An Introduction to Mathematical Finance
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Lecture 2

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Time value of money

It is common sense that NOK1000 to be received after one year are worth less than the same amount today. The main reason is that money due in the future or locked in a fixed term account cannot be spent right away. Therefore, one would expect to be compensated for the postponed consumption. In addition, prices of goods may rise in the meantime and the amount will not have the same purchasing power as it would have at present.

We will be concerned with two questions:

- What is the future value of an amount invested or borrowed today?
- What is the present value of an amount to be paid or received at a certain time in the future?

1 Simple interest

In exchange for the use of a depositor's money, banks pay a fraction of the account balance back to the depositor. This fractional payment is known as *interest.* Alternatively. it is the fee charged to you when borrowing money from a bank. Suppose that an amount is deposited into a bank account. This initial deposit is called the *principal* and it is denoted by P. We shall first consider the case when interest is attracted only by the principal. This is called *simple interest.*

After one year the interest earned will be rP,where $r > 0$ is the *interest rate.* The value of the investment will become

$$V(1) = P + rP = (1 + r) P.$$

After two years, the investment will grow to

$$V(2) = P + rP + rP = (1 + 2r) P.$$

Consider a fraction of a year. Interest is typically calculated on a daily basis: the interest earned in one day will be $\frac{1}{365}rP$. After n days the total value of the investment will become

$$V\left(\frac{n}{365}\right) = P + \frac{n}{365}rP = \left(1 + \frac{n}{365}r\right)P.$$

Rule of simple interest: The future value of the principal at time t, denoted by $V(t)$, is given by

$$V(t) = (1 + rt)P, \tag{1}$$

where time t, expressed in years, is any nonnegative real number. The number $1 + rt$ is called the *growth factor* and it is linear in t.

If the principal P is invested at time s (rather than 0), then the value of the investment at time $t \geq s$ will be

$$V(t) = (1 + (t - s)r)P$$

Example 1. A deposit of NOK1500 held for 20 days and attracting simple interest at a rate 1%. This gives $t = 20/365$ and $r = 0.01$. After 20 days

$$V\left(\frac{20}{365}\right) = \left(1 + \frac{20}{365} \times 0.01\right) \times 1500 \simeq 1500.82.$$

The *rate of return* of an investment starting at time s and finishing at time t will be denoted by $R(s,t)$ and is defined by

$$R(s,t) := \frac{V(t) - V(s)}{V(s)}. \tag{2}$$

In the case of simple interest $R(s,t) = (t - s)r$ and, in particular, the interest rate is equal to the rate of return over one year, $R(s,t) = r$. As a general rule, interest rates will always refer to a period of one year . By contrast, the rate of return reflects both the interest rate and the length of time the investment is held.

Another important problem is to find the initial amount whose value at time t is given. In the case of simple interest the answer is obtained by solving equation (1) for the principal P obtaining

$$V(0) = P = (1 + rt)^{-1}V(t). \tag{3}$$

This quantity is called the *present or discounted value* of $V(t)$ and $(1 + rt)^{-1}$ is called the *discount factor*.

In practice, simple interest is used only for short-term investments (less than a year). It is not realistic description of the value of money in the longer term. Usually, the interest already earned can be reinvested to attract even more interest.

2 Discrete or periodic compounding

Suppose that an amount P is deposited in a bank account earning interest at a constant rate $r > 0$. However, now we assume that the interest earned will be added to the principal periodically (for example: annually, semi-annually, quarterly, monthly, daily). Subsequently, interest will be earned by the principal and all the interest earned so far. In this case, we talk of *discrete or periodic compounding*.

If m interest payments are made per year, the time between two consecutive payments measured in years will be $1/m$, the first interest payment due at time $1/m$. Each interest payment will increase the principal by a factor of $1+\frac{r}{m}$. Hence, the future value of the principal will become

$$V(t) = \left(1+\frac{r}{m}\right)^{tm} P, \tag{4}$$

because there will be tm interest payments during this period. The amount $\left(1+\frac{r}{m}\right)^{tm}$ is also called *the growth factor*. Note that, in this case, it is not a linear function of time.

Example 2. Find the amount to which NOK1500 will grow if compounded quarterly at 1% for 5 years.

We have $m=4$, $r=0.01$ and $t=5$, which yields

$$V(5) = \left(1+\frac{0.01}{4}\right)^{5\times 4} \times 1500 \simeq 1576.81.$$

Proposition 3. *The future value $V(t)$ increases if one of the parameters m, t, r or P increases, the others remaining unchanged.*

Proof. For t, r or P is trivial. To show that $V(t)$ increases when increasing the compounding frequency m, it suffices to show that

$$\left(1+\frac{r}{m}\right)^{m} < \left(1+\frac{r}{k}\right)^{k}, \qquad m<k.$$

By the binomial formula

$$\begin{aligned}
\left(1+\frac{r}{m}\right)^{m} &= \sum_{j=0}^{m} \frac{m!}{j!\,(m-j)!}\left(\frac{r}{m}\right)^{j} \\
&= 1+r+\sum_{j=2}^{m} \frac{r^j}{j!}\frac{m(m-1)\cdots(m-j+1)}{m\times\cdots\times m} \\
&= 1+r+\sum_{j=2}^{m} \frac{r^j}{j!}\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)\cdots\left(1-\frac{(j-1)}{m}\right) \\
&< 1+r+\sum_{j=2}^{m} \frac{r^j}{j!}\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\cdots\left(1-\frac{(j-1)}{k}\right) \\
&< 1+r+\sum_{j=2}^{k} \frac{r^j}{j!}\left(1-\frac{1}{k}\right)\left(1-\frac{2}{k}\right)\cdots\left(1-\frac{(j-1)}{k}\right) \\
&= 1+r+\sum_{j=2}^{k} \frac{r^j}{j!}\frac{k(k-1)\cdots(k-j+1)}{k\times\cdots\times k} \\
&= \sum_{j=0}^{k} \frac{k!}{j!\,(k-j)!}\left(\frac{r}{k}\right)^{j} = \left(1+\frac{r}{k}\right)^{k}.
\end{aligned}$$

□

In this case the formula for the *present or discounted value* of $V(t)$ is given by

$$V(0) = V(t)\left(1+\frac{r}{m}\right)^{-tm}, \tag{5}$$

where $\left(1+\frac{r}{m}\right)^{-tm}$ is the *discount factor.*

The value of an investment at time $0 < t < T$, given the value $V(T)$, assuming periodic compounding with frequency m and interest rate r, is given by

$$V(t) = \left(1+\frac{r}{m}\right)^{-(T-t)m} V(T). \tag{6}$$

The rate of return on a deposit attracting interest compounded periodically is given by

$$R(s,t) = \frac{V(t)-V(s)}{V(s)} = \left(1+\frac{r}{m}\right)^{(t-s)m} - 1. \tag{7}$$

Remark 4. The rate return on a deposit subject to periodic compounding is not additive. Take $m = 1$, then

$$R(0,1) = R(1,2) = r,$$
$$R(0,2) = (1+r)^2 - 1 = 2r + r^2,$$

and clearly these equations imply that

$$R(0,1) + R(1,2) \neq R(0,2).$$

3 Continuous compounding

Continuous compounding is obtained by considering periodic compounding and increasing to infinity the compounding frequency. That is,

$$\begin{aligned} V(t) &= \lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{tm} P \\ &= \lim_{m\to\infty}\left(1+\frac{rt}{tm}\right)^{tm} P \\ &= e^{rt}P, \end{aligned} \tag{8}$$

where we have used that

$$e^x = \lim_{y\to\infty}\left(1+\frac{x}{y}\right)^{y}.$$

The corresponding growth factor is e^{rt}. Formula (8) is a good approximation of the case of periodic compounding when $m$is large. Continuous compounding is simpler and it is the method of compounding used in continuous time models (for example, in the Black-Scholes model).

Proposition 5. *Continuous compounding produces higher future value than periodic compounding with any frequency m (same principal P and interest rate r).*

Proof. Fix $t > 0$. By Proposition (3) we get that

$$\left(1+\frac{r}{m}\right)^{tm} < \left(1+\frac{r}{k}\right)^{tk}, \qquad m < k.$$

Hence, the sequence $\left\{\left(1+\frac{r}{m}\right)^{tm}\right\}_{m\geq 1}$ is monotonically increasing and converges from below to e^{rt}. □

The present value under continuous compounding is given by $V(0) = V(t)\, e^{-rt}$. The discount factor is e^{-rt}. Given the terminal value $V(T)$, the we have

$$V(t) = V(T)\, e^{-r(T-t)}, \qquad 0 \leq t \leq T. \tag{9}$$

The rate of return $R(s,t)$ defined by (2) also fails to be additive. In this case is convenient to introduce the logarithmic rate of return, defined by

$$r(s,t) := \log\left(\frac{V(t)}{V(s)}\right). \tag{10}$$

The following result is trivial.

Proposition 6. *The logarithmic rate of return is additive, that is,*

$$r(s,t) + r(t,u) = r(s,u), \tag{11}$$

for $0 \leq s \leq t \leq u$.

4 Comparing compounding methods

The idea is to compare growth factors over a fixed period of time, usually one year.

Definition 7. We say that two compounding methods are *equivalent* if the corresponding growth factors over a period of one year are the same.

Example 8. Semi-annual compounding at 10% is equivalent to annual compounding at 10.25%. Indeed,

$$\left(1+\frac{0.1}{2}\right)^{2} = 1.1025 \qquad \text{(Semi-annual)}$$
$$\left(1+\frac{0.1}{1}\right)^{1} = 1.1025 \qquad \text{(Annual)}$$

One can switch from one compounding method to other compounding method by changing the interest rate. We shall normally use either annual or continuous compounding.

Definition 9. For a given compounding method with interest rate r, the *effective annual rate* r_e is the one that gives the same growth factor over a one year period under annual compounding. In particular, in the case of periodic compounding with frequency m and rate r the effective annual rate satisfies

$$\left(1+\frac{r}{m}\right)^{m} = 1 + r_e. \tag{12}$$

In the case of continuous compounding with rate r we have

$$e^{r} = 1 + r_e. \tag{13}$$

Remark 10. Two compounding methods are equivalent if and only if the corresponding effective rate r_e and r'_e are equal $r_e = r'_e$.

In terms of the effective rate r_e the future value can be written as

$$V(t) = (1 + r_e)^t P, \qquad t \geq 0.$$

This applies to both, continuous compounding and periodic compounding.

Simple interest does not fit into the scheme for comparing compounding methods. In this case, the future value $V(t)$ is a linear function of time t, whereas it is an exponential if either continuous or periodic compounding applies.

5 Streams of payments

Definition 11. An *annuity* is a sequence of finitely many payments of a fixed amount due at equal time intervals.

Suppose that payments of an amount C are to be made once a year for n years. Assuming that annual compounding at rate r applies, the present value of an annuity is given by

$$V(0) = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \cdots + \frac{C}{(1+r)^n}. \tag{14}$$

The present value factor for an annuity, denoted by $PA(r, n)$, is given by

$$PA(r, n) = \sum_{i=1}^{n} \frac{1}{(1+r)^i}, \tag{15}$$

and obviously $V(0) = C \times PA(r, n)$. The expression for $PA(r, n)$ can be simplified by using the formula

$$a + qa + q^2 a + \cdots + q^{n-1} a = a \frac{1 - q^n}{1 - q}.$$

Setting $a = 1/(1 + r)$ and $q = 1/(1 + r)$ we obtain

$$PA(r, n) = \frac{1 - (1+r)^{-n}}{r}. \tag{16}$$

Remark 12. An initial bank deposit of $C \times PA(r, n)$attracting interest at a rate r compounded annually would produce a stream of n annual payments of C each.

Example 13. A loan of NOK100000 to be paid in 5 equal instalments due at yearly intervals. The instalments include both the interest payable each year (calculated at 15% of the current outstanding balance) and repayment of a fraction of the loan. This is called an *amortized loan.* In order to find the amount of each instalment note that the loan is equivalent to an annuity from the point of view of the lender. Hence,

$$\begin{aligned} 100000 &= C \times PA(0.15, 5) \\ &= C \times \frac{1 - (1.15)^{-5}}{0.15} \\ &\simeq C \times 3.35216, \end{aligned}$$

which yields $C \simeq 29831.5$ NOK.

Definition 14. A *perpetuity* is a sequence of infinitely many payments of a fixed amount due at equal time intervals.

The present value factor for a perpetuity is obtained as follows

$$\begin{aligned} V(0) &= \lim_{n\to\infty} C \times PA(r,n) \\ &= C \times \lim_{n\to\infty} \frac{1-(1+r)^{-n}}{r} \\ &= C \times \frac{1}{r}. \end{aligned}$$

Remark 15. This limit is equivalent to compute the sum of a geometric series. Note that, although you will get an infinite number of fixed payments C, the partial sum of their present values is convergent.

6 Money market

The money market consists of risk free securities, typically bonds.

Definition 16. Bonds are financial assets promising the holder a sequence of guaranteed future payments.

Risk free means here that these payments will be made with certainty.

Definition 17. A *zero-coupon bond* is a bond involving just a single future payment.

The issuing institution of a zero-coupon bond promises to exchange the bond for a certain amount of money F, the face value, on a given day T, the maturity date. Effectively, the person who buys the bond is lending money to the bond issuer/writer. Given an interest rate r and a maturity date, say one year, the present value of the bond should be

$$V(0) = F(1+r)^{-1}. \tag{17}$$

In reality, the opposite happens: bonds are freely traded and their prices are driven by market forces, whereas the interest rate is implied by the bond prices,

$$r = \frac{F}{V(0)} - 1. \tag{18}$$

This formula gives the *implied annual compounding rate.* We will assume form now on that $F = 1$. Bonds can be sold at any time t prior to maturity time T at some price, denoted by $B(t,T)$. Note that $B(T,T) = F = 1$. Again these prices determine the implied interest rates. By applying formulas (6) and (9) with $V(t) = B(t,T)$ and $V(T) = 1$,we get

$$B(t,T) = \left(1 + \frac{r_m}{m}\right)^{-m(T-t)}, \tag{19}$$

and

$$B(t,T) = e^{-r_c(T-t)}. \tag{20}$$

These different implied rates are equivalent, since the bond price does not depend on the compounding method used.

Remark 18. In general, the implied interest rate may depend on the trading time as well as the maturity date T. The dependence on T is called the *term structure*. Actually, the most advanced models for interest rates model $B(t,T)$ as a two parameter stochastic process. In this course we will adopt the simplifying assumption that the interest rate remains constant.

7 Money market account

Investment banks offer the possibility of investing in the money market by buying and selling bonds on behalf of its customers. Actually, when you open a bank account attracting some interest r, the bank trades in the money market to pay you that interest.

Suppose you open a deposit for T years and you set an initial amount $A(0)$. With $A(0)$ the bank buys $A(0)/B(0,T)$ bonds. The value of each bond if sold at time $t < T$ yields

$$B(t,T) = e^{-r(T-t)} = e^{rt}e^{-rT} = e^{rt}B(0,T).$$

As a result, the investment done by the bank gives

$$A(t) = \frac{A(0)}{B(0,T)}B(t,T) = e^{rt}A(0), \qquad t \le T, \tag{21}$$

which is also the value of your bank account. One can extend this procedure for all t by buying new bonds with longer maturities.

This lecture follows very closely the content in Chapter 2 in [1].

References

[1] M. Capiński and T. Zastawniak. Mathematics for Finance. An Introduction to Financial Engineering. (2003)