II: Stationary interface

A, Matched asymptotic expansions (1)

We use a typical example to illustrate the idea of the method of matched asymptotic expansions. To use the perturbation method, there has to be a small parameter in the differential equation. Suppose we are asked to solve the differential equation

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0,$$

subject to boundary conditions $y(x=0)=0$ and $y(x=1)=1$, and ϵ is a small parameter, i.e. $0<\epsilon \ll 1$.

As an initial guess, one may expect

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 + \cdots\cdots$$

the solution is:

$$\boxed{y = \frac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}}$$

Substituting this expression into the differential equation, the leading order terms obey $y_0' + y_0 = 0$.

One can see that, since the highest order of derivative (y'') is multiplied by the small parameter ϵ in the full differential equation, the order of derivative of the differential equation for the leading order terms is reduced by one. The solution for the leading order equation is

$$y_0 = Ae^{-x}$$

which cannot satisfy both boundary conditions $y(0)=0$ and $y(1)=1$. One may then realize that $\varepsilon y''$ cannot be small everywhere. That means y'' can be very large in some region. Now we guess that y'' is very large when x is close to zero. Then $y_0 = Ae^{-x}$ is a valid solution for $x \gg 0$. Imposing the boundary condition $y(1)=1$, we obtain $y_0 = e^{1-x}$. We call it the outer solution and write $y_{out} = e^{1-x}$.

For $x \simeq 0$, we introduce a variable $\bar{x} \equiv x/\varepsilon$, the full differential equation then becomes $y_{\bar{x}\bar{x}} + (1+\varepsilon) y_{\bar{x}} + \varepsilon^2 y = 0$.

Defining $y_{in}(\bar{x})$ as the leading order solution for the equation.

Hence, $y_{in}''(\bar{x}) + y_{in}'(\bar{x}) = 0$. $\Rightarrow$ $y_{in} = Be^{-\bar{x}} + C$ subject to b.c. $y(\bar{x}=0)=0$.

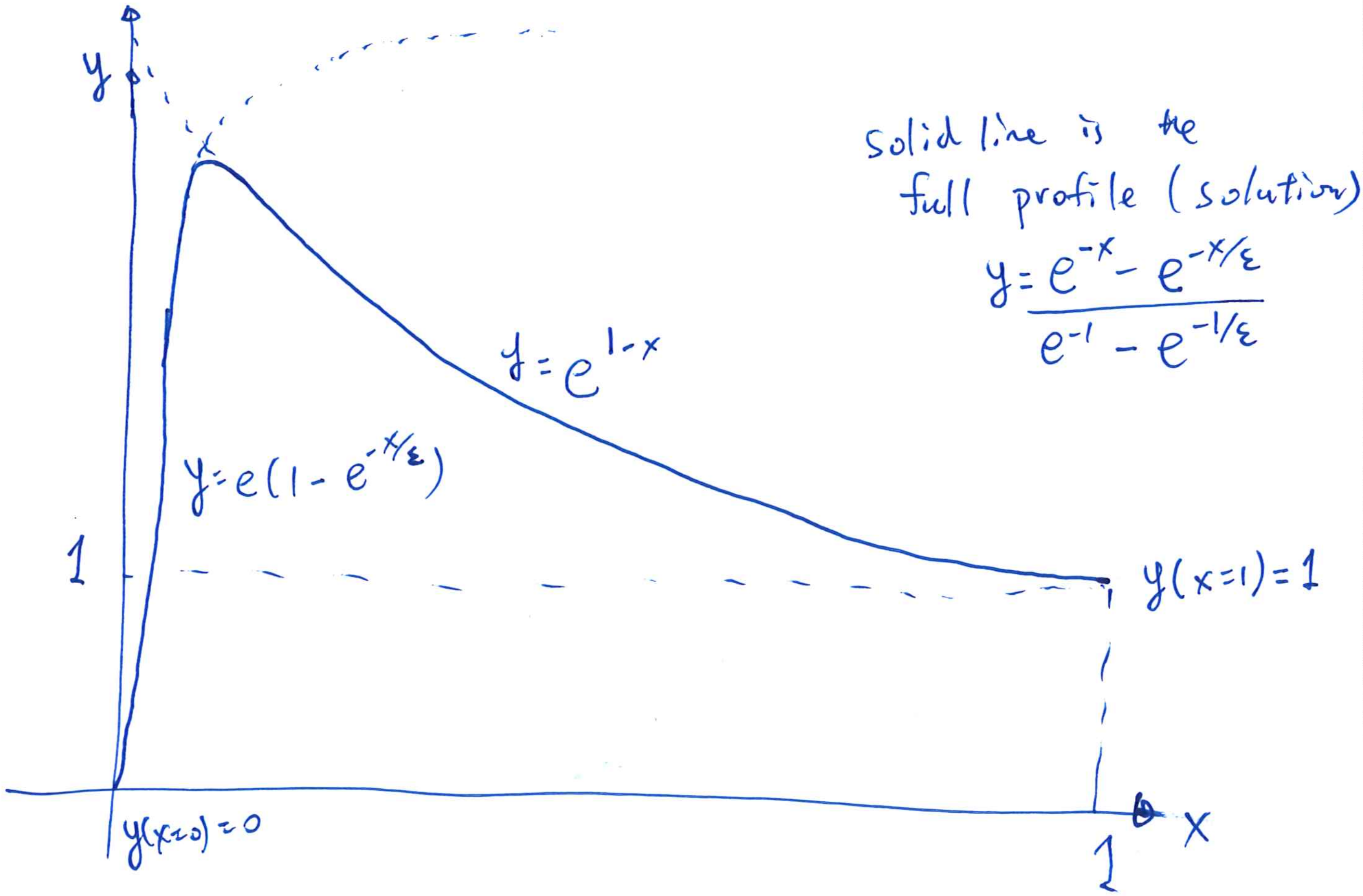
Then $y_{in} = B(e^{-\bar{x}} - 1)$

II: Stationary interface

A. Matched asymptotic expansions (2)

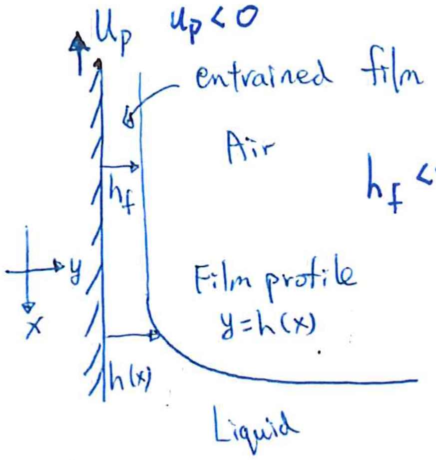
$$\boxed{\begin{array}{ll} y_{out} = e^{1-x} & \text{for } x \gg \varepsilon \\ y_{in} = B\left(e^{-x/\varepsilon} - 1\right) & \text{for } x \approx \varepsilon \end{array}}$$

To determine the unknown coefficient B, we have to match the the out solution and the inner solution.
when $x \to 0$, $y_{out} \to e$; when $\bar{x} = x/\varepsilon \to \infty$, $y_{in} \to -B$.
Matching the solutions gives $B = -e$.



II: Stationary interface

B. The Landau Levich Derjaguin film (1)



$h_f \ll l_c$

A completely wetting ($\theta_e = 0$) solid plate is being withdrawn from a viscous liquid bath. A liquid film is entrained and deposited on the solid surface. Question: How does the thickness of the film h_f scale with the plate speed U_p?

$$u_d = \frac{1}{h}\int_0^h u_x \, dy$$

For a stationary film profile, $\frac{\partial h}{\partial t} = 0 \Rightarrow \frac{\partial u_d h}{\partial x} = 0 \Rightarrow u_d h = q$.

Here q is a constant flux.

Including gravity, $u_x = \frac{\left(-\frac{\partial P}{\partial x} + \rho g\right)}{2\eta}\left[h^2 - (h-y)^2\right] + U_p$. $\quad U_p > 0$

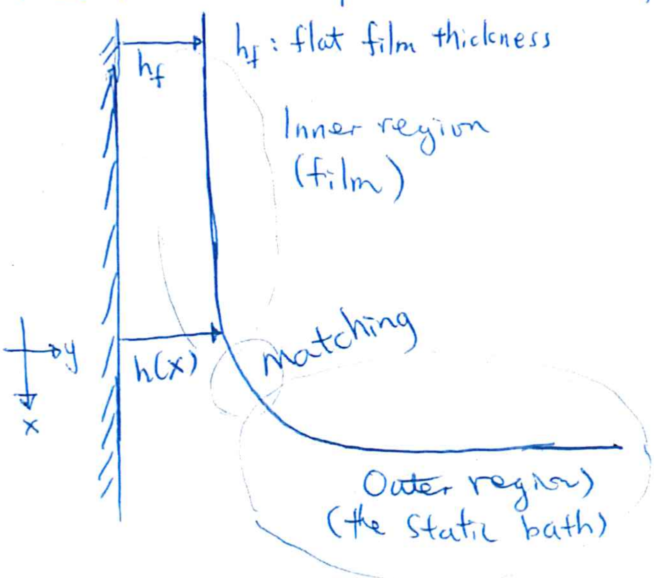
Then $u_d = \frac{h^2}{3\eta}(\gamma h''' + \rho g) + U_p$, since $q = u_d h$, the thin film equation that governs the film profile is:

$$\boxed{h''' = \frac{3Ca\,(1 - q/U_p h)}{h^2} - \frac{1}{l_c^2}}$$

$Ca \equiv \frac{\eta U_p}{\gamma}$, $\quad l_c = \left(\frac{\gamma}{\rho g}\right)^{\frac{1}{2}}$

We consider small Ca, i.e. $Ca \ll 1$, a small parameter, and assume the film is very thin (no influence by gravity).

Then we can separate the interface profile into 2 regions.



(Note that the film indeed has to be very thin, the sketch is just to show different regions)

In the liquid bath, since Ca is small, the interface is not influenced by the flow, we have a static bath (outer region). The interfacial profile is determined by balancing gravity and Laplace pressure

$$\frac{d\kappa}{dx} = \frac{-1}{l_c}$$

Since the slope is not small in the bath, we need a full expression for the curvature

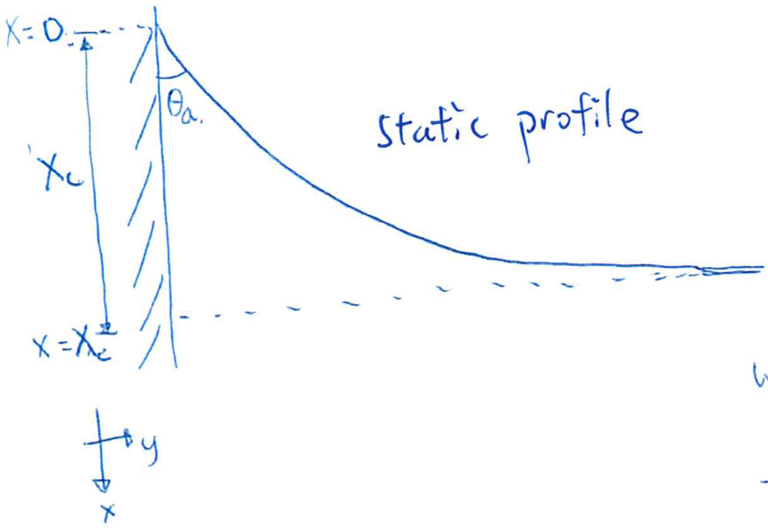
$$\kappa = \frac{h''}{(1 + h'^2)^{3/2}}$$

In the film (inner) region, we assume the film is very thin, so gravity can be neglected, ~~ie~~ we ~~we~~ end up with,

$$h''' = \frac{3Ca\,(1 - q/U_p h)}{h^2}$$

B. The Landau Levich Derjaguin film (2)

The Outer region:



$$\kappa = \frac{h''}{(1+h'^2)^{3/2}} = \frac{X_c - X}{l_c}$$

Rescale the lengths by l_c

$$\bar{h} = \frac{h}{l_c}, \quad \bar{x} = \frac{X_c - X}{l_c}$$

We obtain:

$$\frac{\bar{h}''}{(1+\bar{h}'^2)^{3/2}} = \bar{x}$$

It would be easier to do the integration if we describe the interface by $\bar{x}(\bar{h})$, then the curvature would be

$$\kappa = \frac{\bar{x}''}{(1+x'^2)^{3/2}}$$

$$\Rightarrow \quad \frac{\bar{x}''}{(1+x'^2)^{3/2}} = \bar{x}$$

$$\frac{\bar{x}'\bar{x}''}{(1+\bar{x}'^2)^{3/2}} = \bar{x}'\bar{x}$$

$$\frac{\bar{x}'\frac{d\bar{x}'}{dh}}{(1+\bar{x}'^2)^{3/2}} = \bar{x}'\bar{x}$$

$$\int \frac{\bar{x}'d\bar{x}'}{(1+\bar{x}'^2)^{3/2}} = \int \bar{x}'\bar{x}\, dh$$

$$\frac{-1}{(1+\bar{x}'^2)^{\frac{1}{2}}} = \frac{\bar{x}^2}{2} + c$$

Since $\bar{x}=0$, $\bar{x}'=0 \Rightarrow c=-1$

when $\bar{x} = \bar{x}_c$, then $\bar{x}' = \cot\theta_a$

$$\Rightarrow \bar{x}_c = \sqrt{2}\sqrt{1-\sin\theta_a}$$

Since the curvature $\bar{\kappa} = \bar{x}$

At $\bar{x} = \bar{x}_c$, $\bar{\kappa} = \sqrt{2}\sqrt{1-\sin\theta_a}$

So when $\bar{x}$ is close to $\bar{x}_c$, the profile can be described by

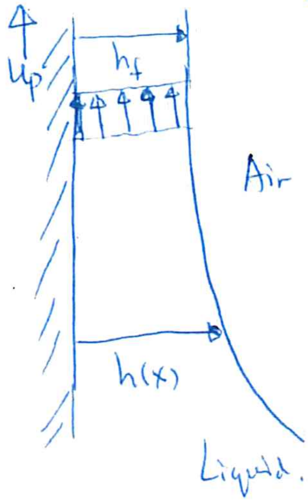
$$\bar{h} = \tan\theta_a(\bar{x}_c - \bar{x}) + \frac{1}{2}\bar{\kappa}_c(\bar{x}_c - \bar{x})^2$$

here $\bar{\kappa}_c = \sqrt{2}\sqrt{1-\sin\theta_a}$.

This outer solution will be used to match the inner solution.

B. The Landau Levich Derjaguin film (3)

The Inner region:



Assume the thickness of the film is small.
i.e. $3Ca/h_f^2 \gg 1/l_c \Rightarrow h_f \ll \sqrt{3Ca} \cdot l_c$

$$h''' = \frac{3Ca\,(1 - q/U_p h)}{h^2}$$

At the flat film region, i.e. $x \to -\infty$, $h \to h_f$.

$h''' = 0 \Rightarrow \frac{q}{U_p} = h_f$ So the flux $q = U_p h_f$

Note that $U_p < 0$

So $$h''' = \frac{3Ca\,(1 - h_f/h)}{h^2}$$

we now rescale the lengths in this way,

$$\hat{h} = \frac{h}{h_f}, \quad \hat{x} = \frac{x}{h_f} \cdot (3Ca)^{\frac{1}{3}}$$

we then end up with,

$$\tilde{h}''' = \frac{1}{\hat{h}^2} - \frac{1}{\hat{h}^3}$$

The boundary conditions are $\hat{x} \to -\infty$, $\tilde{h} \to 1$.
that means close to the flat film, we can write

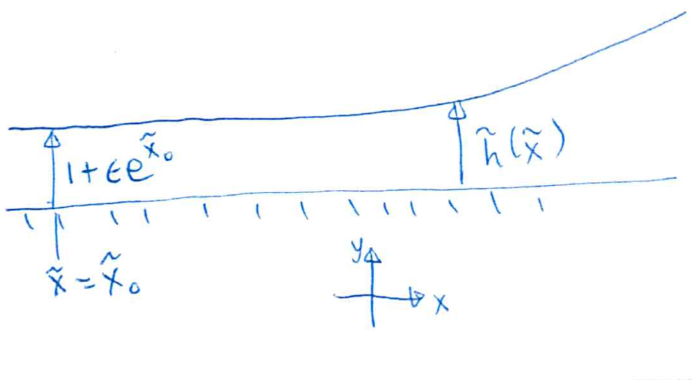
$$\hat{h} = 1 + \epsilon \hat{\delta}(\tilde{x}) \quad \text{where } \epsilon \ll 1.$$

Linearizing the above differential equation, we get.

$\tilde{\delta}'''(\hat{x}) = \tilde{\delta}(\hat{x})$, the only possible solution is $\tilde{\delta}(\hat{x}) = e^{\tilde{x}}$.

the other two cannot match the b.c. $\hat{x} \to -\infty$, $\hat{h} \to 1$.

So $\hat{h} = 1 + \tilde{\delta}(\tilde{x}) = 1 + e^{\tilde{x}}$, Numerically we can solve the equation $\tilde{h}''' = \frac{1}{\hat{h}^2} - \frac{1}{\hat{h}^3}$ with the conditions



$$\tilde{h}(\hat{x}_0) = 1 + \epsilon e^{\tilde{x}_0}$$

$$\tilde{h}'(\hat{x}_0) = \epsilon e^{\tilde{x}_0}$$

$$\hat{h}''(\tilde{x}_0) = \epsilon e^{\tilde{x}_0}$$

B. The Landau Levich Derjaguin film (4)

The asymptotic solution for $\hat{h}(\hat{x})$ when $\hat{x} \to \infty$:

when $\hat{x} \to \infty$, $\hat{h}$ is large, then we have.

$$\hat{h}''' = 0.$$

$$\Rightarrow \hat{h} = \frac{1}{2} a \hat{x}^2 + b \hat{x}$$

Note then a and b are independent of Ca since the rescaled differential equation does not depend on Ca.

This asymptotic solution has to match the outer solution,

Matching: Inner solution: $\hat{h} = \frac{1}{2} a \hat{x}^2 + b \hat{x}$ $\hat{x} \to \infty$

$$\Rightarrow h = \frac{3^{2/3}}{2} a \cdot \frac{Ca^{2/3}}{h_f} x^2 + 3^{\frac{1}{3}} Ca^{\frac{1}{3}} \cdot b x$$

Outer solution: $\bar{h} = \tan\theta_a (\bar{x}_c - \bar{x}) + \frac{1}{2} \bar{\kappa}_c (\bar{x}_c - \bar{x})^2$ $\bar{x} \to \bar{x}_c$

$$\Rightarrow h = \tan\theta_a \, x + \frac{1}{2} \frac{\bar{\kappa}_c}{l_c} x^2$$

Matching $\Rightarrow$

$$\begin{cases} 3^{\frac{1}{3}} Ca^{\frac{1}{3}} \cdot b = \tan\theta_a \\ \frac{3^{2/3}}{2} a \cdot \frac{Ca^{2/3}}{h_f} = \frac{1}{2} \bar{\kappa}_c / l_c \end{cases}$$

For small Ca, $\theta_a \to 0$ and $\bar{\kappa}_c \to \sqrt{2}$

$$\Rightarrow \quad h_f = \frac{3^{2/3}}{\sqrt{2}} \cdot a \cdot Ca^{2/3} \cdot l_c$$

a is determined from the numerical computation.

$$h_f = 0.95 \, Ca^{2/3} \, l_c$$