

8th October, 2023

MEK4350 / MEK9350

Mandatory assignment 1 of 2

Submission deadline

Thursday 19th October 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with \LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Please look at figure 1. The blue curve in the middle is a time series of sea surface elevation recorded by a radar at Ekofisk on November 30th, 2018. This time series shows a phenomenon known as “Three sisters”, three consecutive waves that all satisfy the criterion for being rogue or freak waves. The forward and backward simulations assume long crested waves, which is certainly wrong. You may download the pdf figures [here](#).

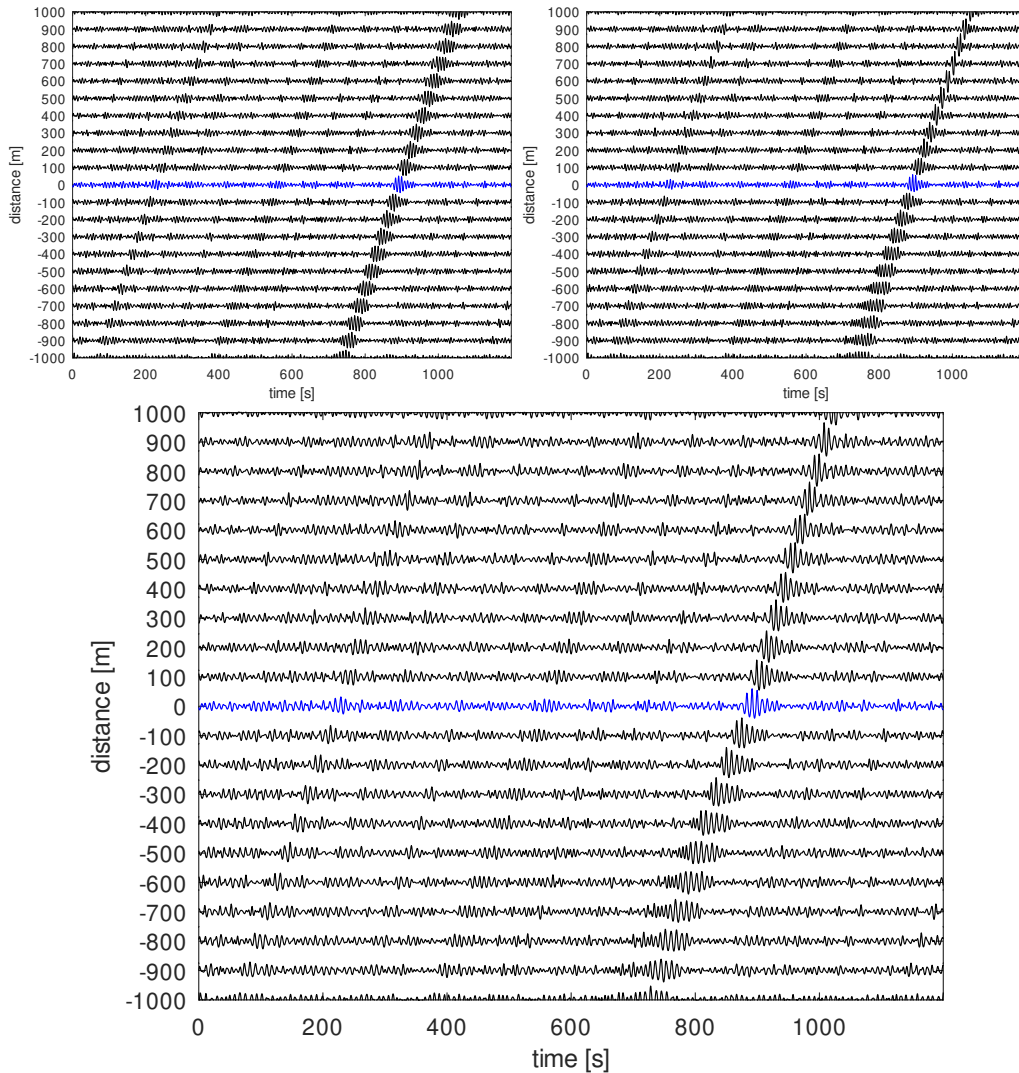


Figure 1: Recorded “Three sisters” time series (blue), with numerically computed propagation backward and forward in space (black) using three different models: Upper left: linear Schrödinger equation. Upper right: cubic nonlinear Schrödinger equation. Bottom: higher order nonlinear Schrödinger equation known as the Dysthe equation.

In order to appreciate how the wave field of the “Three sisters” time series might have appeared upstream and downstream of the location monitored by the radar, numerical simulations were carried out with three different types of Schrödinger equations for spatial evolution. These simulations assume that the waves were perfectly long-crested, and are therefore definitely not a reliable description of reality!

It is clear from the figure that wave modulations propagate with a characteristic velocity. This is the group velocity c_g . It should be possible to indicate this characteristic velocity by some parallel straight lines.

The data from our laboratory experiments on September 20th can be downloaded from the semester page. Please do the following:

1. Present our laboratory measurements in a similar fashion as in figure 1, showing the time series for all the probes at locations along the second axis corresponding to the locations of those probes.
2. Find the characteristic group velocity of the experimental waves.
3. Plot some parallel straight lines, with slope given by the characteristic group velocity, in order to check if the modulations indeed propagate with the group velocity. This may be easier to see for the irregular waves than for the Stokes waves?

In class we discussed the cubic nonlinear Schrödinger (NLS) equation for time evolution

$$\frac{\partial B}{\partial t} + \frac{\omega_c}{2k_c} \frac{\partial B}{\partial x} + \frac{i\omega_c}{8k_c^2} \frac{\partial^2 B}{\partial x^2} + \frac{ik_c^2 \omega_c}{2} |B|^2 B = 0 \quad (1)$$

where B is the complex amplitude of the waves, and the wave field itself can be represented approximately by

$$\eta(x, t) = \text{Re} \left\{ B(x, t) e^{i(k_c x - \omega_c t)} \right\} \quad (2)$$

In class we also discussed the modulational instability of Stokes waves, known as the Benjamin–Feir instability. Assuming a small perturbation of the form

$$B = B_0(1 + \alpha + i\beta) e^{-\frac{1}{2}i\omega_c k_c^2 |B_0|^2 t} \quad (3)$$

where the real quantities α and β are small perturbations, and assuming a plane wave solution

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \end{pmatrix} e^{i(\lambda x - \Omega t)} + \text{c.c.} \quad (4)$$

we found the dispersion relation for the perturbation

$$\Omega = \frac{1}{2}\lambda \pm \sqrt{\frac{1}{8}\lambda^2 \left(\frac{1}{8}\lambda^2 - |B_0|^2 \right)} \quad (5)$$

The unstable growth rate for time evolution of Stokes waves is given by the imaginary value $\text{Im}\Omega$.

We will need the corresponding unstable growth rate for space evolution. This can be obtained by first rewriting the NLS equation (1) into a form suitable for space evolution, and then carrying out the instability analysis for this space evolution NLS equation.

4. Start by assuming the last two terms in (1) are one order smaller than the first two terms and obtain the space evolution NLS equation by iteration.
5. Find the unstable growth rate for spatial evolution of Stokes waves.

Now we shall compare the analytical growth rate found above with the growth rate that can be computed from the laboratory measurements. The idealized model for unstable growth is

$$\hat{\eta}(x, \omega) = A(\omega)e^{(\mu_r(\omega)+i\mu_i(\omega))x} \quad (6)$$

where $\mu_r(\omega)$ is the growth rate for a given frequency ω , $\mu_i(\omega)$ is a phase shift which we do not care to know, $A(\omega)$ is a complex amplitude which we also do not care to know, and $\hat{\eta}(x, \omega)$ is the temporal Fourier transform of $\eta(x, t)$. With the time series measured at two locations x_1 and x_2 we can find the growth rate by

$$\mu_r(\omega) = \frac{1}{x_2 - x_1} \log \left| \frac{\hat{\eta}(x_2, \omega)}{\hat{\eta}(x_1, \omega)} \right| \quad (7)$$

6. Compute the Fourier transform of the experimental time series by `fft($\eta(\mathbf{t})$)` or `ifft($\eta(\mathbf{t})$)` on the computer, compute and plot the growth rate according to equation (7), and plot the analytical growth rate in the same plot for comparison.
7. Can you confirm that the result is better if you:
 - Employ two wave probes that are close to each other and close to the wave paddle?

- Employ a selection of the time series in a domain delimited by the parallel straight lines with slope given by the characteristic group velocity, as suggested at the beginning of this exercise?

Finally, an optional problem for those who may already be familiar with the computation of the power spectrum:

8. In order that the power spectral density $S(x, \omega)$ looks nice and smooth, some amount of smoothing is often applied by default or by request in the routine that estimates $S(x, \omega)$. This is sometimes done by the Matlab function `pwelch`. Can you confirm that such smoothing actually ruins the estimation of the unstable growth rate?