2

The boundary integral equations

2.1 Green's functions of Stokes flow

The Green's functions of Stokes flow represent solutions of the continuity equation $\nabla \cdot \mathbf{u} = 0$ and the singularly forced Stokes equation

$$-\nabla P + \mu \nabla^2 \mathbf{u} + \mathbf{g}\delta(\mathbf{x} - \mathbf{x}_0) = 0 \qquad (2.1.1)$$

where $\mathbf{g}$ is an arbitrary constant, $\mathbf{x}_0$ is an arbitrary point, and δ is the three-dimensional delta function. Introducing the Green's function $\mathbf{G}$, we write the solution of (2.1.1) in the form

$$u_i(\mathbf{x}) = \frac{1}{8\pi\mu} G_{ij}(\mathbf{x}, \mathbf{x}_0) g_j \qquad (2.1.2)$$

where $\mathbf{x}_0$ is the *pole* or the *source point*, and $\mathbf{x}$ is the *observation* or *field point*. Physically, (2.1.2) expresses the velocity field due to a concentrated point force of strength $\mathbf{g}$ placed at the point $\mathbf{x}_0$, and may be identified with the flow produced by the slow settling of a small particle. In the literature of boundary integral methods, the Green's function may appear under the names *fundamental solution* or *propagator*.

It is convenient to classify the Green's functions into three categories depending on the topology of the domain of flow. First, we have the free-space Green's function for infinite unbounded flow; second, the Green's functions for infinite or semi-infinite flow that is bounded by a solid surface; and third, the Green's functions for internal flow that is completely confined by solid surfaces. The Green's functions in the second and third categories are required to vanish over the internal or external boundaries of the flow. As the observation point $\mathbf{x}$ approaches the pole $\mathbf{x}_0$ all Green's functions exhibit singular behaviour and, to leading order, behave like the free-space Green's function. The Green's functions for infinite unbounded or bounded flow are required to decay at infinity at a rate equal to or lower than that of the free-space Green's function.

Taking the divergence of (2.1.2) and using the continuity equation we find

$$\frac{\partial G_{ij}}{\partial x_i}(\mathbf{x}, \mathbf{x}_0) = 0 \tag{2.1.3}$$

Integrating (2.1.3) over a volume of fluid that is bounded by the surface D and using the divergence theorem, we find

$$\int_D G_{ij}(\mathbf{x}, \mathbf{x}_0) n_i(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) = 0 \tag{2.1.4}$$

independently of whether the pole $\mathbf{x}_0$ is located inside, right on, or outside D.

The vorticity, pressure, and stress fields associated with the flow (2.1.2) may be presented in the corresponding forms:

$$\omega_i(\mathbf{x}) = \frac{1}{8\pi\mu} \Omega_{ij}(\mathbf{x}, \mathbf{x}_0) g_j \tag{2.1.5}$$

$$P(\mathbf{x}) = \frac{1}{8\pi} p_j(\mathbf{x}, \mathbf{x}_0) g_j \tag{2.1.6}$$

$$\sigma_{ik}(\mathbf{x}) = \frac{1}{8\pi} T_{ijk}(\mathbf{x}, \mathbf{x}_0) g_j \tag{2.1.7}$$

where $\boldsymbol{\Omega}$, $\mathbf{p}$, and $\mathbf{T}$ are the vorticity tensor, pressure vector, and stress tensor associated with the Green's function. The stress tensor $\mathbf{T}$, in particular, is defined as

$$T_{ijk}(\mathbf{x}, \mathbf{x}_0) = -\delta_{ik} p_j(\mathbf{x}, \mathbf{x}_0) + \frac{\partial G_{ij}}{\partial x_k}(\mathbf{x}, \mathbf{x}_0) + \frac{\partial G_{kj}}{\partial x_i}(\mathbf{x}, \mathbf{x}_0) \tag{2.1.8}$$

It will be noted that $T_{ijk} = T_{kji}$ as required by the symmetry of the stress tensor $\boldsymbol{\sigma}$. When the domain of flow is infinite, we require that all $\boldsymbol{\Omega}$, $\mathbf{p}$, and $\mathbf{T}$ vanish as the observation point is moved to infinity.

Substituting (2.1.2), (2.1.6), and (2.1.8) into (2.1.1) we obtain the equations

$$-\frac{\partial p_j}{\partial x_k}(\mathbf{x}, \mathbf{x}_0) + \nabla^2 G_{kj}(\mathbf{x}, \mathbf{x}_0) = -8\pi\delta_{kj}\delta(\mathbf{x} - \mathbf{x}_0) \tag{2.1.9}$$

and

$$\frac{\partial T_{ijk}}{\partial x_i}(\mathbf{x}, \mathbf{x}_0) = \frac{\partial T_{jki}}{\partial x_i}(\mathbf{x}, \mathbf{x}_0) = -8\pi\delta_{kj}\delta(\mathbf{x} - \mathbf{x}_0) \tag{2.1.10}$$

Furthermore, using (2.1.10) we find

$$\frac{\partial}{\partial x_k}[\varepsilon_{ilm} x_l T_{mjk}(\mathbf{x}, \mathbf{x}_0)] = -8\pi\varepsilon_{ilj} x_l \delta(\mathbf{x} - \mathbf{x}_0) \tag{2.1.11}$$

Integrating (2.1.10) and (2.1.11) over the volume of fluid enclosed by the smooth surface D and using the divergence theorem to convert the volume integral into a surface integral, we obtain the identities

$$\int_D T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_i(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) = \int_D T_{kji}(\mathbf{x}, \mathbf{x}_0) n_i(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) = -\begin{bmatrix} 8\pi \\ 4\pi \\ 0 \end{bmatrix} \delta_{jk} \tag{2.1.12}$$

$$\varepsilon_{ilm} \int_D x_l T_{mjk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) = -\begin{bmatrix} 8\pi \\ 4\pi \\ 0 \end{bmatrix} \varepsilon_{ilj} x_{0,l} \tag{2.1.13}$$

where the unit normal vector $\mathbf{n}$ is directed outside the control volume, and $x_{0,l}$ on the right-hand side of (2.1.13) indicates the l component of $\mathbf{x}_0$. The values -8π, -4π, and 0 on the right-hand sides of (2.1.12) and (2.1.13) apply when the point $\mathbf{x}_0$ is located respectively inside, right on, or outside D. When $\mathbf{x}_0$ is right on D, the integrals in (2.1.12) and (2.1.13) are improper but convergent (see discussion at the end of section 2.3).

In section 3.2 we shall see that the pressure vector $\mathbf{p}$ and the stress tensor $\mathbf{T}$ associated with a Green's function for infinite unbounded or bounded flow represent two fundamental solutions of Stokes flow. Specifically, we shall show that $\mathbf{p}(\mathbf{x}, \mathbf{x}_0)$ represents the velocity field at the point $\mathbf{x}_0$, due to a point source of strength -8π with pole at $\mathbf{x}$. Furthermore, we shall show that

$$u_j(\mathbf{x}_0) = T_{ijk}(\mathbf{x}, \mathbf{x}_0) q_{ik} \tag{2.1.14}$$

where $\mathbf{q}$ is a constant matrix, represents the velocity field due to a singularity called the stresslet with pole at $\mathbf{x}$. The pressure field corresponding to (2.1.14) may be conveniently expressed in terms of a pressure matrix $\mathbf{\Pi}$ as

$$P(\mathbf{x}_0) = \mu \Pi_{ik}(\mathbf{x}_0, \mathbf{x}) q_{ik} \tag{2.1.15}$$

The precise definition and further properties of $\mathbf{\Pi}$ will be discussed in section 3.2.

Adding a number of Green's functions with different poles $\mathbf{x}_n$ we can devise a Green's function with multiple poles, namely

$$\mathbf{G} = \sum_{n=1}^{N} \mathbf{G}(\mathbf{x}, \mathbf{x}_n) \tag{2.1.16}$$

In the limiting case where an infinite number of poles are placed exceedingly close to each other, the sum in (2.1.16) reduces to an integral yielding a line, surface, or volume distribution of point forces. Differentiating the Green's function with respect to the pole $\mathbf{x}_0$ we can derive differential singular solutions representing multipoles of the point force (see section 7.2). For instance, differentiating the Green's function once, we obtain the point force doublet that represents the flow produced by two point forces with opposite strengths and indistinguishable poles.

2.2 The free-space Green's function

To compute the free-space Green's function we replace the delta function on the right-hand side of (2.1.1) with the equivalent expression

$$\delta(\hat{\mathbf{x}}) = -\frac{1}{4\pi}\nabla^2\left(\frac{1}{r}\right) \qquad (2.2.1)$$

where $r = |\hat{\mathbf{x}}|$, $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$. Recalling that the pressure is a harmonic function, and balancing the dimensions of the pressure term with those of the delta function in equation (2.1.1), we set

$$P = -\frac{1}{4\pi}\mathbf{g}\cdot\nabla\left(\frac{1}{r}\right) \qquad (2.2.2)$$

Substituting (2.2.1) and (2.2.2) into (2.1.1) we obtain

$$\mu\nabla^2\mathbf{u} = -\frac{1}{4\pi}\mathbf{g}\cdot(\nabla\nabla - \mathbf{I}\nabla^2)\left(\frac{1}{r}\right) \qquad (2.2.3)$$

Next, we express the velocity in terms of a scalar function H as

$$\mathbf{u} = \frac{1}{\mu}\mathbf{g}\cdot(\nabla\nabla - \mathbf{I}\nabla^2)H \qquad (2.2.4)$$

It will be noted that the continuity equation is satisfied for any choice of H. Substituting (2.2.4) into (2.2.3) and discarding the arbitrary constant $\mathbf{g}$ we obtain

$$(\nabla\nabla - \mathbf{I}\nabla^2)\left(\nabla^2 H + \frac{1}{4\pi r}\right) = 0 \qquad (2.2.5)$$

Clearly, (2.2.5) is satisfied by any solution of Poisson's equation, $\nabla^2 H = -1/(4\pi r)$. Using (2.2.1) we find that H is, in fact, the fundamental solution of the biharmonic equation $\nabla^4 H = \delta(\hat{\mathbf{x}})$. Thus

$$H = -\frac{r}{8\pi} \qquad (2.2.6)$$

Substituting (2.2.6) into (2.2.4) we find

$$u_i(\mathbf{x}) = \frac{1}{8\pi\mu}\mathcal{S}_{ij}(\hat{\mathbf{x}})g_j \qquad (2.2.7)$$

where

$$\mathcal{S}_{ij}(\hat{\mathbf{x}}) = \frac{\delta_{ij}}{r} + \frac{\hat{x}_i\hat{x}_j}{r^3} \qquad (2.2.8)$$

is the free-space Green's function, also called the *Stokeslet*, or the

Oseen–Burgers tensor. The vorticity, pressure, and stress fields associated with the flow (2.2.7) may be written in the standard forms (2.1.5), (2.1.6), and (2.1.7) where

$$\Omega_{ij}(\hat{\mathbf{x}}) = 2\varepsilon_{ijl}\frac{\hat{x}_l}{r^3} \tag{2.2.9}$$

and

$$p_i(\hat{\mathbf{x}}) = 2\frac{\hat{x}_i}{r^3} \tag{2.2.10}$$

Substituting (2.2.7) and (2.2.10) into (2.1.8) we obtain the stress tensor

$$T_{ijk}(\hat{\mathbf{x}}) = -6\frac{\hat{x}_i\hat{x}_j\hat{x}_k}{r^5} \tag{2.2.11}$$

As mentioned in section 2.1, $\mathbf{p}$ and $\mathbf{T}$ represent two fundamental solutions of Stokes flow. Specifically, $\mathbf{p}$ represents the velocity at the point $\mathbf{x}$ due to a point source of strength 8π with pole at $\mathbf{x}_0$, or, equivalently, the velocity at $\mathbf{x}_0$ due to a point source of strength -8π with pole at $\mathbf{x}$, whereas

$$u_j(\mathbf{x}_0) = T_{ijk}(\mathbf{x}-\mathbf{x}_0)q_{ik} = -T_{ijk}(\mathbf{x}_0-\mathbf{x})q_{ik} \tag{2.2.12}$$

where $\mathbf{q}$ is a constant matrix, represents the velocity field due to a stresslet with pole at $\mathbf{x}$. Using the results of section 7.2 we find that the pressure field corresponding to the flow (2.2.12) is given by (2.1.15) where

$$\Pi_{ik}(\mathbf{x}_0, \mathbf{x}) = 4\left(-\frac{\delta_{ik}}{r^3} + 3\frac{\hat{x}_i\hat{x}_k}{r^5}\right) \tag{2.2.13}$$

The associated stress field will be discussed in problem 2.2.2.

Now, as an exercise, we shall compute the surface force exerted on a fluid sphere of radius r centered at the pole of a point force. Using (2.1.7) and (2.2.11) we find

$$f_i(\mathbf{x}) = \sigma_{ik}(\mathbf{x})n_k(\mathbf{x}) = \frac{1}{8\pi}T_{ijk}(\mathbf{x}, \mathbf{x}_0)n_k(\mathbf{x})g_j = -\frac{3}{4\pi}\frac{\hat{x}_i\hat{x}_j}{r^4}g_j \tag{2.2.14}$$

The force acting on the sphere is

$$F_i = \int_{\text{sphere}} f_i(\mathbf{x})\,\mathrm{d}S(\mathbf{x}) = -\frac{3}{4\pi}g_j\frac{1}{r^4}\int_{\text{sphere}} \hat{x}_i\hat{x}_j\,\mathrm{d}S(\mathbf{x}) \tag{2.2.15}$$

Using the divergence theorem we compute

$$\int_{\text{sphere}} \hat{x}_i\hat{x}_j\,\mathrm{d}S(\mathbf{x}) = r\int_{\text{sphere}} \hat{x}_i n_j\,\mathrm{d}S(\mathbf{x}) = r\int_{\text{sphere}} \frac{\partial \hat{x}_i}{\partial \hat{x}_j}\,\mathrm{d}V(\mathbf{x}) = \delta_{ij}\tfrac{4}{3}\pi r^4 \tag{2.2.16}$$

Combining (2.2.15) and (2.2.16) we find $\mathbf{F} = -\mathbf{g}$ independently of the radius of the sphere, in agreement with our previous discussion in section 1.2. The torque with respect to the pole of a point force on any surface that encloses the pole of the point force is equal to zero (see problem 2.2.3).

Problems

2.2.1 An alternative method for deriving the free-space Green's function is by using Fourier transforms. Take the three-dimensional complex Fourier transform of (2.1.1) and the continuity equation to find

$$\hat{G}_{ij} = \frac{4}{(2\pi)^{1/2}} \frac{1}{|\mathbf{k}|^2}\left(\delta_{ij} - \frac{k_i k_j}{|\mathbf{k}|^2}\right) \qquad \hat{p}_j = -\frac{4\mathrm{i}}{(2\pi)^{1/2}} \frac{k_j}{|\mathbf{k}|^2} \tag{1}$$

where the three-dimensional complex Fourier transform of a function $f(\mathbf{x})$ is defined as

$$\hat{f}(\mathbf{k}) = \frac{1}{(2\pi)^{3/2}} \int_{\substack{\text{whole}\\ \text{space}}} f(\mathbf{x}) \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{x})\,\mathrm{d}\mathbf{x} \tag{2}$$

Next, invert (1) using

$$f(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}} \int_{\substack{\text{whole}\\ \text{space}}} \hat{f}(\mathbf{k}) \exp(\mathrm{i}\mathbf{x}\cdot\mathbf{k})\,\mathrm{d}\mathbf{k} \tag{3}$$

to obtain the Stokeslet (Ladyzhenskaya 1969, p. 50).

2.2.2 Show that the stress field associated with the flow (2.2.12) is given by $\sigma_{ik} = 2\mu T^{\mathrm{STR}}_{ijlk}(\hat{\mathbf{x}}) q_{jl}$, where the stress tensor $\mathbf{T}^{\mathrm{STR}}$ is given in (7.2.26).

2.2.3 Using (2.2.14) show that the torque with respect to the pole of a point force on any surface that encloses the pole of the point force is equal to zero. What is the torque with respect to another point in space?

2.3 The boundary integral equation

It is well known that the solution of linear, elliptic, and homogeneous boundary value problems may be represented in terms of boundary integrals involving the boundary values of the unknown function and its derivatives (Stakgold 1968). One example of a boundary integral representation is Green's third identity for harmonic functions (Kellogg 1954, p. 219). Another example is Somigliana's identity for the displacement field in linear elastostatics (Love 1944, p. 245). In the case of Stokes flow, we obtain a boundary integral representation involving the boundary values of the velocity and surface force.

A convenient starting point for deriving the boundary integral equation is the Lorentz reciprocal identity (1.4.4) stating that for any two nonsingular (regular) flows $\mathbf{u}$ and $\mathbf{u}'$ with corresponding stress tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}'$,

$$\frac{\partial}{\partial x_k}(u_i'\sigma_{ik} - u_i\sigma_{ik}') = 0 \tag{2.3.1}$$

Identifying $\mathbf{u}'$ with the flow due to a point force with strength $\mathbf{g}$ located at the point $\mathbf{x}_0$, we obtain

$$u_i'(\mathbf{x}) = \frac{1}{8\pi\mu} G_{ij}(\mathbf{x}, \mathbf{x}_0) g_j \qquad \sigma_{ik}'(\mathbf{x}) = \frac{1}{8\pi} T_{ijk}(\mathbf{x}, \mathbf{x}_0) g_j \tag{2.3.2}$$

Substituting (2.3.2) into (2.3.1), and discarding the arbitrary constant $\mathbf{g}$, we obtain

$$\frac{\partial}{\partial x_k}[G_{ij}(\mathbf{x}, \mathbf{x}_0)\sigma_{ik}(\mathbf{x}) - \mu u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0)] = 0 \tag{2.3.3}$$

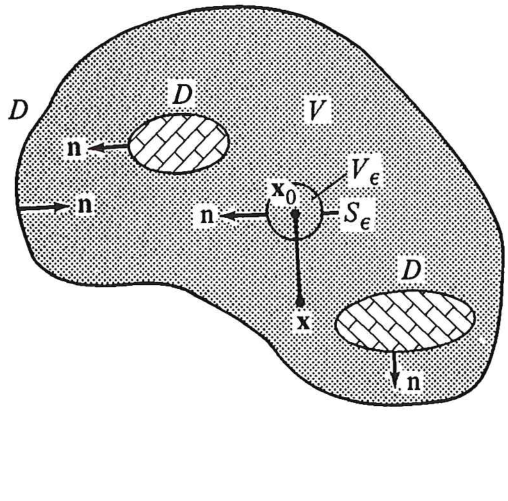
Now, we select a control volume V that is bounded by the closed (simply- or multiply-connected) surface D, as illustrated in Figure 2.3.1. Note that D may be composed of fluid surfaces, fluid interfaces, or solid surfaces. In addition, we select a point $\mathbf{x}_0$ outside V. Noting that the function within the square bracket in (2.3.3) is regular throughout V, integrating (2.3.3) over V, and using the divergence theorem to convert the volume integral over V into a surface integral over D, we obtain

$$\int_D [G_{ij}(\mathbf{x}, \mathbf{x}_0)\sigma_{ik}(\mathbf{x}) - \mu u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0)] n_k(\mathbf{x})\, dS(\mathbf{x}) = 0 \tag{2.3.4}$$

In (2.3.4) as well as in all subsequent equations, the normal vector $\mathbf{n}$ is directed into the control volume V.

Next we select a point $\mathbf{x}_0$ in the interior of V, and define a small spherical volume V_ε of radius ε centered at $\mathbf{x}_0$. The function within the square bracket

Figure 2.3.1. A control volume V within the domain of a flow.



in (2.3.3) is regular throughout the reduced volume $V - V_\varepsilon$. Integrating (2.3.3) over $V - V_\varepsilon$ and using the divergence theorem to convert the volume integral into a surface integral, we obtain

$$\int_{D,S_\varepsilon} [G_{ij}(\mathbf{x}, \mathbf{x}_0)\sigma_{ik}(\mathbf{x}) - \mu u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0)] n_k(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) = 0 \qquad (2.3.5)$$

where S_ε is the spherical surface enclosing V_ε as indicated in Figure 2.3.1. Letting the radius ε tend to zero we find that over S_ε, to leading order in ε, the tensors $\mathbf{G}$ and $\mathbf{T}$ reduce to the Stokeslet and its associated stress tensor, respectively, i.e.

$$G_{ij} \approx \frac{\delta_{ij}}{\varepsilon} + \frac{\hat{x}_i \hat{x}_j}{\varepsilon^3} \qquad T_{ijk} \approx -6\frac{\hat{x}_i \hat{x}_j \hat{x}_k}{\varepsilon^5} \qquad (2.3.6)$$

where $\hat{\mathbf{x}} = \mathbf{x} - \mathbf{x}_0$. Over S_ε, $\mathbf{n} = \hat{\mathbf{x}}/\varepsilon$ and $\mathrm{d}S = \varepsilon^2 \mathrm{d}\Omega$, where Ω is the differential solid angle. Substituting these expressions along with (2.3.6) into (2.3.5) we obtain

$$\int_D [G_{ij}(\mathbf{x}, \mathbf{x}_0)\sigma_{ik}(\mathbf{x}) - \mu u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0)] n_k(\mathbf{x})\, \mathrm{d}S(\mathbf{x})$$

$$= -\int_{S_\varepsilon} \left[\left(\delta_{ij} + \frac{\hat{x}_i \hat{x}_j}{\varepsilon^2} \right) \sigma_{ik}(\mathbf{x}) + 6\mu u_i(\mathbf{x}) \frac{\hat{x}_i \hat{x}_j \hat{x}_k}{\varepsilon^4} \right] \hat{x}_k \, \mathrm{d}\Omega \qquad (2.3.7)$$

As $\varepsilon \to 0$, the values of $\mathbf{u}$ and $\boldsymbol{\sigma}$ over S_ε tend to their corresponding values at the center of V_ε, i.e. to $\mathbf{u}(\mathbf{x}_0)$ and $\boldsymbol{\sigma}(\mathbf{x}_0)$, respectively. Since $\hat{\mathbf{x}}$ decreases linearly with ε, as $\varepsilon \to 0$ the contribution of the stress term within the integral on the right-hand side of (2.3.7) decreases linearly in ε, whereas the contribution of the velocity term tends to a constant value. Thus, in the limit $\varepsilon \to 0$, (2.3.7) reduces to

$$\int_D [G_{ij}(\mathbf{x}, \mathbf{x}_0)\sigma_{ik}(\mathbf{x}) - \mu u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0)] n_k(\mathbf{x})\, \mathrm{d}S(\mathbf{x})$$

$$= -6\mu u_i(\mathbf{x}_0) \frac{1}{\varepsilon^4} \int_{S_\varepsilon} \hat{x}_i \hat{x}_j \, \mathrm{d}S(\mathbf{x}) \qquad (2.3.8)$$

Using the divergence theorem we compute

$$\int_{S_\varepsilon} \hat{x}_i \hat{x}_j \, \mathrm{d}S(\mathbf{x}) = \varepsilon \int_{S_\varepsilon} \hat{x}_i n_j \, \mathrm{d}S(\mathbf{x}) = \varepsilon \int_{V_\varepsilon} \frac{\partial \hat{x}_i}{\partial \hat{x}_j} \mathrm{d}V(\mathbf{x}) = \delta_{ij} \tfrac{4}{3} \pi \varepsilon^4 \qquad (2.3.9)$$

Substituting (2.3.9) into (2.3.8) we finally obtain the desired boundary integral representation

$$u_j(\mathbf{x}_0) = -\frac{1}{8\pi\mu} \int_D \sigma_{ik}(\mathbf{x}) n_k(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0)\, \mathrm{d}S(\mathbf{x})$$

$$+ \frac{1}{8\pi} \int_D u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) \qquad (2.3.10)$$

It will be convenient to introduce the surface force $\mathbf{f} = \boldsymbol{\sigma} \cdot \mathbf{n}$ and rewrite (2.3.10) in the equivalent form

$$u_j(\mathbf{x}_0) = -\frac{1}{8\pi\mu}\int_D f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0)\, dS(\mathbf{x}) + \frac{1}{8\pi}\int_D u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, dS(\mathbf{x}) \qquad (2.3.11)$$

Equation (2.3.11) provides us with a representation of a flow in terms of two boundary distributions involving the Green's function $\mathbf{G}$ and the associated stress tensor $\mathbf{T}$. The densities of these distributions are proportional to the boundary values of the surface force and velocity. The first distribution on the right-hand of (2.3.11) is termed the *single-layer potential,* whereas the second distribution is termed the *double-layer potential.* A detailed discussion of the significance and properties of these potentials will be deferred until Chapter 4.

Now, viewing the double-layer potential as a mere mathematical function, we compute its limiting values as the point $\mathbf{x}_0$ approaches the boundary D either from the internal or from the external side, and obtain two different values. Specifically, if D is a Lyapunov surface, i.e. it has a continuously varying normal vector (see Jaswon & Symm 1977), and the velocity over D varies in a continuous manner, we find

$$\lim_{\mathbf{x}_0 \to D} \int_D u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, dS(\mathbf{x}) = \pm 4\pi u_j(\mathbf{x}_0) + \int_D^{\mathscr{PV}} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, dS(\mathbf{x}) \qquad (2.3.12)$$

where the plus sign applies when the point $\mathbf{x}_0$ approaches D from the side of the flow (indicated by the direction of the normal vector), and the minus sign otherwise (see section 4.3). The superscript $\mathscr{PV}$ indicates the principal value of the double-layer potential, defined as the value of the improper double-layer integral when the point $\mathbf{x}_0$ is right on D. Substituting (2.3.12) with the plus sign into (2.3.11) or with the minus sign into (2.3.4), we find that for a point $\mathbf{x}_0$ that is located right on the boundary D,

$$u_j(\mathbf{x}_0) = -\frac{1}{4\pi\mu}\int_D f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0)\, dS(\mathbf{x}) + \frac{1}{4\pi}\int_D^{\mathscr{PV}} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, dS(\mathbf{x}) \qquad (2.3.13)$$

In summary, equations (2.3.4), (2.3.11), and (2.3.13) are valid when the point $\mathbf{x}_0$ is located outside, inside, or right on the boundary of a selected volume of flow.

In section 3.1 we shall show that the Green's functions satisfy the symmetry property

$$G_{ij}(\mathbf{x}, \mathbf{x}_0) = G_{ji}(\mathbf{x}_0, \mathbf{x}) \tag{2.3.14}$$

which allows us to switch the order of the indices as long as we also switch the order of the arguments, i.e. the location of the observation point and the pole. Substituting (2.3.14) into (2.3.11) we obtain

$$u_j(\mathbf{x}_0) = -\frac{1}{8\pi\mu}\int_D G_{ji}(\mathbf{x}_0, \mathbf{x}) f_i(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) + \frac{1}{8\pi}\int_D u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) \tag{2.3.15}$$

Clearly, the single-layer potential on the right-hand side of (2.3.15) represents a boundary distribution of point forces with strength $-\mathbf{f}$. To understand the significance of the double-layer potential, we decompose the stress tensor $\mathbf{T}$ into its constituents using (2.1.8). Exploiting (2.3.14) we obtain

$$\int_D u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) = -\int_D p_j(\mathbf{x}, \mathbf{x}_0) u_i(\mathbf{x}) n_i(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) + \int_D \frac{\partial G_{ji}(\mathbf{x}_0, \mathbf{x})}{\partial x_k}(u_i n_k + u_k n_i)(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) \tag{2.3.16}$$

In section 3.2 we shall see that when $\mathbf{p}$ corresponds to a Green's function of infinite unbounded or bounded flow, the first integral on the right-hand side of (2.3.16) represents a distribution of point sources. The density of this distribution vanishes over a solid surface or stationary fluid interface where $\mathbf{u} = 0$ or $\mathbf{u}\cdot\mathbf{n} = 0$ respectively. The second integral on the right-hand side of (2.3.16) represents a distribution of symmetric point force dipoles.

Now, inspecting (2.3.15) suggests an expression for the pressure in terms of two boundary distributions corresponding to the single-layer and double-layer potential, namely

$$P(\mathbf{x}_0) = -\frac{1}{8\pi}\int_{\mathscr{C}} p_i(\mathbf{x}_0, \mathbf{x}) f_i(\mathbf{x})\, \mathrm{d}l(\mathbf{x}) + \frac{\mu}{8\pi}\int_{\mathscr{C}} u_i(\mathbf{x}) \Pi_{ik}(\mathbf{x}_0, \mathbf{x}) n_k(\mathbf{x})\, \mathrm{d}l(\mathbf{x}) \tag{2.3.17}$$

where $\mathbf{p}$ and $\mathbf{\Pi}$ express the pressure corresponding to the Green's function and its associated stress tensor defined in (2.1.6) and (2.1.15), respectively.

It will be instructive to apply the boundary integral equation for certain simple flows that are known to be exact solutions to the equations of Stokes flow. For instance, if we are considering rigid body motion then $\mathbf{u} = \mathbf{U} + \boldsymbol{\omega} \times \mathbf{x}$; setting $\mathbf{f} = -P\mathbf{n}$, where P is the constant pressure, and using (2.3.4), (2.3.11), (2.3.13), and (2.1.4) we find

$$\int_D T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) = \begin{bmatrix} 8\pi \\ 4\pi \\ 0 \end{bmatrix} \delta_{ij} \tag{2.3.18}$$

and

$$\varepsilon_{ilm}\int_D x_m T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x})\, \mathrm{d}S(\mathbf{x}) = \begin{bmatrix} 8\pi \\ 4\pi \\ 0 \end{bmatrix} \varepsilon_{jlm} x_{0,m} \tag{2.3.19}$$

for a point $\mathbf{x}_0$ located inside, right on, or outside D, respectively (in the second case the integrals should be interpreted in the principal value sense).

The reader will note that (2.3.18) and (2.3.19) are identical to (2.1.12) and (2.1.13) with the exception of a minus sign due to the opposite orientation of the normal vector (the normal vector in (2.1.12) and (2.1.13) is directed outside the control volume). Two sets of identities similar to (2.3.18) and (2.3.19) may be derived by applying the boundary integral equations for linear and parabolic flow (problem 2.3.4).

To derive the above boundary integral equations, we used the reciprocal identity (1.4.4). Had we used the alternative reciprocal identity discussed in problem (1.4.2), we would have obtained a different but equivalent set of equations. Specifically, for a point $\mathbf{x}_0$ that is located within a selected volume of flow we would have obtained

$$u_j(\mathbf{x}_0) = -\frac{1}{8\pi\mu}\int_D G_{ji}(\mathbf{x}_0, \mathbf{x})\left(-Pn_i + \frac{\partial u_i}{\partial x_k} n_k\right)(\mathbf{x})\, \mathrm{d}S(\mathbf{x})$$
$$+\frac{1}{8\pi}\int_D u_i(\mathbf{x})\left[-p_j(\mathbf{x}, \mathbf{x}_0) n_i(\mathbf{x}) + \frac{\partial G_{ji}(\mathbf{x}_0, \mathbf{x})}{\partial x_k} n_k(\mathbf{x})\right] \mathrm{d}S(\mathbf{x}) \tag{2.3.20}$$

which is the counterpart of (2.3.15) (Happel & Brenner 1973, p. 81). Due to the more direct physical significance of the density of the single-layer potential, equation (2.3.15) is preferable to (2.3.20) in theoretical analyses as well as numerical implementations.

Infinite flow

A number of problems involve flow in completely unbounded or partially bounded domains. Two examples are flow due to the motion of a small particle in an infinitely dilute suspension, and semi-infinite shear flow over a wall containing a depression or projection. In these cases, in order to apply the boundary integral equation, we select a control volume that is confined by a solid or fluid boundary S_B and a large spherical surface S_∞ extending to infinity. If the fluid at infinity is quiescent, the velocity must decay at least as fast as $1/r$, whereas the pressure and stress must decay at least as fast as $1/r^2$, where r is a typical distance from S_B. These scalings become evident by expanding the far pressure field in terms of spherical harmonics, requiring that the pressure at infinity tends to a constant value, and inspecting the corresponding velocity (Lamb 1932, section 335; see