}$ and $\Omega_{j,l}^{\pm}$ does not satisfy the dispersion relation (77), because then the determinant of the coefficient matrix is non-zero.

If the pair $\boldsymbol{K}_{j,l}^{\pm}$ and $\Omega_{j,l}^{\pm}$ does satisfy the dispersion relation then we are forcing the linear system with its own natural wave solution. Our previous experience suggests this will give resonant blow-up unless we introduce slow modulation scales $\boldsymbol{x}_1 = \epsilon \boldsymbol{x}$ and $t_1 = \epsilon t$.

Now we invoke the Fredholm alternative which can be expressed as follows:

> Let M be a matrix, and let X, Y and B be vectors. The matrix system $MX = B$ has a solution for X if and only if any left eigenvector Y with zero eigenvalue, $YM = 0$, is orthogonal to B, i.e. $YB = 0$. This is called a solvability condition.

The left eigenvector with zero eigenvalue is $(\mathrm{i}\Omega^{\pm}_{j,l}, -K^{\pm}_{j,l}\tanh K^{\pm}_{j,l}h)$ and the solvability condition is

$$\mathrm{i}\Omega^{\pm}_{j,l}\hat{F}^{\pm}_{2,j,l} - K^{\pm}_{j,l}\tanh(K^{\pm}_{j,l}h)\hat{G}^{\pm}_{2,j,l} = 0 \tag{93}$$

Finally, if $\boldsymbol{K}^{\pm}_{j,l} = 0$ then we are forcing a second-order solution that is not a propagating wave. In particular we find

$$\eta^{-}_{2,j,j} = 0 \qquad \text{and} \qquad \phi^{-}_{2,j,j} = \frac{\omega_j^2(1 - C_j^2)}{2}\frac{|b_j|^2}{4}t \tag{94}$$

where we have used the notation $C_j = \coth(k_j h)$. This implies a change of pressure without affecting either the velocity field nor the surface elevation.

Summary of what we have found so far

In the absence of resonances, the second-order problem (92) can be solved by matrix inversion. However, we realize that the second-order solution can suffer unbounded resonant growth if three waves can resonate in a triad

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 \qquad \text{and} \qquad \omega_1 + \omega_2 = \omega_3 \tag{95}$$

in which case we anticipate that the complex amplitudes of the leading-order solution should be modulated on the slow scales $\boldsymbol{x}_1 = \epsilon\boldsymbol{x}$ and $t_1 = \epsilon t$.

Similarly, we anticipate that the third-order solution can suffer unbounded resonant growth if four waves can resonate in a quartet

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 = \boldsymbol{k}_4 \qquad \text{and} \qquad \omega_1 + \omega_2 + \omega_3 = \omega_4 \tag{96}$$

in which case we anticipate that the complex amplitudes of the leading-order solution should be modulated on the slow scales $\boldsymbol{x}_2 = \epsilon^2\boldsymbol{x}$ and $t_2 = \epsilon^2 t$.

We should therefore investigate under which conditions the dispersion relation (77) allows triad or quartet resonance before we attempt to solve the second- and third-order problems.

2.2.5 Exercises

1. Expand $\nabla \cdot \boldsymbol{n}$ to third order in steepness $\epsilon = k_c a_c$, where $\boldsymbol{n}$ is the unit normal vector to the surface $z = \eta$.

   Hint: It may be useful to start by showing that for $|\nu| < 1$ we have

$$\frac{1}{1+\nu} = 1 - \nu + \nu^2 - \nu^3 + \ldots$$

2. In order to appreciate the relative importance of equations (48) and (49), let us consider a spherical raindrop with diameter 2 mm, let us cut it horizontally through its center. Compute the air pressure force from above (48) on the upper half of the sphere, the surface tension force (49) acting on the circular cut around the sphere, and the weight of the raindrop.

   You may use the air pressure of one standard atmosphere 101325 Pa and the surface tension between water and air 0.07286 N/m and the water density $\rho = 998.2$ kg/m^3 both at 20°C.

3. The dispersion relation (77) has an inflection point, a minimum phase speed and a minimum group velocity. Find all of these in the limit of infinite depth.

   You may use the acceleration of gravity 9.81 m/s^2 and the surface tension between water and air 0.07286 N/m and the water density $\rho = 998.2$ kg/m^3 both at 20°C.

4. Discuss how the dispersion relation (77) can be approximated for (i) pure capillary waves on deep water, (ii) capillary–gravity waves on deep water, (iii) gravity waves on shallow water.

   Discuss how long waves should be in order to be "deep water" waves, and how short waves should be in order to be "capillary" waves.

2.3 Nonlinear resonance conditions

We inquire if three or four waves can resonantly interact such that the conditions

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 \qquad \text{and} \qquad \omega_1 + \omega_2 = \omega_3 \tag{97}$$

or

$$\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4 \qquad \text{and} \qquad \omega_1 + \omega_2 = \omega_3 + \omega_4 \tag{98}$$

are satisfied. Here ω_n and $\boldsymbol{k}_n$ are related according to the dispersion relation.

In particular we shall be concerned with the dispersion relation for gravity–capillary waves on arbitrary depth

$$\omega^2 = (gk + \frac{\gamma}{\rho} k^3) \tanh(kh). \tag{99}$$

2.3.1 Three-wave resonance of long waves

For non-dispersive waves, $\omega = \alpha k$ where α is a constant, the resonance conditions (97) are satisfied for co-linear wave vectors.

The dispersion relation (99) is approximately non-dispersive for long waves. This is seen by Taylor expansion of (99) around $k = 0$

$$\omega = \sqrt{gh}k + \frac{\sqrt{gh}}{6} \left(\frac{3\gamma}{\rho g} - h^2 \right) k^3 + O(k^5) \tag{100}$$

To the second order in k this equation is non-dispersive.

It is even possible to achieve non-dispersive gravity–capillary waves accurate to fourth order in k by setting $h = \sqrt{\frac{3\gamma}{\rho g}}$. With typical values $g = 9.81$ m/s^2, $\rho = 998.2$ kg/m^3 and $\gamma = 0.07286$ N/m (values for 20°C), we get the target depth $h = 4.7$ mm, which is not interesting for ocean waves, but could be quite interesting for laboratory experiments or for waves that occur on paved roads on a rainy day.

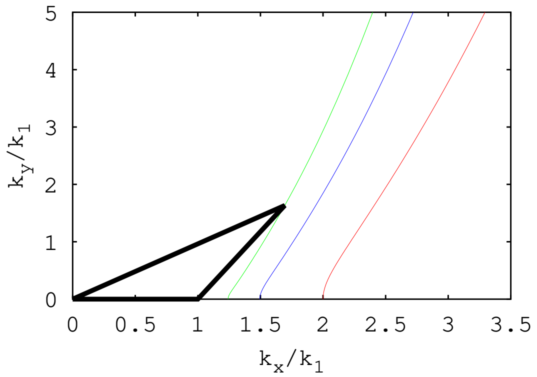

Figure 5: Resonant triad of gravity–capillary waves on infinite depth, normalized against $\boldsymbol{k}_1$ oriented along the first axis, with $\Gamma \equiv \frac{\gamma k_1^2}{\rho g} = 0.5$ (green); 1 (blue); 2 (red).

2.3.2 Three-wave resonance of one long and two short waves

In equation (97) let $\boldsymbol{k}_1 \approx \boldsymbol{k}_3$, then $\boldsymbol{k}_2$ is quite small. We consider wave 1 and wave 3 to be the "short" waves while wave 2 is the "long" wave. In this case there can be three-wave resonance if the phase speed of the long wave is equal to the component of the group velocity of the short waves (e.g. wave 1) in the direction of the long wave. This is seen by Taylor-expanding ω_3 around $\boldsymbol{k}_1$

$$\omega_3 = \omega(\boldsymbol{k}_3) \approx \omega(\boldsymbol{k}_1) + (\boldsymbol{k}_3 - \boldsymbol{k}_1) \cdot \left.\frac{\partial \omega}{\partial \boldsymbol{k}}\right|_{\boldsymbol{k}_1} = \omega_1 + \boldsymbol{k}_2 \cdot \boldsymbol{c}_{g1} \tag{101}$$

thus we must check if the phase speed of the long wave satisfies

$$c_2 \equiv \frac{\omega_2}{k_2} \approx \frac{\boldsymbol{k}_2}{k_2} \cdot \boldsymbol{c}_{g1}. \tag{102}$$

The dispersion relation (99) allows this condition to be satisfied in the limit of long gravity waves when all three waves have the same direction, then this becomes a limiting case of the result in section 2.3.1. For other types of dispersion relations, or interactions between surface and internal waves, one could expect this condition to give more interesting triad resonance configurations.

2.3.3 Three-wave resonance of deep-water capillary–gravity waves

Consider infinite depth and set $\Gamma = \frac{\gamma k_1^2}{\rho g}$. In figure 5 the wave vector $\boldsymbol{k}_1$ is oriented along the first axis. The green, blue and red curves show the locus where wave vectors $\boldsymbol{k}_2$ and $\boldsymbol{k}_3$ should meet for Γ having values 0.5, 1 and 2.

This configuration of resonant triads of gravity–capillary waves on infinite depth was first investigated by McGoldrick (1965).

2.3.4 Three-wave resonance of two unidirectional capillary–gravity waves (Wilton's ripples)

As a special case of the previous triad resonance, if we limit to the special case $\boldsymbol{k}_1 = \boldsymbol{k}_2$ then it can be shown that the resonance condition (97) is satisfied when

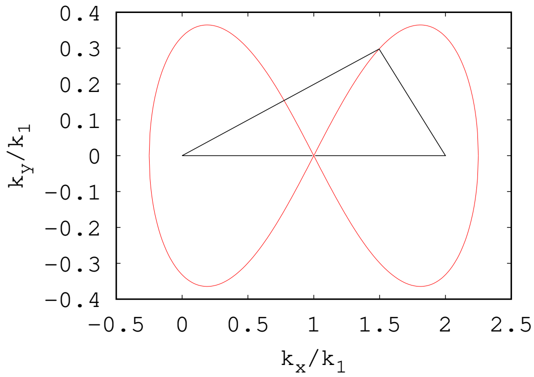

Figure 6: Resonant quartet of gravity waves on infinite depth.

$k = \sqrt{\frac{\rho g}{2\gamma}}$. With typical values $g = 9.81$ m/s^2, $\rho = 998$ kg/m^3 and $\gamma = 0.0728$ N/m (values for 20°C), we get $k_1 = 259$ m^{-1} which corresponds to the long wavelength $\lambda_1 = \frac{2\pi}{k_1} = 2.4$ cm and short wavelength $\lambda_3 = \frac{2\pi}{k_3} = 1.2$ cm.

These waves are commonly called Wilton's ripples, after Wilton (1915), although they were previously described by Harrison (1909).

2.3.5 There are no three-wave resonances for deep-water gravity waves

For the limit of infinite depth and no capillarity, so that the dispersion relation becomes $\omega^2 = gk$, there are no resonant triads of gravity waves.

One way to show this is to let the angle between wave vectors $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$ be θ, eliminate k_3 from equations (97) by the expression $k_3^2 = k_1^2 + k_2^2 + 2k_1k_2\cos\theta$, and derive the following expression for the angle

$$\cos\theta = \frac{2(k_1 + k_2)}{\sqrt{k_1k_2}} + 3. \tag{103}$$

There are obviously no solutions for $\cos\theta \geq 3$.

The nonexistence of resonant triads of gravity waves on deep water was first shown by Phillips (1960).

2.3.6 Quartet resonance of gravity waves

Let us simplify the problem by letting $\boldsymbol{k}_1 = \boldsymbol{k}_2 = (k_1, 0)$. Then let us write $\boldsymbol{k}_3 = (k_x, k_y) = k_1(1 + x, y)$ and $\boldsymbol{k}_4 = k_1(1 - x, -y)$ for non-dimensional variables x and y. In the case of infinite depth the resonance condition (98) requires that

$$\left[(1+x)^2 + y^2\right]^{\frac{1}{4}} + \left[(1-x)^2 + y^2\right]^{\frac{1}{4}} = 2. \tag{104}$$

For $y = 0$ we have the three solutions $x = 0$ and $x = \pm\frac{5}{4}$. The full solution for infinite depth is shown in figure 6 where the first axis corresponds to x and the second axis corresponds to y, and where the four resonating wave vectors are indicated. The red curve is known as the "figure 8 of Phillips" after Phillips (1960).

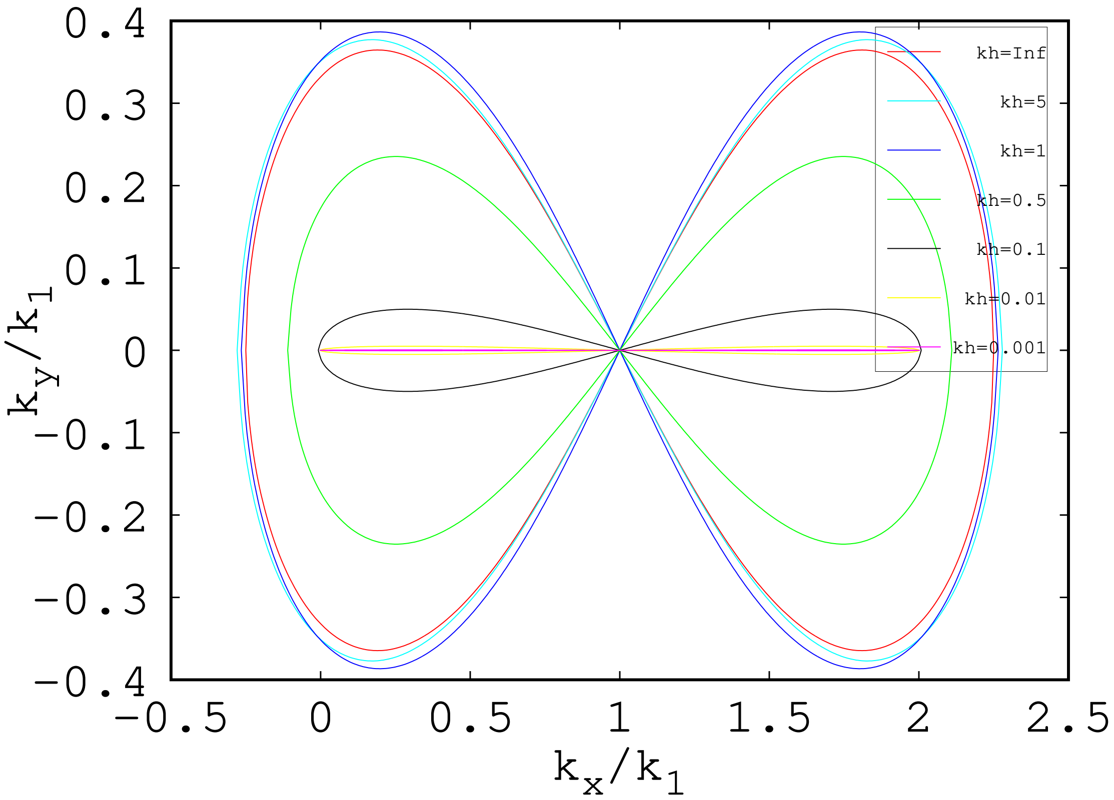

Figure 7: Resonant quartet of gravity waves on various depths.

It is interesting to show how equation (104) and figure 6 are modified as the depth decreases from infinite to small, this is shown in figure 7. It is interesting to notice that these curves do not simply "shrink" as kh decreases, instead the "figure of 8" deforms first by increasing slightly in size before it decreases.

2.3.7 Exercises

1. Show that for non-dispersive isotropic waves, $\omega = \alpha k$ where α is a constant and $k = |\boldsymbol{k}|$, the three-wave resonance conditions (97) are satisfied when the wave vectors point in the same direction.

2. Show analytically that (102) is satisfied for non-dispersive isotropic waves when the wave vectors point in the same direction.

3. Show numerically that (102) is satisfied for one shallow-water gravity wave, $\omega = \sqrt{gh}k$, and two "short" gravity waves on finite depth, $\omega^2 = gk\tanh(kh)$, only in the limit that $kh \to 0$.

4. Show that there can be quartet resonance for gravity–capillary waves on infinite depth with three identical waves $\boldsymbol{k}_1 = \boldsymbol{k}_2 = \boldsymbol{k}_3$ and a fourth wave $\boldsymbol{k}_4$ pointing in the same direction, with resonance conditions $3\boldsymbol{k}_1 = \boldsymbol{k}_4$ and $3\omega_1 = \omega_4$. What are the wavelengths of these waves?

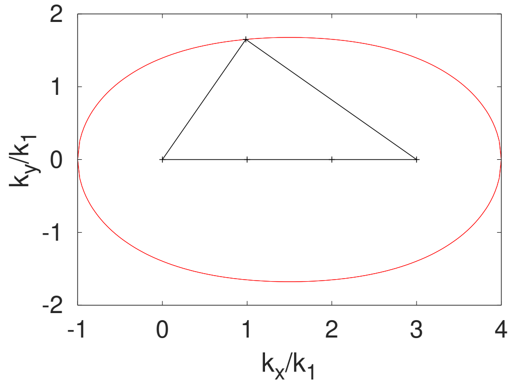

Figure 8: Resonant quintet of gravity waves on infinite depth.

This resonance was also discussed by Harrison (1909) and Wilton (1915).

5. Derive the modification of equation (104) for finite depth needed to produce the curves shown in figure 7.

6. Show that there can be quintet resonance for gravity–capillary waves on infinite depth with four identical waves $\boldsymbol{k}_1 = \boldsymbol{k}_2 = \boldsymbol{k}_3 = \boldsymbol{k}_4$ and a fifth wave $\boldsymbol{k}_5$ pointing in the same direction, with resonance conditions $4\boldsymbol{k}_1 = \boldsymbol{k}_5$ and $4\omega_1 = \omega_5$. What are the wavelengths of these waves?

7. Show that there can be quintet resonance for gravity waves on infinite depth with three waves that are identical $\boldsymbol{k}_1 = \boldsymbol{k}_2 = \boldsymbol{k}_3 = (k_1, 0)$. The two additional waves $\boldsymbol{k}_4$ and $\boldsymbol{k}_5$ do not have to be parallel to the first three.

   Hint: Take inspiration from the derivation of the quartet resonance, and show that you get a picture like that shown in figure 8.

8. Try to show that gravity waves on finite depth have no triad resonance.

   Hint: This is much easier to sketch graphically than to prove analytically.

2.4 Second-order nonlinear non-resonant waves

2.4.1 Second-order nonlinear gravity waves

Having established that there are no resonances at the second order for gravity waves, on water of deep or finite depth, we can proceed to solve the particular solution at the second order.

If the first-order solution is a sum of monochromatic waves

$$\eta_1(\boldsymbol{x}, t) = \sum_j \mathrm{Re}\left\{ b_j \mathrm{e}^{\mathrm{i}(\boldsymbol{k}_j \cdot \boldsymbol{x} - \omega_j t)} \right\} \tag{105}$$

then the second-order forcing will be

$$F_2 = \sum_{j,l} \mathrm{Re}\left\{ \hat{F}^{+}_{2,j,l} \mathrm{e}^{\mathrm{i}(\boldsymbol{K}^{+}_{j,l} \cdot \boldsymbol{x} - \Omega^{+}_{j,l} t)} + \hat{F}^{-}_{2,j,l} \mathrm{e}^{\mathrm{i}(\boldsymbol{K}^{-}_{j,l} \cdot \boldsymbol{x} - \Omega^{-}_{j,l} t)} \right\} \tag{106}$$

and

$$G_2 = \sum_{j,l} \mathrm{Re}\left\{ \hat{G}^{+}_{2,j,l} \mathrm{e}^{\mathrm{i}(\boldsymbol{K}^{+}_{j,l}\cdot\boldsymbol{x} - \Omega^{+}_{j,l} t)} + \hat{G}^{-}_{2,j,l} \mathrm{e}^{\mathrm{i}(\boldsymbol{K}^{-}_{j,l}\cdot\boldsymbol{x} - \Omega^{-}_{j,l} t)} \right\} \tag{107}$$

where $\boldsymbol{K}^{\pm}_{j,l} = \boldsymbol{k}_j \pm \boldsymbol{k}_l$ and $\Omega^{\pm}_{j,l} = \omega_j \pm \omega_l$. The corresponding coefficients are

$$\hat{F}^{+}_{2,j,l} = -\frac{\mathrm{i}g b_j b_l}{8} \left\{ \frac{\boldsymbol{k}_j}{\omega_j} + \frac{\boldsymbol{k}_l}{\omega_l} \right\} \cdot \boldsymbol{K}^{+}_{j,l} \tag{108}$$

$$\hat{F}^{-}_{2,j,l} = -\frac{\mathrm{i}g b_j b_l^*}{8} \left\{ \frac{\boldsymbol{k}_j}{\omega_j} + \frac{\boldsymbol{k}_l}{\omega_l} \right\} \cdot \boldsymbol{K}^{-}_{j,l} \tag{109}$$

$$\hat{G}^{+}_{2,j,l} = \frac{b_j b_l}{8} \left\{ \omega_j^2 + \omega_j\omega_l + \omega_l^2 - \frac{g^2 \boldsymbol{k}_j \cdot \boldsymbol{k}_l}{\omega_j \omega_l} \right\} \tag{110}$$

and

$$\hat{G}^{-}_{2,j,l} = \frac{b_j b_l^*}{8} \left\{ \omega_j^2 - \omega_j\omega_l + \omega_l^2 - \frac{g^2 \boldsymbol{k}_j \cdot \boldsymbol{k}_l}{\omega_j \omega_l} \right\} \tag{111}$$

The particular solution will be according to (90). For non-zero $\boldsymbol{K}^{\pm}_{j,l}$ we need to solve the system

$$\begin{pmatrix} -\mathrm{i}\Omega^{\pm}_{j,l} & -K^{\pm}_{j,l} \\ g & -\mathrm{i}\Omega^{\pm}_{j,l} \coth K^{\pm}_{j,l} h \end{pmatrix} \begin{pmatrix} \hat{\eta}^{\pm}_{2,j,l} \\ A^{\pm}_{2,j,l} \end{pmatrix} = \begin{pmatrix} \hat{F}^{\pm}_{2,j,l} \\ \hat{G}^{\pm}_{2,j,l} \end{pmatrix} \tag{112}$$

We know that the matrix is not singular, the inverse matrix is

$$\frac{1}{{\Omega^{\pm}_{j,l}}^2 \coth(K^{\pm}_{j,l} h) - g K^{\pm}_{j,l}} \begin{pmatrix} \mathrm{i}\Omega^{\pm}_{j,l} \coth(K^{\pm}_{j,l} h) & -K^{\pm}_{j,l} \\ g & \mathrm{i}\Omega^{\pm}_{j,l} \end{pmatrix} \tag{113}$$

and the solution is

$$\hat{\eta}^{\pm}_{2,j,l} = \frac{\mathrm{i}\Omega^{\pm}_{j,l} \coth(K^{\pm}_{j,l} h) \hat{F}^{\pm}_{2,j,l} - K^{\pm}_{j,l} \hat{G}^{\pm}_{2,j,l}}{{\Omega^{\pm}_{j,l}}^2 \coth(K^{\pm}_{j,l} h) - g K^{\pm}_{j,l}} \tag{114}$$

and

$$A^{\pm}_{2,j,l} = \frac{g \hat{F}^{\pm}_{2,j,l} + \mathrm{i}\Omega^{\pm}_{j,l} G^{\pm}_{2,j,l}}{{\Omega^{\pm}_{j,l}}^2 \coth(K^{\pm}_{j,l} h) - g K^{\pm}_{j,l}} \tag{115}$$

Special case $\boldsymbol{k}_j = \boldsymbol{k}_l$ and $\omega_j = \omega_l$

Then $K^{+}_{j,l} = 2k_j$ and $\Omega^{+}_{j,l} = 2\omega_j$ and we get the solution

$$\hat{\eta}^{+}_{2,j,j} = \frac{3 - s_j^2}{8 s_j^3} k_j b_j^2 \tag{116}$$

and

$$\hat{A}^{+}_{2,j,j} = -\mathrm{i}\omega_j \frac{3(1 - s_j^2)}{8 s_j} b_j^2 \tag{117}$$

where we have used the notation $s_j = \tanh(k_j h)$. Notice that for infinite depth, $s_j = 1$, we have

$$\hat{\eta}^+_{2,j,j} = \frac{1}{4} k_j b_j^2 \qquad \text{and} \qquad \hat{A}^+_{2,j,j} = 0 \qquad \text{for} \qquad h \to \infty \tag{118}$$

In this case we further have $K^-_{j,l} = 0$ and $\Omega^-_{j,l} = 0$ and we get the solution

$$\hat{\eta}^-_{2,j,j} = 0 \qquad \text{and} \qquad \hat{\phi}^-_{2,j,j} = \frac{\omega_j^2 (1 - C_j^2)|b_j|^2}{8} t \tag{119}$$

where $C_j = \coth(k_j h)$. This implies a change of pressure without affecting either the velocity field or the surface elevation.

2.4.2 Second-order nonlinear Stokes waves

The Stokes wave corresponds to the leading order solution only being a monochromatic simple-harmonic wave

$$\eta_1 = \frac{1}{2} b e^{i(kx - \omega t)} + \text{c.c.} = \frac{1}{2} a e^{i\chi} + \text{c.c.} \tag{120}$$

and

$$\phi_1 = \frac{\omega}{2k} \frac{\cosh(k(z+h))}{\sinh(kh)} \left(-ia e^{i\chi} + \text{c.c.} \right) \tag{121}$$

where the complex amplitude b is expressed by its magnitude $a = |b|$ and phase $\theta = \arg b$, thus $b = a \exp(i\theta)$, and where we have introduced the phase function $\chi = kx - \omega t + \theta$.

Taking advantage of the previously found second-order solution we have

$$\eta_2 = \frac{3 - s^2}{8 s^3} k a^2 e^{2i\chi} + \text{c.c.} \tag{122}$$

and

$$\phi_2 = -\frac{\omega^2 (1 - s^2)}{4 s^2} t |a|^2 + \frac{3\omega(1 - s^2)}{8 s^3} \frac{\cosh(2k(z+h))}{\sinh(2kh)} \left(-i a^2 e^{2i\chi} + \text{c.c.} \right) \tag{123}$$

where $s = \tanh kh$.

It is remarkable that for deep water, $kh \to \infty$ and $\tanh kh \to 1$ and therefore $\phi_2 = 0$. There is no second-order nonlinear correction to the velocity field on deep water!

We summarize the solution for the surface elevation to the second order, $\eta = \eta_1 + \eta_2$, for a being a constant and real amplitude,

$$\eta = a \cos\chi + \frac{3 - s^2}{4 s^3} k a^2 \cos 2\chi \tag{124}$$

which on deep water reduces to

$$\eta = a \cos\chi + \frac{1}{2} k a^2 \cos 2\chi \tag{125}$$

On shallow water, kh small, we have $\tanh kh \approx kh \ll 1$ which means that the second-order term in (124) can become arbitrarily large. In this case we may want to impose a consistency condition on the Stokes wave expansion that the second-order term should be smaller than the first-order term, which reduces to

$$Ur \equiv \frac{ka}{(kh)^3} < \frac{4}{3} \qquad \text{when } kh \text{ is small.} \tag{126}$$

The quantity Ur is known as the Ursell number. In general, the Stokes wave expansion diverges when the Ursell number is not small.

2.5 Nonlinear resonant waves

2.5.1 Second-order nonlinear Wilton's ripples

Wilton's ripples have second-order nonlinear resonance between a first and a second harmonic wave. We therefore anticipate the need to introduce slow scales $\boldsymbol{x}_1$ and t_1.

As an exercise, if we limit to slow time only, show that the resulting system is

$$\frac{\partial A_1}{\partial t_1} + \boldsymbol{c}_{g1} \cdot \frac{\partial A_1}{\partial \boldsymbol{x}_1} = \alpha A_2 A_1^* \tag{127}$$

$$\frac{\partial A_2}{\partial t_1} + \boldsymbol{c}_{g2} \cdot \frac{\partial A_2}{\partial \boldsymbol{x}_1} = \beta A_1^2 \tag{128}$$

2.5.2 Third order nonlinear Stokes waves

Since we know that there are resonances at the third order for gravity waves on any depth, we anticipate the need to introduce slow scales $\boldsymbol{x}_2$ and t_2.

It turns out that in this special case it is enough to introduce the slow time t_2 only, thus we set $b = b(t_2)$ in the first-order solution above.

The solvability condition that ends up being imposed at the third order is

$$\frac{\partial b}{\partial t_2} = \mathrm{i}\alpha |b|^2 b \tag{129}$$

for some constant α, and which has the solution

$$b = a\mathrm{e}^{\mathrm{i}\alpha a^2 t_2} \tag{130}$$

thus the magnitude a remains constant while the phase changes slowly. This means that the solutions (124) and (125) remain the same to second order with only a small change of the phase function

$$\chi = kx - \omega t + \alpha a^2 t_2 + \theta.$$

We could now have proceeded to find the constant α above, as well as the particular solutions for η_3 and ϕ_3, ..., however, that is not very useful for two reasons:

First, it turns out that the limitation to slow modulation on scale $t_2 = \epsilon^2 t$ is not representative for realistic ocean waves, and second, it turns out that the Stokes wave is unstable for infinitesimal perturbations of waves close to the leading-order monochromatic wave.

Therefore we now turn to the extremely important topic of wave modulations and instabilities.

2.6 Linear modulation of narrow-banded waves

Looking back to where the dispersion relation came from, we notice that associating ω with the operation $\mathrm{i}\frac{\partial}{\partial t}$ and $\boldsymbol{k}$ with the operation $-\mathrm{i}\frac{\partial}{\partial \boldsymbol{x}}$, we have

$$(\omega - \tilde{\omega}(\boldsymbol{k}))\,\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = \left(\mathrm{i}\frac{\partial}{\partial t} - \tilde{\omega}(-\mathrm{i}\frac{\partial}{\partial \boldsymbol{x}})\right)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = 0 \tag{131}$$

where the notation $\tilde{\omega}$ indicates the functional expression for the dispersion relation.

We may now assume that the wave field is characterized by a characteristic wave vector $\boldsymbol{k}_c$ and a characteristic angular frequency ω_c, and that modulation around the characteristic wave can be expressed in terms of slow scales, e.g.

$$\eta_1 = \frac{1}{2}B(\boldsymbol{x}_\delta, t_\delta)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_c\cdot\boldsymbol{x}-\omega_c t)} + \text{c.c.} \tag{132}$$

where B depends on some general slow scales $\boldsymbol{x}_\delta = \delta\boldsymbol{x}$ and $t_\delta = \delta t$ where $0 < \delta \ll 1$ is a small parameter characterizing the bandwidth of the spectrum of η, i.e. the modulation of η typically happens over length $\frac{2\pi}{\delta k}$ and time $\frac{2\pi}{\delta\omega}$.

Note: $\delta = \epsilon^2$ is required for arresting resonant growth at third order, see equation (129), $\delta = \epsilon$ is what typically makes mathematicians happy, while $\delta = \sqrt{\epsilon}$ is more appropriate for realistic ocean waves.

With this two-scale approach we have $\frac{\partial}{\partial \boldsymbol{x}} \to \frac{\partial}{\partial \boldsymbol{x}} + \delta\frac{\partial}{\partial \boldsymbol{x}_\delta}$ and $\frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \delta\frac{\partial}{\partial t_\delta}$. Similarly we may substitute $\boldsymbol{k} = \boldsymbol{k}_c + \Delta\boldsymbol{k}$ and $\omega = \omega_c + \Delta\omega$ in the dispersion relation, where the ratio $\Delta\omega/\omega_c = O(\delta) \ll 1$. We can therefore associate ω_c with the rapid derivative operation $\mathrm{i}\frac{\partial}{\partial t}$ and associate $\Delta\omega$ with the slow derivative operation $\mathrm{i}\delta\frac{\partial}{\partial t_\delta}$.

Writing

$$(\omega_c + \Delta\omega - \tilde{\omega}(\boldsymbol{k}_c + \Delta\boldsymbol{k}))\,B(\boldsymbol{x}_\delta, t_\delta)\mathrm{e}^{\mathrm{i}(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)} = 0 \tag{133}$$

the slow derivatives should act on the complex amplitude B thus giving an evolution equation for B

$$\left(\omega_c + \mathrm{i}\delta\frac{\partial}{\partial t_\delta} - \tilde{\omega}(\boldsymbol{k}_c - \mathrm{i}\delta\frac{\partial}{\partial \boldsymbol{x}_{\boldsymbol{\delta}}})\right)B = 0 \tag{134}$$

We can now do a power series expansion of the dispersion relation with respect to the small parameter δ and arrive at the linear Schrödinger equation

$$\mathrm{i}\frac{\partial B}{\partial t} + \mathrm{i}\boldsymbol{c}_g\cdot\frac{\partial B}{\partial \boldsymbol{x}} + \frac{1}{2}\delta\frac{\partial^2\omega}{\partial\boldsymbol{k}\partial\boldsymbol{k}} : \frac{\partial^2 B}{\partial\boldsymbol{x}\partial\boldsymbol{x}} + O(\delta^2) = 0 \tag{135}$$

where both the group velocity and the second derivative of the frequency are evaluated at the characteristic wave, and where we have been too lazy to write the slow derivatives.

2.6.1 Exercises

Show that if the characteristic wave is oriented in the x-direction, $\boldsymbol{k}_c = (k_c, 0)$, then the linear Schrödinger equation on deep water becomes

$$\frac{\partial B}{\partial t} + \frac{\omega_c}{2k_c}\frac{\partial B}{\partial x} + \frac{\mathrm{i}\omega_c}{8k_c^2}\frac{\partial^2 B}{\partial x^2} - \frac{\mathrm{i}\omega_c}{4k_c^2}\frac{\partial^2 B}{\partial y^2} = 0 \tag{136}$$

2.7 Cubic nonlinear Schrödinger (NLS) equation

Suppose we consider a generalization of the solution discussed above, of the form

$$\eta = \epsilon^2 \bar{\eta} + \frac{1}{2}\left\{B\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_c\cdot\boldsymbol{x}-\omega_c t)} + \epsilon B_2 \mathrm{e}^{2\mathrm{i}(\boldsymbol{k}_c\cdot\boldsymbol{x}-\omega_c t)} + \epsilon^2 B_3 \mathrm{e}^{3\mathrm{i}(\boldsymbol{k}_c\cdot\boldsymbol{x}-\omega_c t)} + \text{c.c.}\right\} \tag{137}$$

and a corresponding equation for the velocity potential,

$$\phi = \epsilon \bar{\phi} + \frac{1}{2}\left\{A\mathrm{e}^{\mathrm{i}(\boldsymbol{k}_c\cdot\boldsymbol{x}-\omega_c t)} + \epsilon A_2 \mathrm{e}^{2\mathrm{i}(\boldsymbol{k}_c\cdot\boldsymbol{x}-\omega_c t)} + \epsilon^2 A_3 \mathrm{e}^{3\mathrm{i}(\boldsymbol{k}_c\cdot\boldsymbol{x}-\omega_c t)} + \text{c.c.}\right\} \tag{138}$$

where B, B_2, B_3, A, A_2, A_3, $\bar{\eta}$ and $\bar{\phi}$ all depend on the slow horizontal and time scales $\boldsymbol{x}_1 = \epsilon\boldsymbol{x}$ and $t_1 = \epsilon t$ (this is the most typical choice for slow scales), and where A, A_2, A_3 and $\bar{\phi}$ in addition depend on the vertical scale.

In equations (137) and (138) the particular scaling for the first terms $\bar{\eta}$ and $\bar{\phi}$ are based on hindsight for the case of deep water, for small depth these terms will be bigger.

We expect that a solvability condition should appear at the third order to arrest unbounded resonant growth due to four-wave quartet resonances (the figure 8 of Phillips). This solvability condition will now be combined with the linear modulations described in section 2.6.

On deep water this solvability condition is the cubic nonlinear Schrödinger (NLS) equation

$$\frac{\partial B}{\partial t} + \frac{\omega_c}{2k_c}\frac{\partial B}{\partial x} + \frac{\mathrm{i}\omega_c}{8k_c^2}\frac{\partial^2 B}{\partial x^2} - \frac{\mathrm{i}\omega_c}{4k_c^2}\frac{\partial^2 B}{\partial y^2} + \frac{\mathrm{i}k_c^2\omega_c}{2}|B|^2 B = 0 \tag{139}$$

where we have been too lazy to write the slow scales with their index $_1$. There are two accompanying relations (for deep water)

$$B_2 = \frac{k_c}{2}B^2 \tag{140}$$

$$A_2 = 0, \tag{141}$$

and the other quantities B_3, A_3, $\bar{\eta}$ and $\bar{\phi}$ are too small to be considered within this truncation level.

2.7.1 Stokes wave

Assuming a solution uniform in space, with amplitude independent of the modulation scales in the x- and y-directions, we see by inspection that we have

$$B = B_0 \mathrm{e}^{-\frac{1}{2}\mathrm{i}\omega_c k_c^2 |B_0|^2 t} \tag{142}$$

which gives us the Stokes wave.

2.7.2 Benjamin–Feir instability

It can be shown that (142) is unstable to slow perturbations in space and time. This instability is a modulational instability known as the Benjamin–Feir instability.

We can carry out this instability analysis by assuming a small perturbation of the form

$$B = B_0(1 + \alpha + \mathrm{i}\beta)\mathrm{e}^{-\frac{1}{2}\mathrm{i}\omega_c k_c^2 |B_0|^2 t} \tag{143}$$

where the real quantities α and β are small perturbations.

After linearization in α and β we get the linear system that can be solved by assuming a plane wave solution

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} \mathrm{e}^{\mathrm{i}(\lambda x + \mu y - \Omega t)} + \text{c.c.} \tag{144}$$

The dispersion relation for the perturbation (the condition for nontrivial solution for $\hat{\alpha}$ and $\hat{\beta}$) is

$$\Omega = \frac{1}{2}\lambda \pm \sqrt{\left(\frac{1}{8}\lambda^2 - \frac{1}{4}\mu^2\right)\left(\frac{1}{8}\lambda^2 - \frac{1}{4}\mu^2 - |B_0|^2\right)} \tag{145}$$

There is instability when the radicand becomes negative. This can be shown to happen within a domain delimited by two straight lines crossing at the origin, $\lambda^2 - 2\mu^2 = 0$, and two branches of a hyperbola, $\lambda^2 - 2\mu^2 = 8|B_0|^2$.

The maximum growth rate occurs along another pair of branches of a hyperbola, $\lambda^2 - 2\mu^2 = 4|B_0|^2$. Notice that we have the same maximum growth rate along this entire hyperbola.

2.7.3 Other solutions of the NLS equation

The NLS equation has localized solutions that propagate with permanent shape, so-called solitons. Suppose we assume such a solution

$$B = f(x - \frac{1}{2}t)\mathrm{e}^{\alpha \mathrm{i} t} \tag{146}$$

Substituting into the NLS equation we get

$$\frac{1}{8}f'' + \alpha f + \frac{1}{2}f^3 = 0 \tag{147}$$

which has solutions of the form $f(\xi) = \gamma\,\mathrm{sech}(\beta\xi)$. A general solution is thus

$$B = B_0\,\mathrm{sech}\left(\sqrt{2}|B_0|(x - \frac{1}{2}t)\right)\mathrm{e}^{-\frac{\mathrm{i}}{4}|B_0|^2 t} \tag{148}$$

If we limit to one horizontal coordinate x, and perform a coordinate transformation

$$t' = -t \qquad x' = \sqrt{8}(x + \frac{1}{2}t)$$

we transform the NLS equation into the standard form

$$\mathrm{i}\frac{\partial B}{\partial t'} + \frac{\partial^2 B}{\partial x'^2} + 2B|B|^2 = 0. \tag{149}$$

A plane wave solution (the Stokes wave) and a soliton can now be written

$$B = \mathrm{e}^{2\mathrm{i}t'} \qquad \text{and} \qquad B = \frac{\mathrm{e}^{\mathrm{i}t'}}{\cosh x'}. \tag{150}$$

A special type of solutions are called "breathers" because they approach the plane wave solution when $t' \to \pm\infty$. One of the most famous breathers is known as the Peregrine-breather (Peregrine, 1983) given by

$$B = \mathrm{e}^{2\mathrm{i}t'}\left[1 - \frac{4(1+4\mathrm{i}t')}{1+4x'^2+16t'^2}\right] \tag{151}$$

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