

BUBBLE DYNAMICS AND CAVITATION

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1 INTRODUCTION

The first analysis of a problem in cavitation and bubble dynamics was made by Rayleigh (1917), who solved the problem of the collapse of an empty cavity in a large mass of liquid. Rayleigh also considered in this same paper the problem of a gas-filled cavity under the assumption that the gas undergoes isothermal compression. His interest in these problems presumably arose from concern with cavitation and cavitation damage. With neglect of surface tension and liquid viscosity and with the assumption of liquid incompressibility, Rayleigh showed from the momentum equation that the bubble boundary $R(t)$ obeyed the relation

$$R\ddot{R} + \frac{3}{2}(\dot{R})^2 = \frac{p(R) - p_\infty}{\rho}, \quad (1.1)$$

where ρ is the liquid density, p_∞ is the pressure in the liquid at a large distance from the bubble, and $p(R)$ is the pressure in the liquid at the bubble boundary. For this Rayleigh problem, $p(R)$ is also the pressure within the bubble. Incompressibility of the liquid means that the liquid velocity at a distance r from the bubble center is

$$u(r, t) = \frac{R^2}{r^2} \dot{R}. \quad (1.2)$$

The pressure in the liquid is readily found from the general Bernoulli equation to be

$$p(r, t) = p_\infty + \frac{R}{r} [p(R) - p_\infty] + \frac{1}{2} \rho \frac{R}{r} \dot{R}^2 \left[1 - \left(\frac{R}{r} \right)^3 \right]. \quad (1.3)$$

While Rayleigh neglected surface tension and liquid viscosity and kept the pressure p_∞ constant, his dynamical equation (1.1) is easily extended to include these effects.

For a spherical bubble, viscosity affects only the boundary condition so that it becomes

$$p(R) = p_i - \frac{2\sigma}{R} - \frac{4\mu}{R} \dot{R}, \quad (1.4)$$

where now p_i is the pressure in the bubble and $p(R)$, as before, is the pressure in the liquid at the bubble boundary. The surface-tension constant and the coefficient of the liquid viscosity are σ and μ , respectively. By allowing p_∞ to be a function of time, one can use Equation (1.1) to describe the experimental observations on cavitation-bubble growth and collapse in a liquid flow (Plesset 1949). Other effects not considered by Rayleigh, such as the stability of the interface, the compressibility of the liquid, the effect of energy flow into or out of the bubble, and the physical conditions within the bubble, are described in the sections that follow.

We may write here a generalized Rayleigh equation for bubble dynamics, in view of Equation (1.4), as

$$R\ddot{R} + \frac{3}{2}(\dot{R})^2 = \frac{1}{\rho} \left\{ p_i - p_\infty - \frac{2\sigma}{R} - \frac{4\mu}{R} \dot{R} \right\}, \quad (1.5)$$

where the pressure in the gas at the bubble wall, p_i , may be a function of the time, and the pressure at infinity, p_∞ , may also be a function of the time. We may also, for future reference, define an equilibrium radius of a gas nucleus for given values of p_i and p_∞ :

$$R_0 = \frac{2\sigma}{p_i - p_\infty}. \quad (1.6)$$

A bubble of this radius will clearly remain at rest if it is initially at rest, although it should be noted that this equilibrium is an unstable one.

The discussion of bubble dynamics divides itself in a natural way on the one hand into those situations in which the bubble interior consists for the most part of permanent gas and on the other hand into those situations in which the bubble interior is composed almost entirely of the vapor of the surrounding liquid. Vapor-bubble dynamics can be of ten simplified by a further subdivision into vapor-bubble dynamics in a subcooled liquid and into vapor-bubble dynamics in a superheated liquid (Plesset 1957). The subcooled-liquid case corresponds to that in which the vapor density is so small that latent heat flow does not affect the motion which is then controlled by the inertia of the liquid. In this sense the liquid may be said to be "cold," and the liquid is usually described as a cavitating liquid. The superheated liquid in a similar way may be described as one in which boiling phenomena occur. In boiling, the vapor-bubble dynamics is controlled by the latent heat flow rather than by the liquid inertia (Plesset 1969). Here the coupling of the energy equation with the momentum equation is essential. The case of the gas bubble also requires in most cases the simultaneous consideration of the momentum equation and the energy equation. A large body of literature has developed on this topic, which is considered in the following section.

2 GAS BUBBLES

We consider in this section the case in which the medium filling the cavity is essentially a permanent, noncondensable gas. We neglect all the effects associated with the vapor of the liquid which necessarily is present in the bubble together with the gas. Clearly this procedure is legitimate as long as the partial pressure of the vapor is small compared with the gas pressure.

Small-Amplitude Oscillations

The small-amplitude (linearized) oscillations of a permanent gas bubble in a liquid were first considered by Minnaert (1933), who showed that a substantial contribution to the sound emitted by running water comes from the free radial pulsations of entrained air bubbles. More recently the same type of problem has been considered in greater detail by several investigators including Devin (1959), Plesset & Hsieh (1960), Chapman & Plesset (1971), and Prosperetti (1976a).

In many instances, the case of small-amplitude forced radial oscillations arises when a bubble is immersed in a sound field of wavelength large compared with the bubble radius. Such a sound field may be introduced as an approximation in the Rayleigh equation by writing

$$p_\infty(t) = P_\infty(1 + \varepsilon \cos \omega t), \tag{2.1}$$

where P_∞ is the average ambient pressure, ω is the sound frequency, and ε is the dimensionless amplitude of the pressure perturbation which is supposed to be small. It is also assumed that the oscillations take place about the equilibrium radius R_0 defined by Equation (1.6), so that one may write

$$R = R_0[1 + x(t)], \tag{2.2}$$

with $x(t)$ a small quantity of order ε . Finally, one must introduce a suitable linearization of the internal pressure $p_i(t)$. Denoting by a dot differentiation with respect to time, we write

$$p_i(t) = p_{i,eq} + \left. \frac{\partial p_i}{\partial x} \right|_{x=0, \dot{x}=0} x(t) + \left. \frac{\partial p_i}{\partial \dot{x}} \right|_{x=0, \dot{x}=0} \dot{x}(t) + \dots, \tag{2.3}$$

where $\left. \partial p_i / \partial x \right|_{x=0, \dot{x}=0}$ represents the component of the internal pressure perturbation in phase with the driving force, and $\left. \partial p_i / \partial \dot{x} \right|_{x=0, \dot{x}=0}$ the component out of phase with it by $\pi/2$. Upon substitution of relations (2.1), (2.2), and (2.3) into Equation (1.5), one finds an equation of the harmonic-oscillator form

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = -\varepsilon\alpha e^{i\omega t}, \tag{2.4}$$

where the constant α is defined by $\alpha = P_\infty / \rho R_0^2$, and the damping constant β and the effective natural frequency ω_0 are given formally by

$$\beta = \frac{2\mu}{\rho R_0^2} - \frac{1}{2\rho} \left. \frac{\partial p_i}{\partial \dot{x}} \right|_{x=0, \dot{x}=0} \tag{2.5}$$

and

$$\omega_0^2 = - \frac{1}{\rho} \frac{\partial p_i}{\partial x} \Big|_{x=0, \dot{x}=0} - \frac{2\sigma}{\rho R_0^3}. \quad (2.6)$$

Further progress is clearly dependent upon the specification of the quantities appearing in Equation (2.3). A simple assumption is that the gas follows a polytropic law of compression with polytropic exponent κ

$$p_i = p_{i,eq} \left(\frac{R_0}{R} \right)^{3\kappa}. \quad (2.7)$$

If this assumption were strictly valid, one would find that energy dissipation arises only from liquid viscosity and compressibility (see below). With this assumption the natural frequency is

$$\omega_0^2 = 3\kappa \frac{p_{i,eq}}{\rho R_0^2} - \frac{2\sigma}{\rho R_0^3}. \quad (2.8)$$

If surface-tension effects are neglected, and if the pressure-volume relationship is taken to be adiabatic so that κ equals γ , the ratio of the specific heats of the gas, Equation (2.8) coincides with the result first obtained by Minnaert (1933).

Clearly, the accuracy of Equation (2.7) can be assessed only by considering the complete set of linearized conservation equations of mass, momentum, and energy both in the gas and in the liquid. An analysis of this type was first undertaken by Pfriem (1940; see also Devin 1959), who made use of a somewhat artificial Lagrangian formalism. An explicit treatment in terms of the conservation equations of continuum mechanics has been given by Plesset & Hsieh (1960) and more recently by Prosperetti (1976a). The case of the free oscillations has been analyzed by Chapman & Plesset (1971).

The results of these studies are two-fold. In the first place, it is found that the effective polytropic exponent κ exhibits a strong dependence on the driving sound frequency ω . For the case $\gamma = 7/5$ (diatomic gas) this dependence is illustrated in Figure 1 in terms of the dimensionless frequency $G_1 = MD_g\omega/\gamma R_g T_\infty$ and of another dimensionless parameter $G_2 = R_0^2\omega/D_g$. Here D_g denotes the thermal diffusivity of the gas, M its molecular weight, R_g the universal gas constant, and T_∞ the absolute temperature of the liquid at a distance from the bubble. The physical basis for the behavior depicted in Figure 1 can be clarified by noting that essentially three length scales are involved in the problem, namely, the bubble radius R_0 , the wavelength of sound in the gas $\lambda_g = 2\pi(\gamma R_g T_\infty/M)^{1/2}/\omega$, and the thermal penetration depth in the gas, $L_{th} = (D_g\omega)^{1/2}$. The thermal penetration depth in the liquid is so small that in practice it can be taken to be zero with a negligible error. In terms of these three fundamental length scales, it is seen that, essentially, $G_1 \sim (L_{th}/\lambda_g)^2$ and $G_2 \sim (R_0/L_{th})^2$. If $(G_1 G_2)^{1/2}$ (i.e. R_0/λ_g) is small, the pressure within the bubble is spatially uniform, and one will observe an isothermal behavior, $\kappa \cong 1$ when $R_0 \ll L_{th}$ (i.e. G_2 is small); and one will observe an adiabatic behavior $\kappa \cong \gamma$ when $R_0 \gg L_{th}$ (i.e. G_2 is large). In the first case, the oscillations are too slow to maintain an appreciable temperature gradient in the bubble, whereas in the second case they are

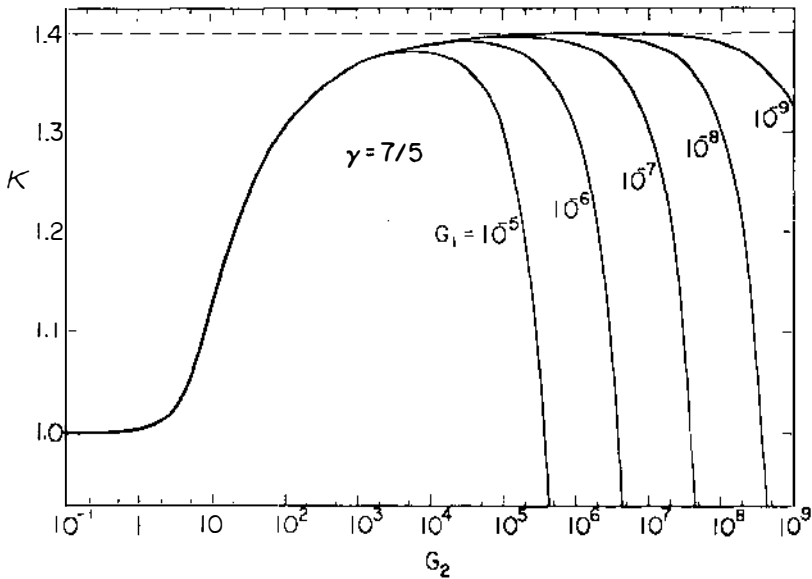


Figure 1 The effective polytropic exponent κ for the small-amplitude forced oscillations of a gas bubble containing a diatomic gas ($\gamma = 7/5$). The numbers labeling the curves denote different values of the dimensionless frequency $G_1 = MD_g\omega/\gamma R_g T_\infty$. The quantity G_2 is defined by $G_2 = \omega R_0^2/D_g$ (from Prosperetti 1976a).

so fast that most of the gas contained in the bubble is practically thermally insulated from the liquid. However, as was first pointed out by Plesset (1964) and by Plesset & Hsieh (1960), for frequencies so large that λ_g is of the order of R_0 or smaller [i.e. $(G_1 G_2)^{1/2}$ is of order 1 or larger], pressure nonuniformities develop in the bubble and a polytropic pressure-volume relationship loses its thermodynamic meaning. As an associated effect, the polytropic exponent κ takes on values outside the range $1 < \kappa < \gamma$, and may become even negative (see Prosperetti 1976a). The interesting question of the thermodynamic behavior at high frequency of the bubble as a whole, however, is still meaningful. As discussed by Plesset (1964) and Prosperetti (1976a), one is led to the conclusion of an overall isothermal behavior, caused by the establishment in the interior of the bubble of a temperature distribution consisting of many standing waves of short wavelength.¹ In view of Equation (2.8), the frequency dependence of the polytropic exponent reflects itself in a frequency dependence of the effective natural frequency, in the sense that the pole of the oscillation amplitude determined from Equation (2.4) is a function of the driving frequency. The frequency at which resonance oscillations take place does remain well defined (see Prosperetti 1976a).

¹ Clearly, at very high frequencies one cannot consider the wavelength of the sound in the liquid large compared with the bubble radius. The bubble oscillations then will not be purely radial, but the physical content of this statement remains nevertheless applicable.

The second important result obtained by the analysis of the complete (linearized) fluid-mechanical problem concerns the damping of the radial oscillations. Figure 2 presents the results of Chapman & Plesset (1971) for the logarithmic decrement $\Lambda = 2\pi\beta/\omega_0$ of an air bubble in water. It is seen that, in the range $0.1 \text{ cm} > R_0 > 4 \times 10^{-4} \text{ cm}$, the thermal component, represented formally by the second term in Equation (2.5), dominates the viscous and acoustic contributions to the energy dissipation. The acoustic contribution can be computed by attributing a slight compressibility to the liquid, and the result is the addition of

$$\beta_{\text{acoustic}} = \frac{1}{2}\omega \frac{\omega R_0/c}{1 + (\omega R_0/c)^2} \approx \frac{1}{2} \left(\frac{\omega R_0}{c} \right) \omega \quad (2.9)$$

to the damping coefficient of Equation (2.5). This quantity determines the fraction of the work performed by the sound field on the bubble that is dissipated as sound waves radiated into the liquid. Results for the case of forced oscillations of an air bubble in water are shown in Figure 3 for two values of the equilibrium radius R_0 (the frequency corresponding to the resonant frequency of the bubble is indicated by an open circle in these figures). Except for extremely small bubbles for which viscosity is very important, the low-frequency damping is dominated by thermal effects, and the high-frequency damping by acoustic effects. The analytic expression for the thermal damping constant is rather involved and is not given here; the reader is referred to Prosperetti (1976a).

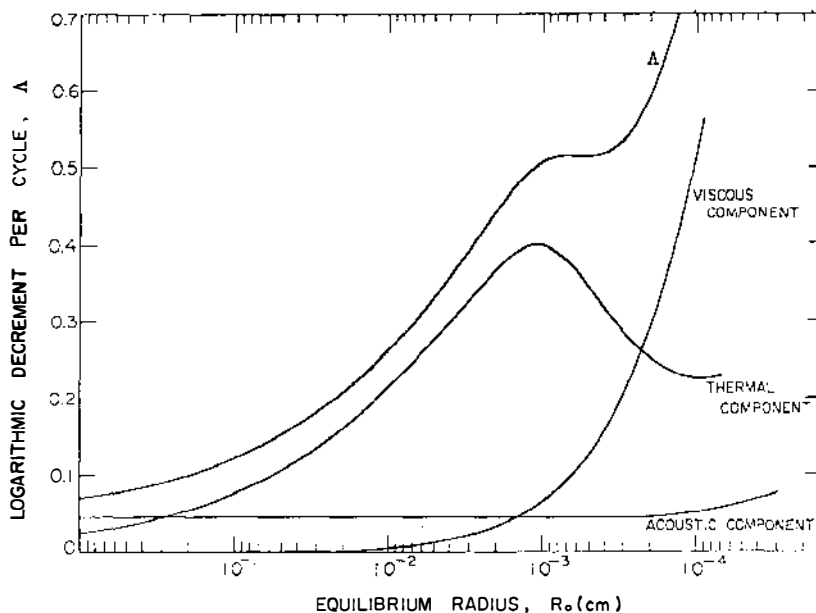


Figure 2 The logarithmic decrement for small-amplitude free oscillations of an air bubble in water as a function of the equilibrium radius R_0 (from Chapman & Plesset 1971).

As a final point, we may remark that Plesset & Hsieh (1960) have considered the small-amplitude oscillations as an initial-value problem and have shown that the solution does indeed approach the solution of the steady-state problem represented by Equation (2.4) as $t \rightarrow \infty$.

Considerable experimental work on gas-bubble oscillations has been reported in the literature. The data for the natural frequency can be said to agree well with the theory (Lauer 1951, Koger & Houghton 1968, Jensen 1974), whereas a large scatter appears in the data for the damping of the oscillations, as can be seen for example in Figure 2 of Koger & Houghton (1968). While this situation may certainly be ascribed to a large extent to the quality of the data (for discussions, see e.g. Devin 1959, Kapustina 1970, van Wijngaarden 1972), it appears that the situation has not yet been completely clarified. Some references to the original works are contained in the papers referenced above; in a recent work Ceschia & Iernetti (1974) made use of the subharmonic threshold to measure the damping of the oscillations and report acceptable agreement with theory for bubbles of different gases in water. An exception is hydrogen, for which anomalous behavior is reported. A different use of the subharmonic threshold, which appears capable of yielding very accurate data, has recently been proposed by Prosperetti (1976b).

Nonlinear Oscillations

With the assumption of polytropic behavior and with the neglect of thermal and acoustic dissipation, the dynamical Equation (1.5) takes the following form in an oscillating pressure field :

$$R\ddot{R} + \frac{3}{2}(\dot{R})^2 = \frac{1}{\rho} \left[p_{i,eq} \left(\frac{R_0}{R} \right)^{3\kappa} - P_\infty (1 - \eta \cos \omega t) - \frac{2\sigma}{R} - 4\mu \frac{1}{R} \dot{R} \right], \quad (2.10)$$

where, in general, the dimensionless pressure amplitude η is not necessarily small. The numerical studies that can be found in the literature have shown the extreme richness of this equation which appears to lie beyond the capabilities of the available analytical techniques.

Among the earliest numerical work, the contribution of Noltingk & Neppiras (1950, 1951) should be mentioned. With the hypothesis of isothermal behavior ($\kappa = 1$) and with the neglect of viscous effects they were able to show the apparently explosive behavior of the solutions of Equation (2.10) which are sometimes found to consist of a very rapid growth followed by a violent collapse to very small values of the radius within a single period of the driving force. Basing themselves on these numerical results, they deduced some conclusions regarding the mechanism and the conditions for acoustic cavitation that have been used as a guide in much subsequent research on this subject.

Noltingk and Neppiras's work was somewhat limited by the capabilities of the computer at their disposal, but later Flynn (1964) and Borotnikova & Soloukin (1964) published several radius-versus-time curves that illustrate the complexity of the possible responses and their sensitivity to the parameters of the problem. Figure 4, from Borotnikova & Soloukin (1964), shows that the explosive behavior can be observed after several oscillations, as had already been conjectured by Willard (1953),

both for bubbles driven below and above resonance. These examples give an indication of the importance of the initial conditions in the subsequent bubble motion, an area in which little research has been conducted.

The most extensive numerical investigation of Equation (2.10) is that undertaken by Lauterborn (1968, 1970a, 1970b, 1976), who has collected an impressive amount of results, still in part unpublished. Figure 5 from his 1976 paper illustrates the steady-state response $X_M = (R_{max} - R_0)/R_0$ (where R_{max} is the maximum value of the radius during the steady oscillations) as a function of the ratio ω/ω_0 for several values of the dimensionless pressure amplitude η . Here ω_0 is the resonant frequency for the linearized oscillations given by Equation (2.8). These calculations refer to an air bubble of radius $R_0 = 10^{-3}$ cm in water under a static pressure $P_\infty = 1$ bar. The complicated pattern of the resonances that gradually unfolds as the pressure amplitude is increased indicates very clearly the importance of nonlinear effects for large-amplitude bubble oscillations.

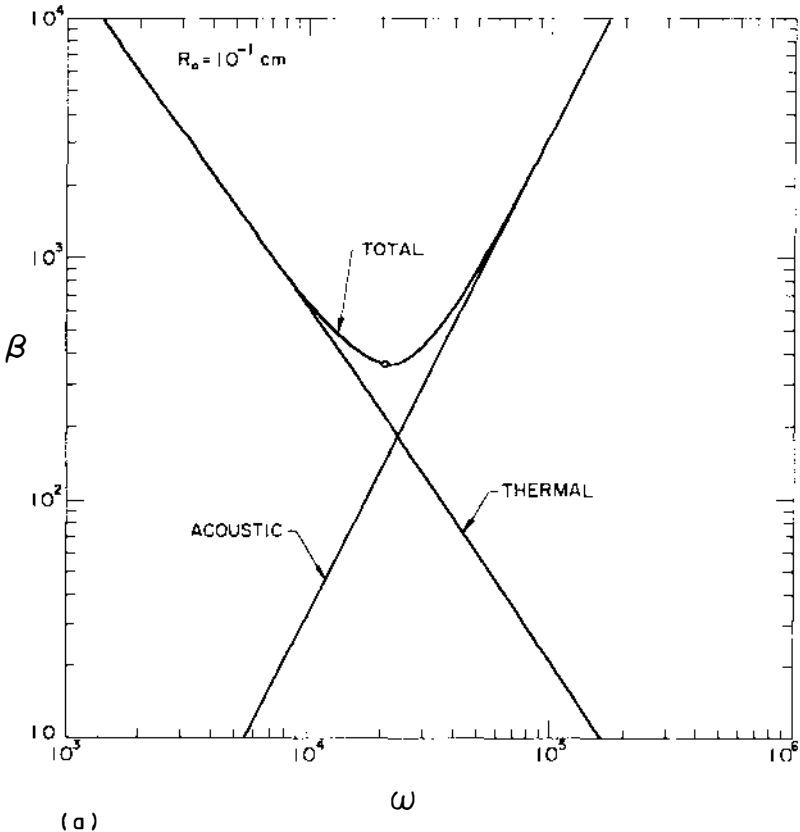
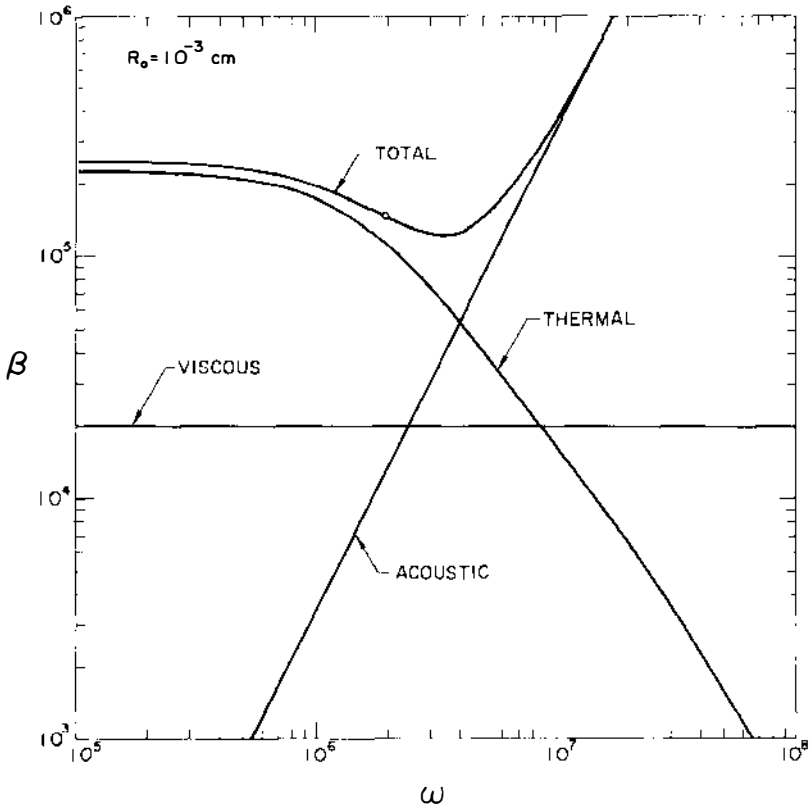


Figure 3 The damping coefficient β for the small-amplitude forced oscillations of

In Figure 5 the broken vertical lines represent the points at which the steady-state response is observed to jump from some low value to a higher one. This behavior is caused by an instability of the small-amplitude oscillation in a frequency region in which the response curve is really multivalued (see Figure 6). This behavior was found analytically by Prosperetti (1974, 1975, 1976b), who gave a description of small-amplitude forced oscillations including some nonlinear effects. Although his theory is quantitatively accurate only for relatively small values of η [$\eta \lesssim 0.25(1 + 2\sigma/R_0 P_\infty)$] and can account for only a few of the resonances of the system, it gives an insight into the complicated effects of the initial conditions on the ensuing oscillatory motion. An example of his results is shown in Figure 7, which depicts the behavior of the first subharmonic amplitudes $u(t), v(t)$ [with $u(t) \cos \frac{1}{2}\omega t - v(t) \sin \frac{1}{2}\omega t$, the subharmonic component in the oscillation] in the (u, v) plane for three different sets of initial conditions. The numbers along the curves represent time in units of ωt , and the dashed lines are the separatrices for the undamped forced oscillations (see



(b)

an air bubble in water as a function of the driving frequency (from Prosperetti 1976a).

Prosperetti 1975 for details). It is seen that it is not possible to express in a simple way the relation between the initial conditions and the value of the amplitudes in the steady-state oscillations (which is the point into which the curves spiral: the origin for curves *a* and *b*, and a nonzero value for curve *c*). One must expect that a much more complicated pattern of this type would be applicable to all the resonances of the complete Equation (2.10), particularly for large-amplitude oscillations for which a small change in the initial conditions can cause markedly different transients and steady-state motions. Such a behavior is indeed reported in the numerical studies by Lauterborn (1976).

In addition to furnishing a suitable example to illustrate the importance of initial conditions, subharmonic bubble oscillations are interesting per se in view of the apparent connection between a signal at half the frequency of the sound field and the development of cavitation in an acoustically irradiated liquid. A subharmonic component in the spectrum of a liquid undergoing acoustic cavitation was first reported by Esche (1952), and later by Negishi (1961). Subsequently De Santis et al (1967), Vaughan (1968), Neppiras (1969a,b), Mosse & Finch (1971), and Coakley (1971) established an apparent causal connection between the onset of cavitation and the appearance of the subharmonic signal. Neppiras (1969a,b), experimenting with single bubbles, showed that they were indeed capable of emitting at a frequency equal to one half of the driving frequency, and advanced the hypothesis that these

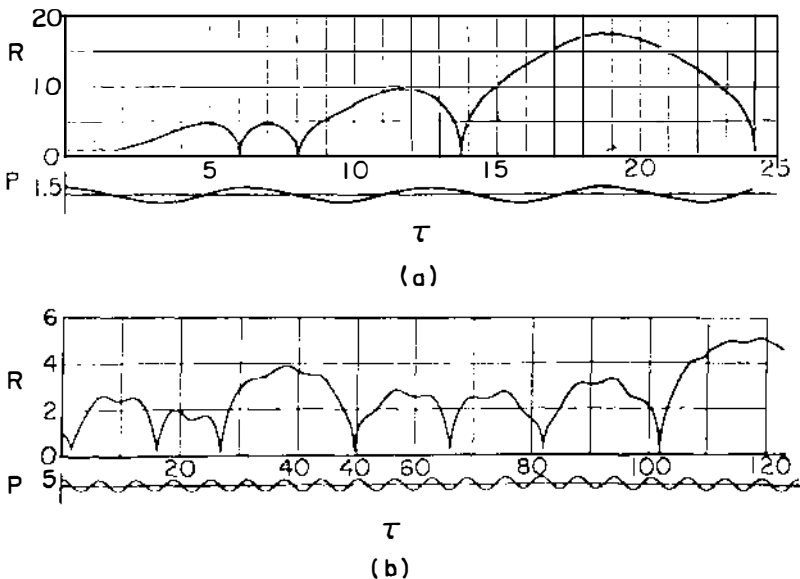


Figure 4 Two examples of numerical results for the transient motion of a gas bubble in an oscillating pressure field. For Figure 4a the conditions are $\eta = 1.5$, $\omega/\omega_0 = 0.154$; for Figure 4b $\eta = 5$, $\omega/\omega_0 = 1.54$. The ordinate scale is the dimensionless radius R/R_0 ; the abscissa scale is the dimensionless time ωt (from Borotnikova & Soloukin 1964).

obtained an exact solution of Equation (2.10) in terms of elliptic integrals. Nonlinear effects on the small-amplitude motion have been analyzed by Prosperetti (1975), who included surface tension and dissipation. Lauterborn (1968, 1976) reported extensive numerical results on large-amplitude free oscillations including viscous dissipation.

It should be pointed out that in all the work on nonlinear oscillations mentioned above no detailed analysis of the gas contained in the bubble was attempted. There-

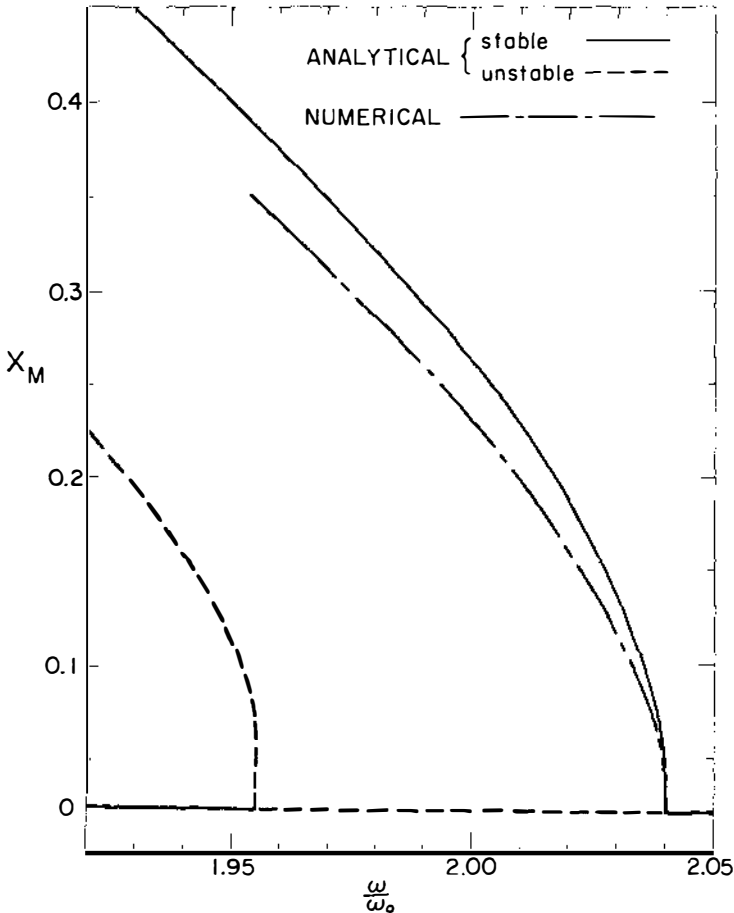


Figure 6 The response curve for the steady subharmonic oscillations of an air bubble of equilibrium radius $R_0 = 10^{-3}$ cm in water. The pressure amplitude is $\eta = 0.3$, and the ambient pressure $P_\infty = 1$ bar. Only viscous damping is taken into account, and the polytropic exponent is $\kappa = 1.33$. The numerical results shown have been obtained by Lauterborn (1974) (from Prosperetti 1974).

fore, all the results described here have been derived with the assumption of a polytropic relationship and sometimes with an assumption of an effective viscosity to account in an approximate way for the effect of the thermal and acoustic dissipation processes. Very recently Flynn (1975a) has derived a somewhat complex formulation of the general problem of cavitation dynamics that includes a detailed analysis of the gas behavior. This formulation has been applied by him to the study

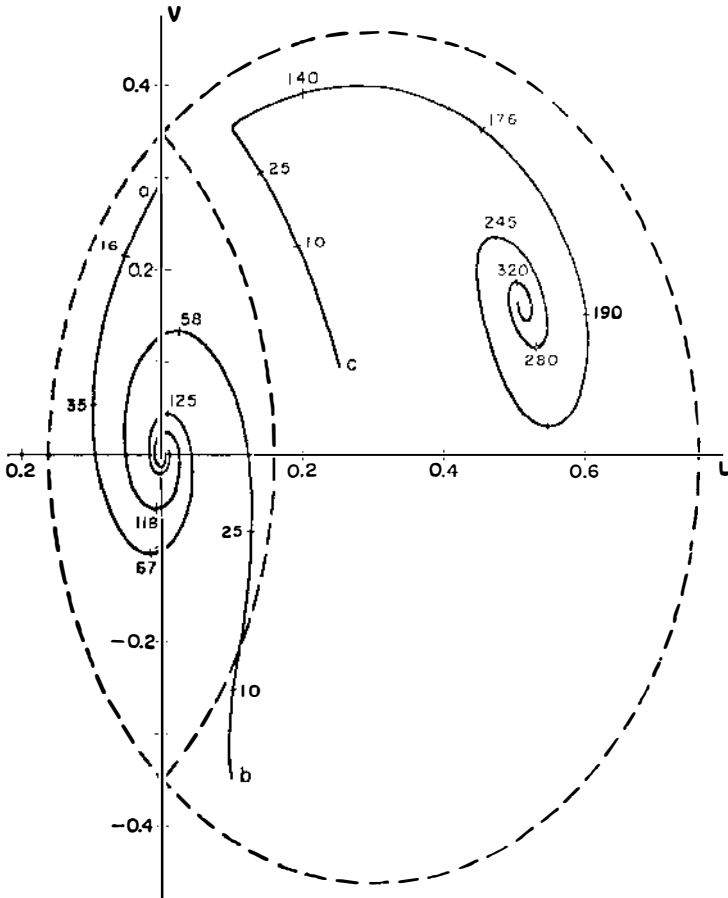


Figure 7 Phase-plane behavior of the transient subharmonic amplitudes u, v [where $u(t) \cos \frac{1}{2}\omega t - v(t) \sin \frac{1}{2}\omega t$ is the subharmonic component of the bubble oscillations] for different initial conditions. The driving frequency is $\omega = 1.8\omega_0$, the pressure amplitude $\eta = 0.4$, the ambient pressure $P_\infty = 1$ bar, and the polytropic exponent $\kappa = 1.33$. The numbers on the curves denote time in units of ωt . The steady oscillations will contain a subharmonic component for the case of the curve labeled c , but not for curves a or b (from Prosperetti 1975).

of large-amplitude free oscillations, and some interesting results on the energy dissipation of the cavity have been obtained (Flynn 1975b).

For large bubbles, such as those produced by underwater explosions, the most significant damping mechanism is acoustic energy radiation in the liquid, and therefore Equation (2.10) cannot be used. For this extreme case, Keller & Kolodner (1956) have obtained another equation that accounts for liquid compressibility in large-amplitude free oscillations. These results have recently been completed and extended to the case of plane and cylindrical bubbles by Epstein & Keller (1972). Other studies of compressibility effects in the motion of gas bubbles are discussed below.

Mass-Diffusion Effects

The mass-diffusion processes taking place across the bubble-liquid interface play a significant role in the behavior of gas bubbles because they may ultimately determine the presence or absence of bubbles in a liquid. The key to these processes is furnished by Henry's law, which establishes a connection between the partial pressure of a gas acting on a liquid surface, p_g , and the equilibrium (or saturation) concentration of gas in the liquid which we denote by c_s :

$$c_s = ap_g \tag{2.11}$$

Here a is a constant characteristic of the particular gas-liquid combination and is primarily a function of temperature. Equation (2.11) is valid also at a nonplanar interface.

If we consider first a situation in which the ambient pressure is fixed and equal to P_∞ , then it is clear that unless the gas concentration c at the bubble surface satisfies Equation (2.11) (where we now interpret p_g as the gas pressure in the bubble and neglect for simplicity its vapor content), the bubble will not be in equilibrium, and it will either grow or shrink according to whether $c > c_s$ or $c < c_s$. The mathematical formulation for this process clearly consists of the diffusion equation in the liquid

$$\frac{\partial c}{\partial t} + \frac{R^2}{r^2} \dot{R} \frac{\partial c}{\partial r} = \frac{\alpha}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial c}{\partial r} \right), \tag{2.12}$$

where α is the coefficient of diffusion of the gas in the liquid, subject to the boundary condition (2.11) at the (moving) bubble surface. Strictly speaking, the pressure p_g in (2.11) should be determined from the dynamical Rayleigh equation. The problem can be substantially simplified by observing that the bubble-wall velocities that are induced by mass diffusion alone are usually quite small in view of the relatively low value of α .² Indeed, a reasoning based on purely dimensional considerations suggests that

$$R\dot{R} = \alpha \frac{c_\infty - c_s}{\rho_g}, \tag{2.13}$$

² For air-water at 20°C, for example, $\alpha \approx 2 \times 10^{-5} \text{ cm}^2 \text{ sec}^{-1}$. An order-of-magnitude estimate for this quantity is given by the familiar result due to Einstein, $\alpha = 6\pi\mu a k_B T$ where k_B is the Boltzmann constant, and a is an equivalent radius of the gas molecule.

where c_∞ is the dissolved mass concentration at a large distance from the bubble and ρ_g is the density of the gas within the bubble. For typical values, one might take $(c_\infty - c_s)/\rho_g \sim 10^{-2}$, $R_0 \sim 10^{-3}$ cm; Equation (2.13) then gives an estimate of $\dot{R} \sim 10^{-4}$ cm sec $^{-1}$ for the growth (or dissolution) velocity of an air bubble in water. These very low values allow one to disregard the inertial effects associated with the bubble motion and to neglect the convective term in Equation (2.12).

An approximate solution based on these simplifications was obtained some time ago by Epstein & Plesset (1950), who included surface-tension effects. Their result, with the neglect of the latter, takes the form

$$R\dot{R} = \alpha \frac{c_\infty - c_s}{\rho_g} [1 + R(\pi\alpha t)^{-1/2}]. \quad (2.14)$$

The second term in this equation reflects the transient effect of the buildup of a diffusion boundary layer adjacent to the bubble surface that quickly becomes large compared with the bubble itself. Indeed, from (2.13) we find that, asymptotically, $R \sim [2\alpha(c_\infty - c_s)t/\rho_g]^{1/2}$ from which $R(\pi\alpha t)^{-1/2} \sim [2(c_\infty - c_s)/\pi\rho_g]^{1/2} \ll 1$. The physical meaning of the result of Equation (2.14) can be clarified by noting that if m_g denotes the gas content in the bubble so that $dm_g/dt = 4\pi R^2 \rho_g dR/dt$, elimination of dR/dt formally reduces it to the expression for the heat flux at the surface of a perfectly conducting sphere in an infinite medium (Carslaw & Jaeger 1959). Experimental evidence in favor of Equation (2.14) was reported by Krieger et al (1967), who proposed a method for the measurement of diffusion coefficients of gases in liquids based on the observation of bubble-dissolution rates. A systematic perturbation approach to the solution of the problem has been formulated by Duda & Vrentas (1969a,b), and an interesting treatment of the case of more than one gas has recently been given by Ward & Tucker (1975). Birkhoff et al (1958) have obtained a similarity solution for the complete problem including convection effects, and the same solution was obtained by Scriven (1959). Szekely and co-workers (1971, 1973) have given a detailed analysis that includes dynamic, viscous, and surface-kinetics effects.

The physical situation is quite different if the bubble is immersed in an oscillating pressure field, and it is readily seen that in this case it may grow even in an undersaturated solution. The mechanism giving rise to this effect may be explained in physical terms in the following way. Suppose that the amplitude of the forced oscillations is large enough so that the gas-liquid solution at the bubble surface becomes undersaturated during the compression half-cycle and supersaturated during the expansion half-cycle. Corresponding to these conditions there will be a mass exchange of alternating direction between the bubble and the liquid. If the interface were plane, the average flux over one oscillation would clearly be zero. However, as a consequence of the spherical geometry, on the average the surface area during the mass inflow is greater than during the mass outflow, so that a net increase in the mass of gas contained in the bubble results. In view of its second-order nature, this effect has been called "rectified mass diffusion" (Blake 1949). An additional consequence of the spherical geometry, however, combines with the area effect just mentioned, namely, the fact that during the expansion half-cycle the thickness of the diffusion layer adjacent to the bubble surface decreases, while it

increases during the compression half-cycle. As a result concentration gradients (and hence mass-flow rates) during the mass inflow half-cycle are enhanced over those prevailing during the mass outflow part of the oscillation. A quantitative analysis of the convective effect shows it to be dominant, as was demonstrated by Hsieh & Plesset (1961), who gave the first rigorous treatment of the rectified diffusion process based on a linearized formulation of the problem. For the case in which the liquid is saturated in the absence of the bubble, neglecting surface-tension effects and assuming isothermal behavior, they find

$$R\dot{R} = \frac{2}{3}\alpha \frac{c_s}{\rho_g} \left(\frac{P_{\max} - P_0}{P_0} \right)^2, \quad (2.15)$$

where P_{\max} and P_0 are the maximum and the average value of the bubble internal pressure, respectively.

If P_{\max} is taken to be a constant, Equation (2.15) is readily integrated and for large times predicts a $t^{1/2}$ dependence of the bubble radius. The fact that in practice bubbles are not seen to grow indefinitely, as this result would imply, is a consequence of the instability of the spherical shape, which, as is discussed below, is an important effect for relatively large bubbles (Hsieh & Plesset 1961). A significant feature of Equation (2.15) is that it shows bubble growth by rectified diffusion to be a very slow process relative to the period of the sound fields commonly encountered. For instance, for an air bubble in water at 20°C at one atmosphere, with $(P_{\max} - P_0)/P_0 = 0.25$, Equation (2.15) predicts a doubling time that ranges from 1.1×10^6 sec for an initial radius of 10^{-1} cm to 1.1×10^2 sec for an initial radius of 10^{-3} cm.

When the liquid is not saturated, the bubble will eventually disappear if the mass flux caused by rectified mass diffusion is not sufficient to balance the loss of mass required by Henry's law. The value, η_{th} , of the pressure amplitude at which the two fluxes are equal corresponds to the threshold for the growth of the bubble by rectified diffusion, and was estimated very simply by Strasberg (1961) by equating Equation (2.13) and Equation (2.15). His result, which includes surface-tension effects, is

$$\eta_{th} = \left(\frac{3}{2}\right)^{1/2} \left(1 + \frac{2\sigma}{R_0 P_0} - \frac{c_\infty}{c_s} \right)^{1/2}. \quad (2.16)$$

Here, and in the following, c_s is the saturation concentration at the average pressure of the sound field only for bubbles driven so much below resonance that inertial effects in their motion are insignificant. As was pointed out by Safar (1968), inertial effects introduce a correction in Equation (2.16) which for isothermal behavior becomes

$$\eta_{th} = \left(\frac{3}{2}\right)^{1/2} \left(1 - \frac{\omega^2}{\omega_0^2} \right) \left(1 - \frac{c_\infty}{c_s} + \frac{2\sigma}{R_0 P_0} \right)^{1/2} \left(1 + \frac{4\sigma}{3R_0 P_0} \right) \left(1 + \frac{2\sigma}{R_0 P_0} \right)^{-1/2}, \quad (2.17)$$

where ω_0 is given by Equation (2.8) with $\kappa = 1$.

A major approximation contained in the above results consists in the neglect of the nonlinear effects associated with the bubble oscillations. This aspect of the problem was analyzed by Eller & Flynn (1965), who made use of a method

developed by Plesset & Zwick (1952) to describe the growth of vapor bubbles. While we return to this method in greater detail below, it may be mentioned here that its applicability is limited to those situations in which appreciable concentration gradients are present only in a layer adjacent to the bubble surface that is thin compared with the bubble radius. Clearly, for the threshold problem under consideration here, this condition takes the form $R_0 \gg (\alpha/\omega)^{1/2}$, and hence is usually met in practice. Eller and Flynn's result is

$$\frac{c_\infty}{c_s} - \left(1 + \frac{2\sigma}{R_0 P_0}\right) \frac{\langle R/R_0 \rangle}{\langle (R/R_0)^4 \rangle} = 0, \tag{2.18}$$

where the brackets denote the average of the enclosed quantity over one period of oscillation. The pressure amplitude of course enters in this equation through the averages, which can be computed either numerically or by an analytic perturbation scheme (Eller & Flynn 1965; Eller 1969, 1972, 1975). For the case in which $c_\infty = c_s$ and $\sigma/R_0 P_0 \ll 1$, Eller gives the result

$$\eta_{th} = \left(1 - \frac{\omega^2}{\omega_0^2}\right) \left(\frac{3\sigma}{R_0 P_0}\right)^{1/2} \left(1 - \frac{\omega^2}{8\omega_0^2}\right)^{-1/2}, \tag{2.19}$$

which is seen to coincide with Safar's modification of the Hsieh-Plesset theory except for the last factor, which is introduced by the nonlinear effects in the second-order approximation. A similar result has been derived from other techniques by Skinner (1970).

Both Equations (2.17) and (2.19) appear to predict a vanishing threshold at resonance. This feature of course is a consequence of the neglect of damping effects which have recently been analyzed by Eller (1975). He found a very pronounced minimum in the threshold curve at resonance and considered the effect of this behavior on the evolution in time of a population of bubbles with given initial radius.

Experimental observations of threshold values were first reported by Strasberg (1959, 1961) at 25 k Hz, and more recently by Eller at 26.6 k Hz (1969) and at 11 k Hz (1972) and by Gould (1974) at 20 k Hz. It is found that the data given by Strasberg at 25 k Hz and by Eller at 26.6 k Hz tend to fall somewhat below the theoretical predictions. Although the discrepancy is not serious for bubble radii greater than about 4×10^{-3} cm, a definite disagreement appears to exist for smaller radii. The data at 11 k Hz are in better agreement with theory, provided that the bubbles are taken to oscillate adiabatically rather than isothermally (Eller 1972). No data for bubbles smaller than 5×10^{-3} cm are reported, however. The data of Gould (1974) are also generally in good agreement with Equation (2.19) for a polytropic form of the pressure-volume relation. The smallest bubble observed by him had a radius of 5.5×10^{-3} cm.

A rigorous analysis of nonlinear effects in the theory of bubble growth by rectified diffusion is more complex than for the threshold calculation because now, in addition to the oscillatory boundary layer of thickness $(\alpha/\omega)^{1/2}$, one must take into account the layer in which the gas concentration is depleted because of the net mass diffusion into the bubble. It is easy to convince oneself that the thickness of

this layer is of the order of the bubble radius or greater so that the applicability of Eller and Flynn's method is not immediately evident. The problem has recently been considered by Skinner (1972), who obtained a solution with the aid of multiple time- and length-scale expansions. Surprisingly, his results are practically equal to those previously derived by Eller (1969) by the same approach as that used for the threshold condition by Eller & Flynn (1965). A comparison with data (Eller 1969), however, shows that the bubble growth through resonance is much sharper than the theory predicts. Other large discrepancies between theory and experiment for bubble growth rates have also been reported by Eller (1969), who observed rates twenty times larger than the theoretical results. Recent experiments by Gould (1974) appear to substantiate Eller's conject

streaming taking place in the vicinity of the bubble surface. He observed the appearance of surface oscillations on the bubble and of a streaming motion in its vicinity together with large increases in growth velocity above the theoretical values. Growth rates measured in the absence of streaming, however, tend to be smaller than the predicted values. Apparently, the connection between surface oscillations and streaming, although experimentally well documented (Kolb & Nyborg 1956, Elder 1959, Gould 1966), has not yet found a proper theoretical treatment. The studies of Davidson (1971) and of Davidson & Riley (1971) refer to the case in which the streaming is produced by an oscillatory motion of the center of a spherical bubble and is therefore not directly applicable to surface-oscillation-induced streaming.

Acoustic Cavitation and Applications

In the terminology of Blake (1949; see also Flynn 1964) we may distinguish between gaseous and vaporous acoustic cavitation. The first description refers to the phenomena associated with bubbles that have predominantly a noncondensable gas content; the second, to bubbles that have predominantly a vapor content. While we have discussed several features of the dynamics of gaseous cavities above, and we shall discuss the behavior of vapor cavities in the next section, it should be realized that at present it is not yet possible to relate all of the specific features of experimentally observed cavitation phenomena with the available experimental and theoretical work on single gas or vapor bubbles. Although it is generally agreed that cavitation damage, white noise, sonoluminescence, chemical reactions, and other features of cavitation are associated with violent bubble motion, at present not only a quantitative understanding of these phenomena is lacking but sometimes even the physical mechanisms through which they take place are obscure.

Violent collapse, as can be seen in Figure 4, is certainly a possible behavior for a gas bubble in an oscillating pressure field, and estimates of the temperatures and pressures reached by the bubble content toward the end of the collapse are of the order of several thousands of degrees Kelvin and of thousands of atmospheres, respectively (see e.g. Flynn 1964; 1975a,b). Unfortunately, during the later stages of the collapse the spherical shape of the bubble becomes highly unstable (see below), and the duration of the collapse itself is so short that adequate theoretical or

experimental investigations appear very difficult. These same considerations can be applied also to vapor-bubble collapse in a highly subcooled liquid.

No less obscure is the beginning of a vapor- or gas-bubble life. This problem is essentially that of nucleation, and the uncertainties associated with it have a significant bearing not only on acoustic cavitation but, perhaps more importantly, on boiling heat transfer. Indeed, while it is observed that homogeneous-nucleation theory based on thermodynamic fluctuations describes very well the behavior of certain organic liquids, such as pentane, hexane, heptane, ether, and benzene (see e.g. Cole 1974, Blander & Katz 1975), the same conclusion cannot be drawn for such an important liquid as water for which the presence of impurities appears to play a determinant role. We cannot enter here into any detail on these matters, and we refer the reader to some pertinent recent papers (Apfel 1970, Gavrilo 1970, Holl 1970, Keller 1972). A useful summary of previous work is given by Flynn (1964).

Once nucleation has taken place, a gas bubble will grow by rectified diffusion over many periods of the driving pressure oscillations until it breaks up. The lifetime of a vapor bubble, however, will in general be much shorter, not more than a few acoustic cycles, and its growth will be determined more by dynamic effects than by a change in the mass content of the cavity. Some aspects of this process are dealt with in the following section.

The description of cavitation events is made even more difficult by the apparent importance of cooperative effects of many bubbles in several phases of the process (Willard 1953). Space limitations prevent us from entering into any detail here. The reader is referred to the works of Willard (1953), Strasberg (1959), Noltingk (1962), Flynn (1964), Coakley (1971), Hinsch, Bader & Lauterborn (1974), and Nyborg (1974). Work on sonoluminescence is described in Negishi (1961), Taylor & Jarman (1968, 1970), Margulis (1969), Saksena & Nyborg (1970), and Coakley & Sanders (1973).

Finally, for some industrial applications of acoustic cavitation the reader is referred to Neppiras (1965). A growing area of interest appears to be in biomedical applications (Rooney 1970, 1972; Nyborg 1974; Hill 1972).

3 CAVITATION BUBBLES AND VAPOR BUBBLES

In this section we consider the analysis of some features of the bubbles that are composed predominantly of vapor. As was suggested in a preceding section, these bubbles also usually originate from small gas nuclei. However, if a bubble grows to many times its initial size so rapidly that mass-diffusion effects are negligible, the gas content plays an unimportant role in the dynamics except possibly near the end of a violent collapse.

We are not concerned here with the problem of boiling heat transfer, which has recently been considered by Rohsenow (1971). Rather, we limit our discussion to the growth and collapse of vapor bubbles in unbounded liquids. In the presence of boundaries, the free surface loses its spherical symmetry. Some of these boundary effects are considered in the next section.

Preliminary Considerations

In the study of the dynamics of vapor bubbles it is helpful to make a distinction between two limiting cases according to the importance of thermal effects. The physical basis for this distinction rests on the combined effects of the strong temperature dependence of the equilibrium vapor density and of the rate of change of the equilibrium vapor pressure with temperature. To illustrate this point we may consider the case of a bubble in water that grows to a radius R in a time t . The amount of thermal energy required to fill the bubble with vapor in thermodynamic equilibrium with the water is $(4/3)\pi R^3 L \rho_v^e(T)$, where L is the latent heat of evaporation and $\rho_v^e(T)$ is the equilibrium vapor density at the water temperature T . This energy is made available through a drop ΔT in the temperature of a surrounding liquid layer of thickness of the order of the diffusion length given by $(Dt)^{1/2}$, where $D = k/\rho c_l$ is the thermal diffusivity in the liquid. The heat energy supplied is, therefore, $4\pi R^2 (Dt)^{1/2} \rho c_l \Delta T$. When these two expressions are equated, one obtains an estimate of the temperature drop ΔT as

$$\Delta T \approx \frac{1}{3} \frac{R \rho_v^e(T) L}{(Dt)^{1/2} \rho c_l}. \quad (3.1)$$

For water at 15°C , with $R = 0.1$ cm and $t = 10^{-3}$ sec, $\rho_v = 1.3 \times 10^{-5}$ gm $^{-3}$, one finds $\Delta T \sim 0.2^\circ\text{C}$. This temperature drop causes a decrease in the vapor pressure of the order of 1%, and hence has an insignificant effect on the growth of the bubble. For water in the neighborhood of 100°C , however, the equilibrium vapor density is about 46 times its value at 15°C , and one finds $\Delta T \sim 13^\circ\text{C}$, with a corresponding decrease in vapor pressure of roughly 50%. Clearly, the bubble-growth dynamics will be drastically altered with a longer growth time to the same radius. Thermal effects rather than inertial effects now dominate the growth process. We refer to the first case as "cavitation" bubbles and to the second one as "boiling" or "vapor" bubbles. A similar difference in behavior will be observed in the collapse process. For a cavitation bubble the internal pressure will remain practically constant until the latest stages of the collapse, while for a boiling bubble a much greater effect from the vapor pressure will be observed.

The Dynamics of a Cavitation Bubble

With the neglect of viscous effects and by use of the identity

$$\frac{1}{2} \frac{1}{R^2 \dot{R}} \frac{d}{dt} (R^3 \dot{R}^2) = R \ddot{R} + \frac{3}{2} \dot{R}^2, \quad (3.2)$$

Equation (1.5) can be integrated once if the ambient pressure is taken to be independent of time and if, in the light of the above discussion, the internal pressure $p_i = p_v$ is also regarded as a constant. The result is

$$\dot{R}^2 - \left(\frac{R_0}{R}\right)^3 \dot{R}_0^2 - \frac{2}{3} \frac{p_v - p_\infty}{\rho} \left[1 - \left(\frac{R_0}{R}\right)^3\right] - \frac{2\sigma}{\rho R} \left[1 - \left(\frac{R_0}{R}\right)^2\right], \quad (3.3)$$

where the subscript zero denotes the initial conditions for the growth. If $p_v > p_\infty$,

it is seen that for $R \gg R_0$ the velocity is approximately equal to its asymptotic value

$$R = \left(\frac{2 p_v - p_\infty}{3 \rho} \right)^{1/2}, \tag{3.4}$$

which can also be obtained by equating the kinetic energy of the flow to the work performed by the pressure forces. It is shown below that Equation (3.4) gives also the growth velocity of a boiling bubble at high superheats or low ambient pressures in the early and intermediate stages. If $p_v < p_\infty$ but an initial impulse is imparted to the bubble wall, Equation (3.3) predicts that the bubble would reach a maximum radius that, with the neglect of surface tension, is given by

$$R = \left[1 + \frac{2}{3} \rho \dot{R}_0^2 (p_\infty - p_v)^{-1} \right] R_0. \tag{3.5}$$

Under the same assumptions one can obtain an expression similar to (3.3), valid for the collapse of the bubble starting from some initial radius R_i . If the initial velocity is taken to vanish, one has

$$\dot{R}^2 = \frac{2 p_\infty - p_v}{3 \rho} \left[\left(\frac{R_0}{R} \right)^3 - 1 \right] + \frac{2 \sigma}{\rho R} \left[\left(\frac{R_i}{R} \right)^2 - 1 \right]. \tag{3.6}$$

This equation would predict that the velocity approaches infinity as $R^{-3/2}$ as $R \rightarrow 0$, which is unacceptable. Therefore, one must conclude that the approximations under which Equation (3.6) is obtained will eventually break down during the collapse process. We return to this point below. Let us observe here that, in the absence of surface-tension effects, one can compute from Equation (3.6) the time t_0 required for complete collapse of the cavity:

$$t_0 = \frac{\Gamma(5/6)}{\Gamma(1/3)} \left[\frac{3 \pi \rho}{2(p_\infty - p_v)} \right]^{1/2} R_i \simeq 0.915 \left(\frac{\rho}{p_\infty - p_v} \right)^{1/2} R_i, \tag{3.7}$$

which is the result of Rayleigh (1917). An extension to the collapse of a closed cavity of arbitrary shape has been given by Miles (1966), who, for oblate and prolate spheroids, finds a correction to (3.7) of the order of the fourth power of the eccentricity (the quantity R_i is defined as the radius of a sphere of equivalent volume for this problem).

Let us now return to the question of the dynamics of the bubble in the later stages of the collapse. An important feature of this problem that should be kept in mind in the following discussion is that the spherical configuration of the bubble surface is unstable during the collapse, so that analyses based on the assumption of spherical symmetry cannot be rigorously correct. Nevertheless, this assumption furnishes an order-of-magnitude estimate of the several quantities involved, and it gives a qualitative picture of the phenomenon. It is in this spirit that the following considerations should be interpreted. The most obvious reason for the nonphysical behavior of (3.6) for $R \ll R_i$ is the neglect of liquid compressibility which becomes important as soon as bubble-wall velocities become comparable with the speed of sound in the liquid. A very successful modification of the Rayleigh equation that takes into account liquid compressibility was obtained by Gilmore (1952; see also Plesset 1969) on the basis of the Kirkwood-Bethe (1942) approximation. This

approximation consists in assuming that the quantity $r(h + \frac{1}{2}u^2)$, with u the liquid velocity and h the enthalpy, is propagated unaltered along the outgoing characteristics, $dr = (u + c) dt$; that is,

$$\frac{D}{Dt} [r(h + \frac{1}{2}u^2)] + c \frac{\partial}{\partial r} [r(h + \frac{1}{2}u^2)] = 0. \quad (3.8)$$

In this equation $c = c(p)$ denotes the speed of sound in the liquid, and $D/Dt = \partial/\partial t + u\partial/\partial r$ is the material derivative. The derivatives with respect to r in the second term of Equation (3.8) can be eliminated with the aid of the momentum equation, $Du/Dt = -\partial h/\partial r$, and of the continuity equation,

$$\frac{\partial u}{\partial r} = -\frac{1}{c^2} \frac{Dh}{Dt} - \frac{2u}{r}.$$

The result then leads to the following equation for the bubble wall

$$\left(1 - \frac{1}{C} \dot{R}\right) R \ddot{R} + \frac{3}{2} (\dot{R})^2 \left(1 - \frac{1}{3C} \dot{R}\right) = H \left(1 + \frac{1}{C} \dot{R}\right) + \left(1 - \frac{1}{C} \dot{R}\right) \frac{R}{C} \dot{H}. \quad (3.9)$$

H and C denote here the values of the quantities h and c at the bubble wall. If surface tension and viscosity are omitted from the boundary conditions and the bubble internal pressure p_v is taken to be a constant, Equation (3.9) can be integrated analytically once to obtain

$$\log \frac{R}{R_i} = -2 \int_{\dot{R}_0}^{\dot{R}} \frac{U(U-C) dU}{U^3 - 3CU^2 + 2HU + 2HC}. \quad (3.10)$$

The bubble-wall velocity, \dot{R} , has been written as U in this equation. The initial conditions correspond to a bubble of radius R_i that at $t = 0$ undergoes a step increase in the ambient pressure. The initial "release" velocity \dot{R}_0 for this situation is easily computed from (3.9) and found to be given by $\dot{R}_0 \approx (p_v - p_\infty)/\rho_\infty c_\infty$, where ρ_∞ and c_∞ are the values of the liquid density and sound speed at large distance from the bubble. In the approximation $|H| \ll C^2$, Equation (3.10) gives the following correction to Equation (3.6):

$$\left(\frac{R_i}{R}\right)^3 = \left(1 - \frac{1}{3} \frac{\dot{R}}{C}\right)^4 \left(1 + \frac{3}{2} \frac{\rho_\infty}{p_\infty - p_v} \dot{R}^2\right), \quad (3.11)$$

from which we find $\dot{R} \propto R^{-1/2}$ as $R \rightarrow 0$, in contrast to the incompressible approximation that was shown to give $\dot{R} \propto R^{-3/2}$.

The accuracy of Equation (3.9) can be assessed through a comparison with the full solution of the problem which can be obtained only numerically. Such an analysis was conducted by Hickling & Plesset (1964), who considered both empty cavities (i.e. essentially $p_\infty - p_v = \text{constant}$) and cavities with a very small gaseous content that undergo isentropic or isothermal compression. An example of their results is shown in Figure 8, which shows that Equation (3.9) is accurate up to bubble-wall Mach numbers of the order of 5. Their analysis included also the details of the liquid flow, with the computation continued past the rebound point of the gas bubble up to the formation of the shock wave in the liquid. They found

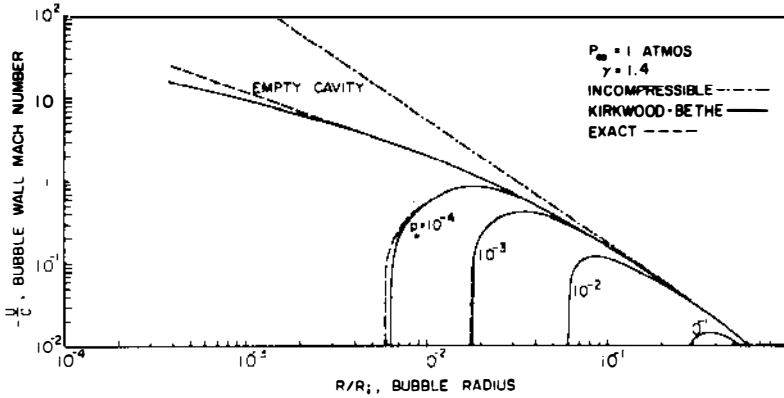


Figure 8 The bubble-wall Mach number as a function of the dimensionless radius R/R_i for a gas bubble collapsing in a compressible liquid (from Hickling & Plesset 1964).

that the peak pressure has an approximate $1/r$ dependence during the rebound, with pressures in the cavity of the order of tens of kilobars. Results of the same order of magnitude were obtained by Flynn (1975b). While this estimate of the internal pressure (and hence of the pressure distribution in the neighboring liquid) may be off by as much as an order of magnitude, it appears likely that the $1/r$ dependence would not be greatly affected by a more accurate description of the latest stages of the collapse. It may be of interest to observe that the Kirkwood-Bethe approximation gives good results for collapse up to relatively large Mach numbers since the collapse motion leads to an expansion wave in the liquid. For bubble growth at very large velocities, the same accuracy could not be expected since this motion would lead to the propagation of a shock wave into the liquid.

Following Benjamin (1958), Jahsman (1968) has given an analytic treatment of the collapse of a gas bubble in a compressible liquid based on a perturbation expansion in terms of a small parameter that is essentially the liquid compressibility. His results show that the Kirkwood-Bethe assumption is verified for the zero and first-order terms of the expansion, but breaks down in higher orders. In this connection mention should be made also of a work by Hunter (1960), who, guided by a numerical investigation, developed a similarity solution for the collapse of an empty cavity in the neighborhood of the collapse point in the region where $c \gg c_\infty$; he found that in the latest stages of the collapse $\dot{R} \propto R^{-0.8}$. It should be remarked that the early work on this subject, summarized by Cole (1948, see also Trilling 1952), was motivated by interest in underwater explosions. Most of the later contributions discussed above were undertaken to investigate the mechanism of cavitation damage, since it was conjectured that the very strong pressure pulses radiated by a collapsing cavity might be responsible for it. The current understanding of cavitation damage, however, is quite different, as is discussed in the following section.

In addition to the neglect of compressibility effects, a second obvious reason for

the failure of Equation (3.6) to model in an acceptable way the physical behavior of a collapsing bubble is the neglect of the variation of p_v that is brought about not only by the increasing importance of even small quantities of noncondensable gas in the cavity as $R \rightarrow 0$ but also by the fact that condensation of the vapor cannot keep up with the bubble-wall motion when its velocity becomes of the order of the speed of sound in the vapor. Some remarks on this aspect of the problem are made below.

A quantitative experimental confirmation of the collapse time predicted by Equation (3.7) was given by Lauterborn (1972a), who realized a very close physical approximation to the Rayleigh empty-bubble model by focusing a giant pulse of a Q-switched ruby laser on an interior point of a liquid mass (Lauterborn 1972b). He reported an experimental value of $t_0 = 297 \times 10^{-6}$ sec to be compared with a theoretical value of 300×10^{-6} sec for a 0.378-cm bubble in water. It should be remarked that the final stage of the collapse is so rapid that Equation (3.7) is negligibly affected by the liquid compressibility, the behavior of the bubble content, or the deviation from spherical shape discussed below.

The first investigation of the behavior of a cavitation bubble in a time-dependent pressure field was carried out some time ago by the integration of the Rayleigh equation with constant p_i for a bubble entrained in a liquid flowing past a submerged object (Plesset 1949). For the ambient pressure $p_\infty(t)$ he made use of the experimentally determined pressure distribution along the object in the absence of the bubble at the point occupied by the bubble at time t , and he obtained a very good agreement with the data. Another interesting study of cavitation-bubble dynamics in a time-varying pressure field has been made by Hsieh (1970), who obtained approximate analytic expressions for the maximum radius that a cavity would attain in a viscous liquid subject to a transient pressure pulse. From his results he concluded that the empirical relation $P_{th} \propto \mu^{0.2}$, obtained by Bull (1956) with a stress-wave technique, between liquid viscosity and threshold pressure for cavitation inception appears to be due more to a coincidental distribution of nuclei in the different liquids tested than to other more basic features of the cavitation process.

The Growth of Vapor Bubbles

As mentioned above, a vapor bubble will designate a bubble in the dynamics of which thermal effects play a dominant role. It is supposed that the nucleus from which the bubble will eventually grow is a small spherical cavity of radius R_0 in a liquid at uniform temperature T_∞ . At equilibrium, the internal pressure in the nucleus will be the equilibrium vapor pressure corresponding to the liquid temperature, $p_v^e(T_\infty)$, and the equilibrium radius is

$$R_0 = \frac{2\sigma}{p_v^e(T_\infty) - p_\infty} \quad (3.12)$$

where as usual p_∞ denotes the ambient pressure, which is here taken to be constant. The pressure p_∞ corresponds to a well-determined equilibrium temperature, a boiling temperature, which we denote by T_b . Equation (3.12) implies $T_\infty > T_b$; the difference $\Delta T = T_\infty - T_b$ is termed the liquid superheat. This simple model for the

vapor nucleus is certainly highly idealized, but it can be shown that initial conditions do not affect significantly the growth of the bubble.

The process of bubble growth in superheated liquids can readily be described in physical terms as follows. When the equilibrium situation depicted by (3.12) is disturbed, the bubble starts to grow very slowly under the restraining effect of surface tension. If the initial superheat is sufficiently large, the growth velocity will eventually reach the asymptotic value for a cavitation bubble, as given by (3.4), before the rate of vapor inflow (which is proportional to R^2) is so large as to produce a substantial cooling of the surrounding liquid. At this point, both inertial and thermal effects limit the growth rate. The growth rate then begins to decrease making inertial effects less and less important until the radius has grown so large that the growth process is limited only by the rate at which heat can be supplied to the bubble wall. The velocity of growth during this asymptotic stage is readily estimated by noting that the bubble internal pressure will have decreased nearly to p_∞ , so that the bubble surface temperature is T_b . The heat flow into the bubble from the liquid is then approximately $4\pi R^2 k_l \Delta T / (D_l t)^{1/2}$, where D_l and k_l are the liquid thermal diffusivity and conductivity, respectively, and t is the time from inception of the growth. This heat flux will be balanced by the absorption of the latent heat necessary to vaporize the liquid into the bubble, which is given by $4\pi R^2 L \rho_v^e(T_b) \dot{R}$ where L is the latent heat and $\rho_v^e(T_b)$ is the equilibrium vapor density corresponding to the boiling temperature. Equating the above quantities, we find

$$\dot{R} = \left(\frac{3}{\pi}\right)^{1/2} \frac{k_l}{L \rho_v^e(T_b)} \frac{T_\infty - T_b}{(D_l t)^{1/2}} \tag{3.13}$$

The factor $(3/\pi)^{1/2}$ has been inserted in (3.13) so as to give the result that is obtained in the asymptotic analysis of Plesset & Zwick (1954). Integration of Equation (3.13) leads to $R \propto t^{1/2}$, a widely used result in bubble-growth theory. It should be emphasized that Equation (3.13) expresses only an asymptotic relationship that is valid only for times large enough so that the growth velocity predicted by it is smaller than the inertia-controlled value (3.4). Indeed, this remark enables one to make a distinction between a cavitation and a boiling bubble since the characteristic growth rate for cavitation-bubble growth is (3.4) and the characteristic growth rate for the boiling-bubble growth is (3.13) (in this connection, see also Brennen 1973).

The complete mathematical formulation of the problem of spherical-bubble growth is similar to the one indicated above for the mass-diffusion problem inasmuch as one has a partial differential equation for the energy (or temperature) in the liquid,

$$\frac{\partial T}{\partial t} + \frac{R^2}{r^2} \dot{R} \frac{\partial T}{\partial r} = \frac{D_l}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right), \tag{3.14}$$

that is coupled to the Rayleigh equation. For the time being we assume that the internal pressure in the bubble is given by the equilibrium vapor pressure corresponding to the bubble surface temperature as determined by (3.14). Some comments

on nonequilibrium effects are made below. The initial and boundary conditions for Equation (3.14) are

$$T(r, 0) = T_\infty, \tag{3.15a}$$

$$T(r, t) \rightarrow T_\infty \quad \text{as } r \rightarrow \infty, \tag{3.15b}$$

$$-4\pi R^2 k_t \frac{\partial T}{\partial r} = L \frac{d}{dt} \left(\frac{4}{3} \pi \rho_v^e R^3 \right), \quad \text{at } r = R(t), \tag{3.15c}$$

where the last equation is an expression of the heat balance at the bubble boundary.

A general solution to the problem posed by (3.14) and (3.15) was obtained by Plesset & Zwick (1952) under the assumption that appreciable temperature gradients are established only in a thin layer surrounding the cavity radius. Their result for the bubble surface temperature $T_s(t)$ is

$$T_s(t) = T_\infty - \left(\frac{D_t}{\pi} \right)^{1/2} \int_0^t \frac{R^2(\theta) \frac{\partial T}{\partial r}(R, \theta)}{\left[\int_\theta^t R^4(\lambda) d\lambda \right]^{1/2}} d\theta, \tag{3.16}$$

where $\partial T(R, t)/\partial r$ is an arbitrarily specified temperature gradient at $r = R(t)$. Upon substitution of (3.15c) for this quantity and with neglect of the temperature dependence of L , ρ_v , k_t , and with the approximation of the pressure-temperature relation by a straight line, Plesset & Zwick (1954) obtained the following formulation of the problem:

$$p^{7/3} \frac{d^2 p}{du^2} + \frac{7}{6} p^{4/3} \left(\frac{dp}{du} \right)^2 = 3 \left[1 - \mu \int_0^u (u-v)^{-1/2} \frac{dp}{dv}(v) dv - \frac{2\sigma}{p^{1/3}} \right], \tag{3.17}$$

where $p = R^3/R_0^3$ is the normalized volume and u is a new dimensionless time scale defined as

$$u(t) = \frac{\alpha}{R_0^4} \int_0^t R^4(\theta) d\theta. \tag{3.18}$$

The constants α and μ are defined by

$$\alpha = \left\{ \left[\rho_v^e(T_\infty) - p_\infty \right] / \rho_l \right\}^{1/2} / R_0, \tag{3.19}$$

$$\mu = \frac{1}{3} \left(\frac{2\sigma D_t}{\pi} \right)^{1/2} \frac{L}{\rho_v^e(T_b) k_t} (T_\infty - T_b)^{-1} \left\{ \rho_l \left[\rho_v^e(T_\infty) - p_\infty \right] \right\}^{-1/2}.$$

From (3.17) and (3.18) they then proceeded to obtain various asymptotic expansions for the radius-time dependence, the leading term of which, for large times, is given by (3.13) (see also Plesset & Zwick 1955). The experimental data of Dergarabedian (1953, 1960) obtained for low superheats (up to about 6°C) in water, carbon tetrachloride, benzene, and other organic liquids agree very well with the results of Plesset and Zwick, but for these cases the asymptotic stage is reached so fast that essentially only the asymptotic results (3.13) could be tested with the data. The same can be said of the work of Niino et al (1973), who obtained superheats of up to

14°C. A close experimental simulation with high superheats of the theoretical conditions to which the complete Plesset-Zwick theory applies is rather difficult because large thermal gradients in the liquid can be easily generated, and further because observation of the bubble in the inertia-controlled phase of the growth, when the radius is small and the growth velocity large, is difficult. The best support of the theory (at least as a mathematical approximation to a more complete set of equations) can be found in comparing its predictions with the results of some extensive numerical computations performed by Dalle Donne & Ferranti (1975), who solved the complete energy equation, Equation (3.14), simultaneously with the Rayleigh equation. Their investigation was primarily motivated by doubts about the validity of the thin-thermal-layer assumption for liquids of high thermal conductivity such as liquid sodium. A detailed comparison of the Plesset-Zwick theory with Dalle Donne and Ferranti's results is available elsewhere (Plesset & Prosperetti 1976, 1977). In summary it can be said that the agreement with Equation (3.15) is extremely good up to liquid-sodium superheats of 300°C provided that the liquid and vapor properties are evaluated at the boiling temperature as indicated in Equations (3.19). An example of this comparison is given in Figure 9. The physical reason for this result is simply that at very high superheats inertial effects play a

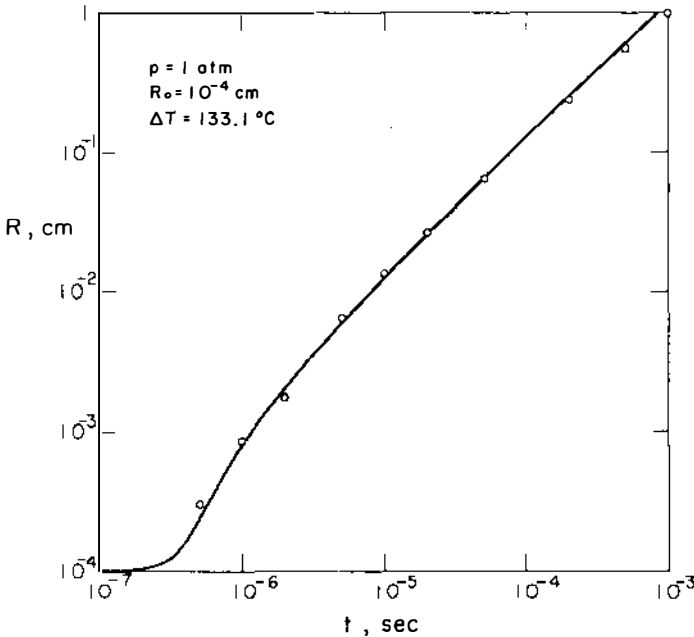


Figure 9 Comparison of the results computed with the aid of the Plesset-Zwick formulation [Equation (3.17), full line] with those obtained by Dalle Donne & Ferranti (1975) by means of numerical integration of the complete equation (3.14) (open circles) for the growth of a vapor bubble in superheated sodium (from Plesset & Prosperetti 1976).

dominant role essentially until the bubble surface has cooled to the boiling temperature.

Plesset & Prosperetti (1976) have also shown that for $p \gg 1$ Equation (3.17) admits a scaling such that a universal bubble-growth law for any liquid and any superheat is obtained in the form

$$S = S(\tau)$$

where

$$S = \mu^2 R/R_0, \quad \tau = \alpha \mu^2 t. \quad (3.20)$$

Several examples of growth velocities $\dot{R}(t)$ scaled according to (3.20) (i.e. graphs of $dS/d\tau$ versus τ) for different values of the parameter μ are shown in Figure 10. The dots on the curves mark the point where $R/R_0 = 10$; it is seen that approximately from this point on, all the curves fall on a single line which demonstrates the validity of the scaling law (3.20). Similar results are obtained for the radius. In the dimensionless variable of Equation (3.20) the inertia-controlled growth velocity (3.4) has the constant value $dS/d\tau = (2/3)^{1/2}$, and the thermally controlled one given by (3.13) is $dS/d\tau = \pi^{-1}(3\tau)^{-1/2}$. This curve is also shown in Figure 10, and it is apparent that it describes a substantial part of the growth only for relatively small subcooling (large values of μt). It is also shown by Plesset & Prosperetti (1976) that the analytic expression for bubble growth obtained by Mikic, Rohsenow & Griffith (1970) can be written in terms of the scaled variables S, τ provided that the pressure-temperature relationship is approximated not by a tangent but by a chord (see Theofanous & Patel 1976). This expression is

$$S = \frac{2}{\pi^2} \left(\frac{3}{2}\right)^{3/2} \left[\left(\frac{1}{2}\pi^2\tau + 1\right)^{3/2} - \left(\frac{1}{2}\pi^2\tau\right)^{3/2} - 1 \right], \quad (3.21)$$

and its derivative is also plotted in Figure 10. It is seen that Equation (3.21) is a good approximation except in the region where inertial and thermal effects are of comparable importance.

In addition to the studies mentioned, other attempts have been made at deducing a scaling law for vapor-bubble growth. We note here those of Birkhoff et al (1958), and of Scriven (1959), who showed that with the assumption $R = \beta t^{1/2}$, the energy equation (3.14) admits a similarity variable that reduces it to an ordinary differential equation that can be solved in closed form. Insertion of the result into the boundary condition (3.15c) determines then the constant β as a function of the several parameters of the problem. For low superheats, it is found that the value of β coincides with the one obtained upon integration of (3.13). The limitations of this approach should be clear from the discussion given here. In particular, it is seen from Figure 10 that a $t^{1/2}$ dependence of the radius cannot account for dynamical effects, so that the applicability of such an analysis must be restricted to low superheats. In the problem of mass diffusion, however, which has the same mathematical structure, but for which inertial effects are in general much less pronounced, the similarity approach is quite appropriate. A very useful feature of it for the application to the mass-diffusion problem is that the diffusion equation is solved exactly without any assumption concerning the thickness of the boundary layer.

Vapor-Bubble Collapse

The theoretical modeling of the collapse of vapor bubbles under conditions in which thermal effects play a significant role is more difficult than the analysis of their growth, and no entirely satisfactory theoretical results are available. The principal difficulty here is that, unlike the growth case, the thickness of the liquid layer in which substantial temperature gradients are present cannot in general be taken to be small compared with the bubble radius for all times. An approach that makes use of the Plesset-Zwick result (3.16) for the surface temperature has been followed by Florschuetz & Chao (1965), but the applicability of that expression to the collapse case is open to doubt, especially for those cases in which large thermal effects result in small collapse velocities. Among the notable features of the results of Florschuetz and Chao is a nonmonotonic decrease in the bubble radius when both inertial and thermal effects are important in the collapse. The experimental evidence in favor of this point appears to be rather unclear for the collapse of spherical bubbles in unbounded liquids (Levenspiel 1959, Akiyama 1965, Hewitt & Parker 1968), possibly because shape deformations during the collapse introduce errors in the precise determination of the radius. It may be mentioned, however, that no such behavior was observed by Gunther (1951) for the collapse of hemispherical bubbles on a heated solid wall. Florschuetz and Chao also reported experiments performed with the aid of a free-fall tower to minimize the effects of translational motion and initial buoyancy-induced deformations. They observed that the shape of all of the bubbles deviated from the spherical during the collapse. While for small subcoolings ($\Delta T_i \lesssim 13^\circ\text{C}$) they appeared to oscillate in the lowest mode between prolate and oblate spheroidal configurations, for larger subcoolings much more marked deviations occurred with the formation of jets and large deformations beginning early in the collapse process.

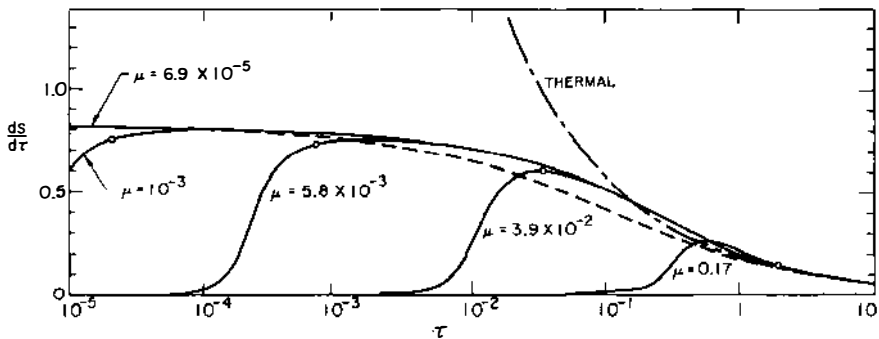


Figure 10 Dimensionless growth velocity versus dimensionless time for vapor-bubble growth, for several values of the parameter μ , which shows the scaled growth behavior. The open circles denote the points where the bubble radius equals ten times its initial value. The dashed line is the Mikic-Rohsenow-Griffith (1970) result (3.21), and the line labeled "thermal" is the asymptotic result of Plesset & Zwick (1954), Equation (3.13) (from Plesset & Prosperetti 1976).

Bubble collapses at high subcoolings (up to 60°C) in water have been obtained by Board & Kimpton (1974), whose data however appear to be affected by strong wall effects. Very recently Delmas & Angelino (1976) have studied the collapse of rising bubbles with subcoolings ΔT_i between 7°C and 42°C in water. They reported two different collapse modes, one for large bubbles and high subcoolings, for which inertial effects are important, the other for smaller bubbles and lower subcoolings. The bubbles belonging to the first class appear to occupy a well-defined region in the $(R_i, \Delta T_i)$ -plane. They exhibit large collapse velocities and rapid ($< 1.4 \times 10^{-4}$ sec) fragmentation upon attainment of the minimum radius. The initial conditions giving rise to the second collapse mode are less predictable. The bubbles present an irregular surface and selected condensation sites. Fragmentation takes place earlier than in the other case and the smaller bubbles to which it gives rise continue to condense. These results are interesting, but it is difficult to estimate the effect of the velocity of rise of the bubbles on which the authors did not make any comment.

Other Topics in the Dynamics of Vapor Bubbles

In recent years research in the acoustic cavitation process in cryogenic liquids (see e.g. Mosse & Finch 1971, Neppiras & Finch 1972, Akulichev 1974) and in the development of ultrasonic bubble chambers (Akulichev et al 1970; Brown et al 1970; Hilke 1973; Shestakov & Tkachev 1973, 1975) have prompted the theoretical investigation of the behavior of vapor bubbles in oscillating pressure fields (Finch & Neppiras 1973, Wang 1973). In these studies, the energy equation in the liquid and the Rayleigh equation have been solved under the assumption of small-amplitude oscillations, allowing for the evaporation-condensation coupling expressed by Equation (3.15c). It has been found that when the liquid temperature is sufficiently close to the saturation temperature, there are two values of the bubble radius that give rise to resonant oscillations for a fixed driving frequency. One of them corresponds to that found for a permanent gas bubble, while the other one is smaller, approximately by an order of magnitude for the case of water at one atmosphere and at frequencies of the order of 10^4 Hz. The resonance peaks are broader than for gas bubbles, and they merge to give a large oscillation amplitude for values of the radius spanning one or two orders of magnitude. In addition to this effect of the evaporation of the liquid, Wang (1973) has also considered the simultaneous diffusion of a permanent gas into the bubble.

In these studies, the thermodynamic state of the vapor in the bubble has been assumed to correspond to conditions of thermodynamic equilibrium at the instantaneous bubble-wall temperature. In a certain sense, this assumption corresponds to the validity of Henry's law, Equation (2.11), in the case of rectified diffusion of mass into the bubble. Therefore, one may expect a rectified heat-transfer effect into the bubble by means of which the average vapor content of the bubble increases with time. Such a process has indeed been demonstrated theoretically (Wang 1974), but no experimental verification of the theory is available. It should be noticed that this effect would be of first (rather than second) order in the oscillation amplitude if thermodynamic nonequilibrium effects for the vapor were accounted for. Whether this increased order of magnitude would give rise to a contribution

bigger than the rectified effect depends strongly on the accommodation coefficients for evaporation and condensation. This aspect of the problem has not as yet been investigated.

More generally, the question of effects associated with a state of thermodynamic nonequilibrium between liquid and vapor is of course an important one in the dynamics of vapor bubbles. An estimate of the difference between the equilibrium vapor density ρ_v^e and the actual vapor density ρ_v is readily obtained by a mass balance over the bubble

$$\frac{d}{dt} (\frac{4}{3}\pi R^3 \rho_v) = 4\pi R^2 J, \tag{3.22}$$

where J is the net mass flux at the bubble wall which, on the basis of the elementary kinetic theory of gases, can be written as

$$J = \alpha \left(\frac{R_g T}{2\pi M} \right)^{1/2} (\rho_v^e - \rho_v). \tag{3.23}$$

Here R_g is the gas constant, M is the molecular weight of the vapor, and α is the accommodation coefficient for evaporation, taken to be equal to that for condensation. Continuity of temperature across the bubble surface has also been assumed in (3.23). Substitution of (3.23) into (3.22) and neglect of the term involving $d\rho_v/dt$ gives then (Plesset 1964):

$$\frac{\rho_v^e - \rho_v}{\rho_v^e} = \frac{\dot{R}}{\alpha(R_g T/2\pi M)^{1/2} + \dot{R}}. \tag{3.24}$$

In view of the fact that the sound speed in the vapor is given by $c = (\gamma R_g T/M)^{1/2}$, this equation shows that the nonequilibrium correction is of the order of the Mach number of the bubble-wall motion in the vapor whenever $\alpha \simeq 1$, and therefore negligible except perhaps near the end of violent collapses. Much more significant effects may be expected, however, for small values of α . Nonequilibrium analyses of vapor-bubble growth have been given by Bornhorst & Hatsopoulos (1967) and by Theofanous et al (1969a). As one might expect, very small values of α give rise to decreased growth rates, but the results corresponding to $\alpha = 1$ are in very good agreement with the equilibrium behavior. Theofanous et al (1969b) have treated in a similar way the case of the collapse of vapor bubbles, although their assumption of a thin thermal layer may be questionable in this case for the reasons already stated.

In addition to the neglect of thermodynamic equilibrium, the formulation of vapor-bubble dynamics given here has been simplified in other respects. For instance, the equation of continuity at the bubble surface gives

$$\rho_l(u_l - \dot{R}) = \rho_v(u_v - \dot{R}) \tag{3.25}$$

(where u_l and u_v are the liquid and vapor velocities at the liquid-vapor interface, respectively), from which

$$\dot{R} = \frac{\rho_l}{\rho_l - \rho_v} u_l + \frac{\rho_v}{\rho_l - \rho_v} u_v. \tag{3.26}$$

Clearly, setting $u_l = \dot{R}$, as was implicitly done in the derivation of the Rayleigh equation, is legitimate only if $\rho_v \ll \rho_l$. While this relation is normally satisfied by liquids at temperatures sufficiently below the critical point, its applicability in some special situations (such as cryogenic liquids) should not be automatically taken for granted. Similarly, the condition of continuity of normal stresses, Equation (1.4), and of conservation of energy at the interface, Equation (3.15c), in principle should contain other terms of order ρ_v/ρ_l . For a more detailed exposition of these matters the reader is referred to Scriven (1959) and Hsieh (1965). The latter author gives also a useful unified treatment of several mathematical aspects of the whole subject of bubble dynamics.

4 THE DYNAMICS OF NONSPHERICAL BUBBLES

It is obvious that as soon as the property of spherical symmetry is lost, the analysis of the various aspects of bubble dynamics becomes exceedingly complex both from the theoretical and the experimental standpoint (Hsieh 1972b). This situation is unfortunate, in view of the practical importance of the effects associated with deviations from the spherical shape and in view of the multitude of factors that promote such deviations. Among these one may mention the inherent dynamical instability of contracting bubbles, the proximity of solid boundaries or of free surfaces, and buoyancy effects. Such important phenomena as bubble breakup or coalescence and cavitation damage are dominated by these effects which are also found to increase heat and mass-transfer rates.

The first problem that is encountered in the dynamics of nonspherical bubbles is that of the specification of the bubble shape. An obvious possibility, which is used in practically all the work described below, is the use of an expansion in terms of spherical harmonics $Y_n^m(\theta, \varphi)$,

$$S(r, \theta, \varphi, t) = r - R(t) - \sum_{n,m} a_{nm}(t) Y_n^m(\theta, \varphi), \quad (4.1)$$

where $S(r, \theta, \varphi, t) = 0$ is the equation of the bubble surface, $R(t)$ is the instantaneous average radius, and $a_{nm}(t)$ is the amplitude of the spherical harmonic component of order n and degree m . It should be emphasized, however, that (4.1) is not the only possible choice, nor is it the most convenient for all problems.

A basic result in the dynamics of nonspherical bubbles has been obtained by Plesset (1954), who solved the equation of motion of an incompressible, inviscid, unbounded fluid with a free surface given by (4.1) in the small-amplitude approximation $|a_{nm}|/R \ll 1$. In this approximation the equations for R and the a_{nm} 's are uncoupled, the first one being just the Rayleigh equation. The equations for the a_{nm} 's do not contain the index m (which accordingly will be dropped in the following) and are found to be

$$\ddot{a}_n - 3(\dot{R}/R)\dot{a}_n + (n-1)[(n+1)(n+2)\sigma/\rho_l R^3 - \dot{R}/R]a_n = 0. \quad (4.2)$$

If the surface-tension term is neglected, this equation reduces to one also given by Birkhoff (1954). It is easily seen that $n = 1$ corresponds to a translation of the bubble

center. In this case Equation (4.2) gives just $R^3 \dot{a}_1 = \text{constant}$, which expresses the constancy of the total liquid momentum which is proportional to the bubble volume times its translational velocity. If all the other a 's vanish this happens of course to be an exact result independent of the small-amplitude approximation. This situation, however, is of little interest to us here, and we shall restrict our considerations to the case $n \geq 2$ in the following.

The introduction of viscous effects complicates the matter substantially, giving to Equation (5.2) an integro-differential structure (Prosperetti 1976c) except in the case where the viscous-diffusion length is small compared with the bubble radius, for which it becomes

$$\ddot{a}_n + [3\dot{R}/R + 2(n+2)(2n+1)v_l/R^2]\dot{a}_n + (n-1)[(n+1)(n+2)\sigma/\rho_l R^3 + 2(n+2)v_l\dot{R}/R^3 - \dot{R}/R]a_n = 0, \quad (4.3)$$

v_l denoting the liquid kinematic viscosity. Both Plesset (1954) and Prosperetti (1976c) have also considered the dynamical effects of the fluid contained in the bubble. The expressions given above are the limiting forms obtained when its density and viscosity vanish.

The Stability of Spherical Growth and Collapse

Even with the neglect of the surface-tension term (which has always a stabilizing effect, but which is important only for very small bubble radii), the qualitative behavior of the solution of Equation (4.2) is not readily understood a priori. Indeed, the stability characteristics do not depend only on the acceleration of the interface, as in the familiar plane case of the Rayleigh-Taylor instability, but also on its velocity. This situation is clearly a consequence of the geometry, since the divergence of the streamlines during bubble growth has a stabilizing effect, while the reverse occurs during collapse. In the limiting case of very large acceleration and of small velocity, such as take place for instance during the early stages of the growth of an underwater-explosion bubble, Equation (4.2) predicts an instability of the spherical shape, which is indeed known to exist (Cole 1948, illustration facing p. 247). Another example of the occurrence of this situation is furnished by the large-amplitude radial pulsations of a permanent gas bubble, for which the radial acceleration becomes very large and positive near the position of minimum radius, with the consequent possibility of large deformations and breakup. A similar situation might be expected to take place also in the early stages of the growth of a cavitation or boiling bubble, but here the duration of this stage of large acceleration and small velocity is so short that the instability does not have the time to develop.

The quantitative investigation of the stability for a bubble expanding or collapsing under a fixed pressure difference $P = p_i - p_\infty$ (cavitation bubble) has been carried out by Plesset & Mitchell (1956), who were able to obtain closed-form solutions of Equation (4.2). The qualitative features of their results can be readily understood with the aid of Equations (3.4) and (3.6). For the growth case we have $\dot{R}^2 \simeq (2/3)P/\rho$, from which one readily obtains $a_n \rightarrow \text{constant}$ as $R \rightarrow \infty$. Hence, although on the basis of the plane Rayleigh-Taylor case one would expect an unstable behavior

during the growth, in practice no significant deviations from the spherical shape take place because the acceleration is large only for a small fraction of the process, and its destabilizing influence is very effectively counterbalanced by the stretching of the bubble surface produced by the divergence of the streamlines. It may also be remarked that, in spite of the factor $(n-1)$ multiplying the radial acceleration in Equation (4.2), high growth rates for instabilities of large order n do not take place because of viscous effects [cf Equation (4.3)].

For the study of the collapse of a cavitation bubble it is expedient to set $b_n = (R_i/R)^{3/2} a_n$, so that Equation (4.2) takes the form

$$\ddot{b}_n - \left[(3/4)(\dot{R}/R)^2 + (n + \frac{1}{2})\ddot{R}/R - (n-1)(n+1)(n+2)\sigma/\rho_l R^3 \right] b_n = 0. \quad (4.4)$$

Now, from Equation (3.6) we have during collapse that $\dot{R}^2 \simeq -(2/3)(P/\rho_l)(R_i/R)^3$ from which the coefficient of b_n in this equation is seen to have the form $-nc^2R^{-5}$, where c is a constant. An approximation of the W.K.B. type gives then

$$a_n \simeq R^{-1/4} \exp\left(\pm icn^{1/2} \int R^{-5/2} dt\right),$$

whence it is seen that a_n increases proportionally to $R^{-1/4}$ and oscillates with increasing frequency as $R \rightarrow 0$ (Plesset & Mitchell 1956; Birkhoff 1954, 1956). The decisive role played by the sinklike converging nature of the flow in the development of this instability is apparent from the second term of Equation (4.2).

Clearly, Equation (4.4) implies the breakup of the bubble when the amplitude of the oscillations becomes of the order of R . The results of the linearized approximation, however, cannot be expected to be quantitatively valid up to this point. A fully nonlinear numerical calculation of the shape of an empty bubble during the collapse has been carried out by Chapman & Plesset (1972). They have obtained local bubble-wall velocities of the order of hundreds of meters per second at the points where the breakup of the bubble takes place. They have also shown that the linearized result is surprisingly accurate for most of the duration of the collapse. The greatest error in the description of the bubble shape is caused by the growth of harmonic components of different orders from those initially present, which is induced by the nonlinear couplings and which cannot be predicted on the basis of the linear theory.

The instability of the spherical shape of collapsing vapor bubbles at high sub-coolings is well documented in the literature (see e.g. Florschuetz & Chao 1965, Figure 11; and Lauterborn 1974). However, no reduction of the data in terms of spherical harmonic components has yet been attempted.

Bubble Collapse in the Vicinity of a Rigid Boundary

We have seen that, in principle, deviations from the spherical shape for a bubble in an unbounded liquid can take place only through the amplification of preexisting small perturbations. No such initial perturbation, however, is necessary for the occurrence of deformations of a bubble near a boundary, because the asymmetry of the flow that the boundary itself introduces is sufficient to give rise to highly distorted bubble shapes. It is possible to get an insight into these effects by observing

that, with the neglect of viscosity, the potential problem can be solved by the method of images that replaces the rigid boundary by an image bubble equal to the real one and located symmetrically to it with respect to the boundary. From this observation one is led to expect that the portion of the bubble farther from the wall acquires a greater velocity than the one near the wall, because there the velocity of the flow induced by the image bubble adds to the collapse velocity, while in the second case it decreases the collapse velocity. This asymmetry leads to the formation of a high-velocity jet directed toward the wall, while the overall characteristics of the sinklike flow of the image attracts the bubble toward the wall (Lauterborn & Bolle 1975).

That jet formation during bubble collapse could be responsible for cavitation damage was suggested as early as 1944 by Kornfeld & Suvorov, but their conjecture was not explored further, and the accepted explanation remained that of the large pressures associated with the latest stages of bubble collapse that had originally been put forward by Rayleigh (1917). The first experimental demonstration of jet formation was obtained by Naudé & Ellis (1961), and definitive conclusions about the mechanism of cavitation damage was later reached by Benjamin & Ellis (1966). This latter paper contains also an extensive summary of the history of the subject and of related work. Additional considerations on the magnitude of the stresses induced by jet impingement can be found in Plesset & Chapman (1971) and in Plesset (1974). Direct measurement of jet velocities is rather difficult. Operating at low ambient pressure, Benjamin & Ellis (1966) obtained velocities of 10 m sec^{-1} , which scale to velocities an order of magnitude greater at an ambient pressure of one atmosphere. The value of 120 m sec^{-1} is reported by Kling & Hammitt (1972) and Lauterborn & Bolle (1975), and is consistent with the numerical results of Plesset & Chapman (1971).

As might be expected, an analytical treatment of the problem of bubble collapse in the vicinity of a rigid boundary is very difficult. An early perturbation approach is that of Rattray (1951), who was successful in predicting the possibility of jet formation and also in predicting the elongation of the bubble in a direction perpendicular to the boundary in the early stages of the collapse. The first complete theoretical analysis of the collapse of an empty cavity in the neighborhood of a rigid wall was obtained numerically by Plesset & Chapman (1971), whose results have been experimentally confirmed by Lauterborn & Bolle (1975). Figure 11, from the work of these authors, shows a comparison of the theoretical results (*continuous lines*) with the experimental ones (*open circles*). The comparison is quite satisfactory, particularly in view of the difficulty in establishing a correspondence between theoretical and experimental initial conditions. The case of a bubble collapsing attached to a wall has also been worked out numerically by Plesset & Chapman (1971).

Surface Oscillations

Before considering some of the phenomena associated with the interplay between radial motion and surface deformations of permanent gas bubbles, we discuss the shape oscillations of a bubble of fixed radius R_0 . The governing equation for this

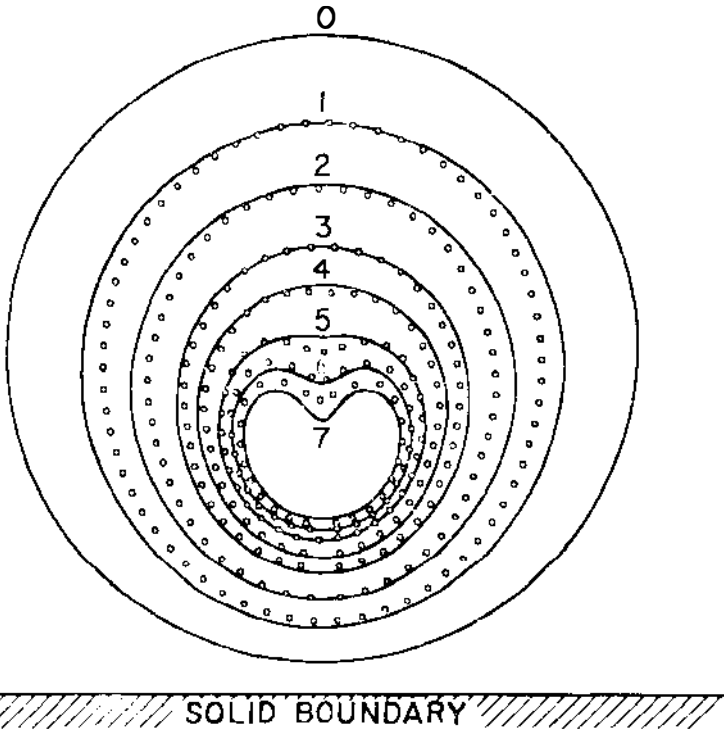


Figure 11 Comparison of the experimental results of Lauterborn & Bolle (1975, open circles) with the theoretical ones of Plesset & Chapman (1971) for the collapse of a cavitation bubble in the vicinity of a rigid boundary (from Lauterborn & Bolle 1975). Bubble shapes corresponding to later values of time than those shown here are given by Plesset & Chapman (1971).

situation is readily obtained from (4.2) or (4.3), and one finds the following expression for the natural frequency of the n th mode

$$\omega_{\sigma n}^2 = (n-1)(n+1)(n+2)\sigma/R_0^3\rho_l \tag{4.5}$$

The familiar dispersion relation for capillary waves on a flat liquid surface is readily obtained from this equation by observing that the wavelength λ is given by $\lambda = 2\pi R_0/n$ and by letting R_0 and n tend to infinity with λ held fixed. For small viscosity, one obtains from (4.3) the viscous damping constant β_n of the oscillations as $\beta_n = (n+2)(2n+1)v_l/R_0^2$ (cf Lamb 1932, p. 475, p. 641).

An understanding of the general behavior of the surface deformations for a bubble undergoing radial pulsations can be obtained in the limit of small-amplitude oscillations by writing $R = R_0(1 + \delta \sin \omega t)$ and then retaining only linear terms in δ (Hsieh & Plesset 1961; Hsieh 1972a, 1974a). Equation (4.4) takes then the form of a Mathieu equation,

$$\ddot{b}_n + (\omega_{0n}^2 + \alpha_n \sin \omega t) b_n = 0, \quad (4.6)$$

with $\alpha_n = [(n+1/2)\omega^2 - 3\omega_{0n}^2]\delta$ and ω_{0n}^2 given by (4.5). It is well known from the theory of the Mathieu equation that, depending on the magnitudes of the constants α_n and ω_{0n}^2 , the solutions of (4.6) may have the form of modulated oscillations the amplitude of which grows exponentially with time. Although Equation (4.5) itself is not valid unless $|b_n| \ll R_0$, this behavior does indicate that parametric excitation of the instability of the spherical shape is possible, and describes the early stages of its development. It may be observed that the most easily excited surface oscillation is the one corresponding to $n = 2$, the threshold of which has a minimum in the vicinity of $\omega = 2\omega_0$. The acoustic emission from a bubble executing surface oscillations in this mode would contain a subharmonic component, and indeed this possibility for the origin of the subharmonic component in acoustically cavitating liquids has also been considered. However, the observed intensity of the signal is incompatible with this hypothesis because, as pointed out by Strasberg (1956), bubbles executing shape oscillations are inefficient sound sources because of the rapid decrease of the velocity potential with distance from the bubble center.

Approximations to Equation (4.4) of a higher order in δ than (4.6) have been carried out by Eller & Crum (1970) in an attempt to explain the observed instability of the position of a pulsating bubble in a sound field first reported by Benjamin and Strasberg (see Benjamin 1964). Although the connection between the onset of surface distortions and the observed erratic translatory motion of the bubble is not obvious, the instability thresholds computed from (4.4) do exhibit a fair agreement with the experimental data for the onset of the bubble translatory motion. More direct experimental observations on the onset of surface oscillations are not very readily obtained in low-viscosity liquids for several reasons. Gould (1974) reported some results that have a large scatter but show agreement with theory in the general trend. Storm (1974) has studied large bubbles trapped in a gel, but the interpretation of his observations is somewhat obscured by the complex rheological characteristics of the suspending medium. Results of studies with large bubbles trapped on a plate have also been reported (Gould 1966, Howkins 1965, Blue 1967), but here a difficulty is encountered in accounting for the presence of the rigid boundary in the theory (in this connection see also Strasberg 1953).

Numerical results for stability thresholds for freely oscillating gas bubbles have been obtained by Strube (1971), who solved simultaneously the Rayleigh equation for a freely pulsating gas bubble and Equation (4.4). He presents results on maximum bubble deformations (which frequently appear shortly after the instant of minimum radius) and gives a chart of the stability boundaries. No work of this type is available for forced oscillations. Experiments on the distortion of a translating gas bubble subject to a pressure step were reported by Smulders & Van Leeuwen (1974) and compared with a theoretical analysis by Hermans (1973). Several results drawn from the theory are confirmed on a qualitative basis although a quantitative comparison was not attempted by the investigators.

As the last topic of this article, we mention a variational approach to nonspherical bubble dynamics formulated by Hsieh (1974b). The difficulties encountered in the accurate description of the shape of nonspherical bubbles are such that there

certainly is a strong case for attempting a variational attack. The results obtained so far, however, are rather limited, and it appears that additional research is necessary before the variational formulation can fulfill its promises of effectiveness and relative simplicity.

5 CONCLUDING REMARKS

It is clear that, in view of the very large number of papers on the subject, even the present list of references is quite incomplete. Further, in view of the fact that the present article is the first one on this subject to appear in the *Annual Review of Fluid Mechanics*, we felt that a relatively detailed exposition of the most important established results was in order even at the expense of omitting some more recent developments such as the dynamics of bubbles in non-Newtonian liquids. We have also tried to cover the fundamental physical aspects of the problems with little or no mention of practical developments. Thus we have been very concise in our coverage of the literature on bubble dynamics in connection with boiling heat transfer. We have also given only a summary of the processes and results for acoustic cavitation and flow cavitation. Finally, we have neglected entirely the phenomena related to the translatory motion of bubbles that have recently been considered elsewhere by Harper (1972).

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