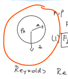
Wednesday, 27 January 2021 08:15

Creeping flow in axisymmetric systems

Overview/motivation



ρ_s, ρ

$F = \frac{4}{3}\pi a^3 g(\rho_s - \rho)$

(1) $\boxed{F_{drag} = 6\pi a \mu u}$ "Stokes law"

$u = \frac{2}{9}\frac{a^2 g(\rho_s - \rho)}{\mu}$ "terminal velocity" ← Ex. 6

Reynolds $Re = \frac{\rho u a}{\mu} \ll 1$

In this case, Navier-Stokes reduce to

(2) $\boxed{\begin{array}{l} \nabla^2\vec{u} = \nabla p \\ \nabla\cdot\vec{u} = 0 \end{array}}$ "creeping flow eq." "continuity eq." } Stokes eqs.

We will derive a general solution to Eq.(2) for flow past an arbitrary axisymmetric body. Eq.(1) will follow for the particular case of a sphere

Steps

- represent Eq.2 for axisymmetric flow in terms of the "stream function ψ" as $E^4\psi = 0$, E is a differential operator
- solve $E^4\psi = 0$

Axisymmetric flow in terms of ψ

Fundamental theorem of vector calculus ("Helmholtz decomposition"): Any continuously differentiable vector field $\vec{u}$ can be expressed in terms of three scalar (ϕ, ξ, χ)

$$\vec{u} = \underbrace{\nabla\phi}_{\text{irrotational}} + \underbrace{\nabla\times(\xi\nabla\chi)}_{\text{divergence free}}$$

$$\vec{u} = \nabla\phi + \nabla\times\vec{A} \quad \text{with } \nabla\cdot\vec{A} = 0$$

ϕ: "velocity potential"; $\vec{A}$: "vector potential for vorticity"

Vorticity

$$\vec{\omega} = \nabla\times\vec{u}$$
$$= \cancel{\nabla\times(\nabla\phi)} + \nabla\times(\nabla\times\vec{A})$$
$$= 0 + \cancel{\nabla(\nabla\cdot\vec{A})} \boxed{- \nabla^2\vec{A} = \vec{\omega}}$$

vector identies

$$\nabla\times\nabla\times\vec{f} = \nabla(\nabla\cdot\vec{f}) - \nabla^2\vec{f}$$

$\nabla\cdot$

$$0 = \nabla^2(\nabla\cdot\vec{f}) - \nabla\cdot(\nabla^2\vec{f})$$

Express Stokes in terms of ϕ and $\vec{A}$

$$\nabla\cdot\vec{u} = \boxed{\nabla^2\phi = 0}$$

$$\nabla^2\vec{u} = \cancel{\nabla(\nabla\cdot\vec{u})} - \nabla\times(\nabla\times\vec{u})$$

$$= \nabla\times\nabla^2\vec{A} = \nabla p$$

$$\nabla\times\nabla\times\nabla^2\vec{A} = \nabla\times\nabla p = 0$$

$$= \cancel{\nabla(\underbrace{\nabla\cdot\nabla^2\vec{A}}_{\nabla^2(\nabla\cdot\vec{A})})} \boxed{- \nabla^4\vec{A} = 0}$$

so far, we have reformulated but not simplified the Stokes eqs.

Simplifying $\vec{u}$ for axisymmetric flow

$\nabla\phi$ only nonzero if $\nabla\phi\cdot\vec{n} = (\vec{u} - \nabla\times\vec{A})\cdot\vec{n} = 0$ at the boundary

in general not true; hence, $\nabla\phi = 0$

$$\vec{u} = \nabla\times\vec{A}$$

For 2D flow $\vec{u} = (u_x, u_y, 0)$ choose $\vec{A} = \psi(x,y)\vec{e_z}$

$$\vec{u} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right)$$

ψ is a scalar function known as the "stream function"

$\vec{u}\cdot\nabla\psi = 0 \Rightarrow$ lines of $\psi = cst$ are tangent everywhere to $\vec{u}$

For axisymmetric flow

spherical coordinates (r, θ, ϕ)

demand $\vec{u} = (u_r, u_\theta, 0)$

$$\vec{A} = \frac{\psi(r,\theta)}{r\sin\theta}\vec{e_\phi}$$

$$u_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\psi}{r\sin\theta}\right) = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

$$\vec{\omega} = \nabla\times\vec{u}$$
$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial\theta}\right]\vec{e_\phi}$$
$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]\vec{e_\phi}$$

$$-\vec{\omega} = \left[\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]\vec{e_\phi}$$

$(r,\theta,\phi) \to (r,\eta,\phi)$

$$= \left[\ \ldots + \frac{\sin\theta}{r^3}\frac{\partial^2\psi}{\partial\eta^2}\right]\vec{e_\phi}$$

$$-\vec{\omega} = \frac{1}{r\sin\theta}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2\psi}{\partial\eta^2}\right]\vec{e_\phi}$$

$$-\vec{\omega} = \frac{1}{r\sqrt{1-\eta^2}}\left[\frac{\partial^2\psi}{\partial r^2} + \frac{1-\eta^2}{r^2}\frac{\partial^2\psi}{\partial\eta^2}\right]\vec{e_\phi}$$

$$\boxed{-\vec{\omega} = \frac{1}{r\sqrt{1-\eta^2}}E^2\psi\,\vec{e_\phi}}$$

Eq. 7.48 Leal

where
$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\eta^2}{r^2}\frac{\partial^2}{\partial\eta^2}$$
$$= \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

$\eta \equiv \cos\theta \quad -1 \leq \eta \leq 1$

$\sin\theta = \sqrt{1-\eta^2}$

$$\frac{\partial}{\partial\theta} = \frac{\partial\eta}{\partial\theta}\frac{\partial}{\partial\eta} = -\sin\theta\frac{\partial}{\partial\eta} = -\sqrt{1-\eta^2}\frac{\partial}{\partial\eta}$$

$\nabla\times\nabla\times\vec{\omega} = 0$

$$\nabla\times\nabla\times\vec{\omega} = \nabla(\nabla\cdot\vec{\omega}) - \nabla^2\omega$$
$$= \cancel{\nabla(\nabla\cdot(\nabla\times\vec{u}))} - \nabla^2(\nabla\times\vec{u})$$
$$= -\nabla\times\nabla^2\vec{u}$$
$$= -\nabla\times\nabla p = 0$$

$$\nabla\times\vec{\omega} = 0$$
$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,\omega_\phi)\vec{e}_r - \frac{1}{r}\frac{\partial}{\partial r}(r\,\omega_\phi)\vec{e}_\theta$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\cancel{\sin\theta} \times \frac{-1}{r\cancel{\sqrt{1-\eta^2}}} E^2\psi\right)\vec{e}_r - \frac{1}{r}\frac{\partial}{\partial r}\frac{-1}{\sqrt{1-\eta^2}} E^2\psi)\vec{e}_\theta$$

$$\nabla\times\vec{\omega} = -\frac{1}{r^2}\frac{\partial}{\partial\eta}(E^2\psi)\vec{e}_r + \frac{1}{r\sqrt{1-\eta^2}}\frac{\partial}{\partial r}(E^2\psi)\vec{e}_\theta$$

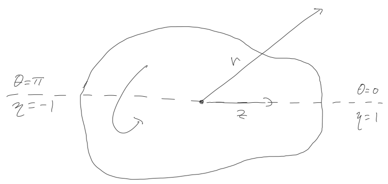
$$\nabla\times\nabla\times\vec{\omega} = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\cancel{r}\frac{1}{\cancel{r}\sqrt{1-\eta^2}}\frac{\partial}{\partial r}E^2\psi\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2}\frac{\partial}{\partial\eta}E^2\psi\right)\right]\vec{e}_\phi = 0$$

$$= \frac{1}{r}\left[\frac{1}{\sqrt{1-\eta^2}}\frac{\partial^2}{\partial r^2}E^2\psi + \frac{\sqrt{1-\eta^2}}{r^2}\frac{\partial}{\partial\eta^2}E^2\psi\right] = 0$$

$$E^2(E^2\psi) = 0$$

$$\boxed{E^4\psi = 0}$$

Solving $E^4\psi = 0$ for axisymmetric flow



consider an axisymmetric body with its c.o.m. at $r=0$

$u_\theta = 0 \qquad \theta = 0, \pi \qquad (\eta = \pm 1)$

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{r\sqrt{1-\eta^2}}\frac{\partial\psi}{\partial r}$$

$\frac{\partial\psi}{\partial r} = 0 \quad$ at $\eta = \pm 1$

impermeable surface $\quad \psi = cst = 0 \quad$ at $\eta = \pm 1$

General solution to $E^4\psi = 0$

$E^2\omega = 0$

$E^2\psi = -\omega$

Strategy is to solve $E^2\omega = 0$ by separation of variables $\boxed{\omega = R(r)\,H(\eta)}$ plug solution into $E^2\psi = -\omega$

$$\frac{d^2R}{dr^2}H + \frac{1-\eta^2}{r^2}\frac{d^2H}{d\eta^2}R = 0$$

$$\frac{r^2}{R}\frac{d^2R}{dr^2} + \frac{1-\eta^2}{H}\frac{d^2H}{d\eta^2} = 0$$

Euler Eq.

Legendre Eq.

r only $\quad$ η only

$-k \qquad k = 0$

$-n(n+1) \qquad n(n+1)$

Euler

$$r^2\frac{dR^2}{dr^2} - R\,n(n+1) = 0$$

insert a general solution $R = r^s$

$$r^2 s(s-1)r^{s-2} - r^s n(n+1) = 0$$

$$s(s-1) - n(n+1) = 0 \qquad \text{roots at } s = n+1,\ s = -n$$

two independent solutions $\quad R = r^{n+1}, \ R = r^{-n}$

Legendre eq

$$(1-\eta^2)\frac{d^2H}{d\eta^2} + H(n+1)n = 0$$

$\frac{d}{d\eta}$, write $Y = \frac{dH}{d\eta}$

$$\frac{d}{d\eta}\left[(1-\eta^2)\frac{dY}{d\eta}\right] + Y n(n+1) = 0$$

$n = 0, 1, 2, \ldots$

Legendre polynomial $P_n(\eta)$ "of the first kind"

$P_n(\eta)$ is regular for $\eta = [-1, 1]$

$Y = P_n(\eta)$

$P_n(\eta)$ is a polynomial of nth order

$P_0(\eta) = 1,$

$P_1(\eta) = \eta$

$P_2(\eta) = \frac{1}{2}(3\eta^2 - 1),$

$\vdots$

require solution for $H(\eta)$ from $Y(\eta)$ to satisfy $\quad \frac{\partial\psi}{\partial r} = 0 \quad \eta = \pm 1$

$\psi = 0, \ \eta = \pm 1$

$H(\eta) = 0 \quad$ at $\eta = \pm 1$

$$\boxed{H(\eta) = \int_{-1}^{\eta} Y(\eta)\,d\eta} = \underbrace{\int_{-1}^{\eta} P_n(\eta)\,d\eta}_{\text{except for } n=0} \equiv Q_n(\eta)$$

$$\omega = R(r)\,H(\eta) = \sum_{n=1}^{\infty} \left(\hat{A}\, r^{n+1} + \hat{C}\, r^{-n}\right) Q_n(\eta)$$

coefficients must be fixed by enforcing the b.c. of a particular problem of interest.

Seek a solution of stream function

$$E^2 \psi = \omega$$
$$= \sum_{n=1}^{\infty} \left(\hat{A}\, r^{n+1} + \hat{C}\, r^{-n}\right) Q_n(\eta)$$

Homogeneous solution ψ_h satisfies $E^2 \psi_h = 0$

$$\psi_h = \sum_{n=1}^{\infty} \left(B_n r^{n+1} + D_n r^{-n}\right) Q_n(\eta)$$

Particular solution, try $\psi_p = r^\lambda Q_n(\eta)$

$$E^2 \psi_p = E^2 r^\lambda Q_n(\eta)$$
$$= \left[\frac{\partial^2}{\partial r^2} + \frac{1-\eta^2}{r^2}\frac{\partial^2}{\partial \eta^2}\right] r^\lambda Q_n(\eta)$$
$$= r^{\lambda-2} Q_n(\eta) + (1-\eta^2)\, r^{\lambda-2} \frac{\partial^2}{\partial \eta^2} Q_n(\eta)$$
$$= r^{\lambda-2} Q_n(\eta) - n(n+1)\, Q_n(\eta)\, r^{\lambda-2}$$
$$= \alpha\, r^{\lambda-2} Q_n(\eta)$$

$$(1-\eta^2)\frac{d^2 H}{d\eta^2} + (n+1)n\, H = 0$$
$$H = Q_n(\eta)$$

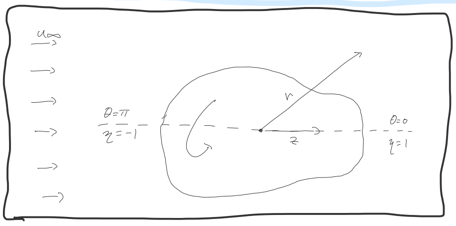
$E^2\psi = \omega$ is satisfied for $\lambda = n+3$ and $\lambda = 2-n$

$$\psi_p = \sum_{n=1}^{\infty} \left(A_n r^{n+3} + C_n r^{2-n}\right) Q_n(\eta)$$

$$\psi = \psi_h + \psi_p = \sum_{n=1}^{\infty} \left(A_n r^{n+3} + B_n r^{n+1} + C_n r^{2-n} + D_n r^{-n}\right) Q_n(\eta)$$

general solution for any axi-sym. flow. $A_n \ldots D_n$ are constants to be fixed for specific particle geometries by employing b.c.'s

Uniform streaming past an arbitrary axi-sym. body



far away from the body $\vec{u} = \vec{i}_z$ for $r \to \infty$

$$\vec{i}_z = \vec{i}_r \cos\theta - \vec{i}_\theta \sin\theta$$
$$= \vec{i}_r\, \eta - \vec{i}_\theta \sqrt{1-\eta^2}$$

$u_r = \eta$, $u_\theta = -\sqrt{1-\eta^2}$ for $r \to \infty$

Last week

$$u_r = -\frac{1}{r\sqrt{1-\eta^2}}\frac{\partial \psi}{\partial r} \Rightarrow \frac{\partial \psi}{\partial r} = (1-\eta^2)\, r \xrightarrow{\text{solve DE's}} \psi = (1-\eta^2)\frac{r^2}{2} + f(\eta) + \text{cst.}$$

$$u_\theta = -\frac{1}{r^2}\frac{\partial \psi}{\partial \eta} \Rightarrow \frac{\partial \psi}{\partial \eta} = -r^2 \eta \longrightarrow \psi = -\frac{r^2\eta^2}{2} + g(r) + \text{cst.}$$

choose $f(\eta) = \text{cst.} = 0$

$g(r) = \frac{r^2}{2}$

Last week, $\psi = \text{cst}$ at $\eta = \pm 1$ (all r)

this cst. is now set to 0, for all r

We can write our solution $\psi = \frac{r^2}{2}(1-\eta^2)$ for $r \to \infty$

$$\boxed{\psi = -r^2\, Q_1(\eta)}$$

In order to satisfy the far field solution,

$A_n = B_{n\geq 2} = 0$, $B_1 = -1$

$$\psi = \underbrace{-r^2 Q_1(\eta)}_{\text{free stream}} + \underbrace{\sum_{n=1} (C_n r^{2-n} + D_n r^{-n}) Q_n(\eta)}_{\text{disturbance flow}}$$

Q_4 : show for general ψ that $F = 4\pi\mu U_c\, l_c\, C_1$

For an arbitrary axi.sym body uniform flow will cause a drag force ; hence, $C_1 \neq 0$

$\Rightarrow$ largest disturbance far field ($r \gg 1$) $\quad \psi' = C_1 r Q_1(\eta)$

$$u'_r = -\frac{1}{r^2}\frac{\partial\psi}{\partial\eta} = -\frac{C_1}{r}\frac{\partial Q_1}{\partial\eta} = -\frac{C_1}{r} P_1(\eta)$$

$$u_\theta = -\frac{1}{r\sqrt{1-\eta^2}}\frac{\partial\psi}{\partial r} = -\frac{C_1 Q_1(\eta)}{r\sqrt{1-\eta^2}}$$ "Stokeslet velocity field"

$\boxed{\text{Special case : uniform streaming past a solid sphere}}$

$u_r = 0$ for $r = 1$ " kinematic condition "

$u_\theta = 0$ for $r = 1$ " no slip .. " $\quad$ Q.5 : full slip

$\boxed{\frac{\partial\psi}{\partial r} = 0 \text{ at } r = 1}$ $\qquad \frac{\partial\psi}{\partial\eta} = 0 \Rightarrow \frac{\partial\psi}{\partial\theta} = 0$ at $r = 1$

$\hookrightarrow \psi = cst$ at the sphere surface.

with $\psi = 0$ at $\eta = \pm 1$ (all r) $\quad \boxed{\psi = 0 \text{ at } r = 1}$

use these b.c.'s (the red boxes above) to fix C_n, D_n

$$\psi|_{r=1} = 0 = -Q_1(\eta) + \sum_{n=1}^{\infty} (C_n + D_n) Q_n(\eta)$$

$$\frac{\partial\psi}{\partial r}\Big|_{r=1} = 0 = -2Q_1(\eta) + \sum_{n=1}^{\infty} [(2-n)C_n - nD_n] Q(\eta)$$

$$0 = \int_{-1}^{1} d\eta \left[-\frac{Q_1(\eta) Q_m(\eta)}{1-\eta^2} + \sum_{n=1}^{\infty} \frac{(C_n + D_n) Q_n(\eta) Q_m(\eta)}{1-\eta^2} \right]$$

$$0 = \int_{-1}^{1} d\eta \left[\frac{-2Q_1(\eta) Q_m(\eta)}{1-\eta^2} + \sum_{n=1}^{\infty} \frac{[(2-n)C_n - nD_n] Q_n(\eta) Q_m(\eta)}{1-\eta^2} \right]$$

use $$\int_{-1}^{1} d\eta \frac{Q_n(\eta) Q_m(\eta)}{1-\eta^2} = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{n(n+1)(2n+1)} & \text{if } m = n \end{cases}$$

for $m = 1$

$$\left.\begin{aligned} 0 &= \frac{-2}{6} + \frac{2(C_1 + D_1)}{6} \\ 0 &= -\frac{4}{6} + \frac{2(C_1 - D_1)}{6} \end{aligned}\right\} \rightsquigarrow \quad D_1 = -\frac{1}{2}, \quad C_1 = \frac{3}{2}$$

for $m > 1$

$$0 = 0 + \frac{2(C_m + D_m)}{m(m+1)(2m+1)} \longrightarrow C_m = -D_m$$

$$0 = 0 + \frac{(2-m)C_m - mD_m}{m(m+1)(2m+1)} \longrightarrow (2-m)\cdot -D_m - mD_m = 0$$

$$2D_m = 0$$

$$\psi = \left(-r^2 + \frac{3}{2}r - \frac{1}{2r}\right) Q_1(\eta) \qquad Q_1(\eta) = \frac{1}{2}(\eta^2 - 1)$$

knowing ψ, we can determine $\vec{u}$

$$u_r = -\frac{1}{r^2}\frac{\partial\psi}{\partial\eta} = -\left(-1 + \frac{3}{2}\frac{1}{r} - \frac{1}{2r^3}\right)\eta$$

$$u_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{r\sin\theta}\frac{\partial}{\partial r}\left(-r^2 + \frac{3}{2}r - \frac{1}{2r}\right) \cdot -\frac{\sin^2\theta}{2}$$

$$= \frac{\sin\theta}{2r}\left(-2r + \frac{3}{2} + \frac{1}{2r^2}\right)$$

$$= -\sin\theta\left(1 - \frac{3}{4}\frac{1}{r} + \frac{1}{4r^3}\right)$$

$\vec{u}' = \vec{u} \cdot u_\infty \qquad r' = r \cdot a \qquad$ (primes for dimensional quantities)

$$u_r' = u_\infty \cos\theta \left(1 - \frac{3a}{2r'} + \frac{1}{2}\left(\frac{a}{r'}\right)^3\right)$$

$$u'_\theta = -u_\infty \sin\theta \left(1 - \frac{3}{4}\left(\frac{a}{r'}\right) - \frac{1}{4}\left(\frac{a}{r'}\right)^3\right)$$

From $\vec{u}'$ follows the pressure $\nabla p' = \mu \nabla^2 \vec{u}'$

$$\frac{\partial p}{\partial r} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_r}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u_r}{\partial\theta}\right) - 2u_r - \frac{2}{\sin\theta}\frac{\partial(u_\theta \sin\theta)}{\partial\theta}\right]$$

$$\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{\partial u_\theta}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u_\theta}{\partial\theta}\right) - \frac{u_\theta}{\sin^2\theta} + 2\frac{\partial u_r}{\partial\theta}\right]$$

integrate w.r.t r, θ

$$p' = P_0 - \frac{3}{2}\frac{\mu u_\infty}{a}\left(\frac{a}{r'}\right)^2 \cos\theta$$

Stokes' law

$$\vec{F}' = \int_S T' \cdot \vec{n} \, dA$$

(integrate the stress vector over body surface S, $\vec{n}$ = outward normal)

$$F_{drag} = \vec{i}_z \cdot \vec{F}$$
$$= \int_S \vec{i}_z \cdot (T \cdot \vec{i}_r) \, dA$$
$$= \int_S (T'_{rr}\cos\theta - T_{\theta r}\sin\theta) \, dA$$

Definitions / Identities

$$\vec{i}_r = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad T = \begin{bmatrix} T_{rr} & T_{r\theta} & T_{r\phi} \\ T_{\theta r} & T_{\theta\theta} & T_{\theta\phi} \\ T_{\phi r} & T_{\phi\theta} & T_{\phi\phi} \end{bmatrix}$$

$$T \cdot \vec{i}_r = \begin{bmatrix} T_{rr} \\ T_{\theta r} \\ T_{\phi r} \end{bmatrix} \qquad \vec{i}_z = \cos\theta \, \vec{i}_r - \sin\theta \, \vec{i}_\theta$$

$$\overleftrightarrow{T} = -p' \overleftrightarrow{\mathbb{1}} + 2\mu \overleftrightarrow{E}, \quad \text{where } \overleftrightarrow{E} = \frac{1}{2}(\nabla u' + \nabla u'^T)$$

$$T_{rr} = -p' + 2\mu \frac{\partial u'_r}{\partial r}, \quad T'_{r\theta} = \mu\left[r \frac{\partial}{\partial r}\left(\frac{u'_\theta}{r}\right) + \frac{1}{r}\frac{\partial u'_r}{\partial\theta}\right], \quad T'_{r\phi} = 0$$

$$\frac{\partial u'_r}{\partial r} = u_\infty\left[\frac{3}{2}\frac{a}{r^2} - \frac{3}{2}\frac{a^3}{r^4}\right]\cos\theta \overset{r=a}{=} 0$$

$$\frac{\partial u'_r}{\partial\theta} = 0$$

$$\frac{\partial(u'_\theta/r)}{\partial r} = u_\infty \sin\theta \frac{\partial}{\partial r}\left[-\frac{1}{r} + \frac{3}{4}\frac{a}{r^2} + \frac{1}{4}\frac{a^3}{r^4}\right]$$
$$= u_\infty \sin\theta \left[\cancel{\frac{1}{r^2}} - \frac{6}{4}\frac{a}{r^3} - \cancel{\frac{a^3}{r^5}}\right]$$

B_1 C_1 D_1

$$\overset{r=a}{=} -\frac{3}{2}\frac{u_\infty \sin\theta}{a^2}$$

$$T'_{rr}\big|_{r=a} = -P_0 + \frac{3\mu u_\infty}{2a}\cos\theta$$

$$T'_{r\theta}\big|_{r=a} = -\frac{3\mu u_\infty}{2a}\sin\theta$$

$$F_{drag} = \int (T'_{rr}\cos\theta - T'_{r\theta}\sin\theta) \, dA$$

$$= \int^{2\pi} d\phi \int^{\pi} d\theta \left[-P_0 \cos\theta + \frac{3\mu u_\infty}{2a}\left(\cancel{\cos^2\theta + \sin^2\theta}\right)\right] a^2 \sin\theta$$

1

$$\boxed{F_{drag} = 6\pi a \mu U_\infty}$$ "Stokes law"

$$\int_0^\pi d\theta \cos\theta \sin\theta = 0$$

$$\int_0^\pi \sin\theta \, d\theta = 2$$

Q_4 : $F_z = 4\pi \mu u_c l_c C_1$

For a sphere $C_1 = \frac{3}{2}$ $\rightarrow$ $F_{drag} = 6\pi a \mu U_\infty$

Finite Re

Kaplan (1957)
Proudman & Pearson (1957)

$$F_z = 6\pi \mu a U_\infty \Big[1 + \underbrace{\frac{3}{8} Re + \frac{9}{40} Re^2 \ln Re}_{} + \text{h.o.}$$

$Re = 0{,}1$, this term equal $0{,}032$