UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: STK-MAT3700/4700 — An Introduction to Mathematical Finance

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This problem set consists of 10 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a (weight 10p)

Let $B(t,T)$ denote the price at time t of a a zero coupon bond with maturity time T. The return of the bond over a period $[s,t]$ with $0 \leq s < t \leq T$ is given by the formula

$$R(s,t) = \frac{B(t,T) - B(s,T)}{B(s,T)}.$$

Moreover the formula for the price of the bond is given by $B(t,T) = e^{-r(T-t)}$. Hence, we get

$$R(s,t) = \frac{e^{-r(T-t)} - e^{-r(T-s)}}{e^{-r(T-s)}} = e^{r(t-s)} - 1.$$

and

$$r = \frac{\log(1 + R(s,t))}{t-s}.$$

In this problem we have $t - s = 1/2$ and $R(s,t) = 0.03$, which yields

$$r = \frac{\log(1 + 0.03)}{1/2} = 2\log(1.03).$$

b (weight 10p)

We will prove that if we suppose that $C^A(0) > C^E(0)$, then we can find an arbitrage opportunity. Since we already know that $C^A(0) \geq C^E(0)$, this will imply that $C^A(0) = C^E(0)$. The arbitrage opportunity is as follows:

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At time $t = 0$, sell an American call and buy a European call, investing the balance in the money market.

Then,

- If the American call is exercised at $t \leq T$:
  - Borrow a share of the stock and sell it for K, closing the short position on the American call.
  - Invest K in the money market.
  - At time T, use the European call to buy the share for K and return it to the owner.

  The risk-less profit will be

$$\left(C^A(0) - C^E(0)\right) e^{rT} + Ke^{r(T-t)} - K > 0.$$

- If the American call is not exercised: You will end up with the European option (nonnegative value) and a risk-less profit of $\left(C^A(0) - C^E(0)\right) e^{rT} > 0$.

c (weight 10p)

Let $0 < K_1 < K_2$. In this strategy you buy a call option with strike K_2 and buy a put option with strike K_1, both with the same expiry time T. The profit of the strangle as a function of the final price of the stock S_T is given by

$$P(S_T) = (S_T - K_2)^+ + (K_1 - S_T)^+ - C^E(0) - P^E(0).$$

In this case, the table of profits is given by

S_T	Profit
$S_T < K_1$	$K_1 - S_T - C^E(0) - P^E(0)$
$K_1 \leq S_T \leq K_2$	$-C^E(0) - P^E(0)$
$S_T > K_2$	$S_T - K_2 - C^E(0) - P^E(0)$

Problem 2

a (weight 10p)

Let B denote the price process for the bank account. We have that $B(0) = 1$ and B(1) $= \frac{7}{6}$. The discounted price processes for the risky assets are given by $S_1^*(0) = S_1(0)/B(0) = 7, S_2^*(0) = 5$, $S_1^*(1) = S_1(1)/B(1) = (10, 6, 5, 10)^T$ and

(Continued on page 3.)

$S_2^*(1)/B(1) = (6,6,4,4)^T$. A risk neutral probability measure $Q=(Q_1,Q_2,Q_3,Q_4)^T$ must satisfy the following conditions

$$\mathbb{E}_Q[S_1^*(1)] = S_1^*(0),$$
$$\mathbb{E}_Q[S_2^*(1)] = S_2^*(0),$$

which are equivalent to the following equations

$$10Q_1 + 6Q_2 + 5Q_3 + 10Q_4 = 7, \quad (1)$$
$$6Q_1 + 6Q_2 + 4Q_3 + 4Q_4 = 5, \quad (2)$$
$$Q_1 + Q_2 + Q_3 + Q_4 = 1 \quad (3)$$

with the following restrictions $Q_1 > 0, Q_2 > 0, Q_3 > 0, Q_4 > 0$. The previous system of equations is equivalent to From (3) we have that $Q_4 = 1 - Q_1 - Q_2 - Q_3$ and substituting this value in (1) and (2) we obtain

$$4Q_2 + 5Q_3 = 3, \quad (4)$$
$$2Q_1 + 2Q_2 = 1. \quad (5)$$

From (5) we get that $Q_2 = \frac{1-2Q_1}{2}$. Substituting this value in (4) we get

$$4\left(\frac{1-2Q_1}{2}\right) + 5Q_3 = 3 \Longleftrightarrow Q_3 = \frac{1+4Q_1}{5},$$

and

$$Q_4 = 1 - Q_1 - \left(\frac{1}{2} - Q_1\right) - \left(\frac{1+4Q_1}{5}\right) = \frac{3-8Q_1}{10}.$$

Hence, setting $Q_1 = \lambda$, we get $Q_\lambda = \left(\lambda, \frac{1-2\lambda}{2}, \frac{1+4\lambda}{5}, \frac{3-8\lambda}{10}\right)^T$. Finally, using the restrictions $Q_i > 0, i = 1, ..., 4$, we have the following conditions on the parameter λ

$$Q_1 = \lambda > 0$$
$$Q_2 = \frac{1-2\lambda}{2} > 0 \Longleftrightarrow \lambda < \frac{1}{2},$$
$$Q_3 = \frac{1+4\lambda}{5} > 0 \Longleftrightarrow \lambda > -\frac{1}{4},$$
$$Q_4 = \frac{3-8\lambda}{10} > 0 \Longleftrightarrow \lambda < \frac{3}{8},$$

which yield that $\lambda \in \left(0, \frac{3}{8}\right)$. Therefore, the set of risk neutral measures M is given by

$$M = \left\{ Q_\lambda = \left(\lambda, \frac{1-2\lambda}{2}, \frac{1+4\lambda}{5}, \frac{3-8\lambda}{10}\right)^T, 0 < \lambda < \frac{3}{8} \right\}$$

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By the first fundamental theorem of asset pricing we know that the market is arbitrage free because the set of risk neutral probability measures is non empty. Alternative parametrizations of M are

$$
\begin{aligned}
M &= \left\{ Q_\lambda = \left(\frac{1-2\lambda}{2}, \lambda, \frac{3-4\lambda}{5}, \frac{8\lambda-1}{10} \right)^T, \frac{1}{8} < \lambda < \frac{1}{2} \right\} \\
&= \left\{ Q_\lambda = \left(\frac{5\lambda-1}{4}, \frac{3-5\lambda}{4}, \lambda, \frac{2-4\lambda}{4} \right)^T, \frac{1}{5} < \lambda < \frac{1}{2} \right\} \\
&= \left\{ Q_\lambda = \left(\frac{3-10\lambda}{8}, \frac{1+10\lambda}{8}, \frac{1-2\lambda}{2}, \lambda \right)^T, 0 < \lambda < \frac{3}{10} \right\}.
\end{aligned}
$$

b (weight 10p)

By the second fundamental theorem of asset pricing we can conclude that the market is not complete because there are infinitely many risk neutral measures in this market. A contingent claim $X = (X_1, X_2, X_3, X_4)^T$ is attainable if there exists a portfolio $H = (H_0, H_1, H_2)^T$ such that $X = H_0 B(1) + H_1 S_1(1) + H_2 S_2(1)$. This translates to the following system of equations

$$
X_1 = \frac{7}{6} H_0 + \frac{35}{3} H_1 + 7 H_2, \tag{6}
$$

$$
X_2 = \frac{7}{6} H_0 + 7 H_1 + 7 H_2, \tag{7}
$$

$$
X_3 = \frac{7}{6} H_0 + \frac{35}{6} H_1 + \frac{14}{3} H_2, \tag{8}
$$

$$
X_4 = \frac{7}{6} H_0 + \frac{35}{3} H_1 + \frac{14}{3} H_2. \tag{9}
$$

From (6) we get that $\frac{7}{6} H_0 = X_1 - \frac{35}{3} H_1 - 7 H_2$. Substituting this expression for $\frac{7}{6} H_0$ in (7),(8) and (9) we obtain

$$
X_2 = X_1 - \frac{14}{3} H_1, \tag{10}
$$

$$
X_3 = X_1 - \frac{35}{6} H_1 - \frac{7}{3} H_2, \tag{11}
$$

$$
X_4 = X_1 - \frac{7}{3} H_2 \tag{12}
$$

From (10) we get that $H_1 = \frac{3}{14}(X_1 - X_2)$ and from (12) we get that $H_2 = \frac{3}{7}(X_1 - X_4)$. Substituting these expressions in (11) we finally get

$$
X_3 = X_1 - \frac{35}{6} \frac{3}{14} (X_1 - X_2) - \frac{7}{3} \frac{3}{7} (X_1 - X_4) \Longleftrightarrow 5X_1 - 5X_2 + 4X_3 - 4X_4 = 0.
$$

Alternatively, since $M \neq \emptyset$, we have that X is attainable if and only if $\mathbb{E}_{Q_\lambda}[X/B(1)]$ does not depend on λ. We have that

$$
\mathbb{E}_{Q_\lambda}[X/B(1)] = \frac{1}{B(1)} \left\{ \lambda \left(X_1 - X_2 + \frac{4}{5} X_3 - \frac{4}{5} X_4 \right) + \frac{X_2}{2} + \frac{X_3}{5} + \frac{3}{10} X_4 \right\},
$$

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and the previous expectation does not depend on λ if and only if

$$X_1 - X_2 + \frac{4}{5}X_3 - \frac{4}{5}X_4 = 0 \Longleftrightarrow 5X_1 - 5X_2 + 4X_3 - 4X_4 = 0.$$

c (weight 10p)

We have that

$$X = \begin{pmatrix} \max\left(0, S_1\left(0,\omega_1\right) - 8, S_1\left(1,\omega_1\right) - 8\right) \\ \max\left(0, S_1\left(0,\omega_2\right) - 8, S_1\left(1,\omega_2\right) - 8\right) \\ \max\left(0, S_1\left(0,\omega_3\right) - 8, S_1\left(1,\omega_3\right) - 8\right) \\ \max\left(0, S_1\left(0,\omega_4\right) - 8, S_1\left(1,\omega_4\right) - 8\right) \end{pmatrix} = \begin{pmatrix} \max\left(0, 7-8, \frac{35}{3} - 8\right) \\ \max\left(0, 7-8, 7-8\right) \\ \max\left(0, 7-8, \frac{35}{6} - 8\right) \\ \max\left(0, 7-8, \frac{35}{3} - 8\right) \end{pmatrix} = \begin{pmatrix} \frac{11}{3} \\ 0 \\ 0 \\ \frac{11}{3} \end{pmatrix},$$

and, therefore, it is not attainable because

$$5X_1 - 5X_2 + 4X_3 - 4X_4 = 5 \times \frac{11}{3} - 5 \times 0 + 4 \times 0 - 4 \times \frac{11}{3} = \frac{11}{3} \neq 0.$$

Hence, there is an interval of arbitrage free prices $\left[V_-\left(X\right), V_+\left(X\right)\right]$, where $V_-\left(X\right)$ is the lower hedging price of X and $V_+\left(X\right)$ is the upper hedging price of X. Moreover, we know that

$$V_-\left(X\right) = \inf_{Q \in M}\left\{\mathbb{E}_Q\left[\frac{X}{B\left(1\right)}\right]\right\} = \inf_{\lambda \in \left(0, \frac{3}{8}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B\left(1\right)}\right]\right\},$$

and

$$V_+\left(X\right) = \sup_{Q \in M}\left\{\mathbb{E}_Q\left[\frac{X}{B\left(1\right)}\right]\right\} = \sup_{\lambda \in \left(0, \frac{3}{8}\right)}\left\{\mathbb{E}_{Q_\lambda}\left[\frac{X}{B\left(1\right)}\right]\right\}.$$

We have that

$$\begin{aligned} \mathbb{E}_{Q_\lambda}\left[\frac{X}{B\left(1\right)}\right] &= \frac{6}{7}\mathbb{E}_{Q_\lambda}\left[X\right] \\ &= \frac{6}{7}\left\{\frac{11}{3} \times \lambda + 0 \times \frac{1-2\lambda}{2} + 0 \times \frac{1+4\lambda}{5} + \frac{11}{3} \times \frac{3-8\lambda}{10}\right\} \\ &= \frac{6}{7}\left\{\frac{11}{3}\lambda + \frac{11}{3}\frac{3-8\lambda}{10}\right\} = \frac{11}{35}\left\{2\lambda + 3\right\}. \end{aligned}$$

The previous computation yields

$$\begin{aligned} V_-\left(X\right) &= \inf_{\lambda \in \left(0, \frac{3}{8}\right)}\left\{\frac{11}{35}\left\{2\lambda + 3\right\}\right\} = \frac{11}{35} \times 3 = \frac{33}{35}, \\ V_+\left(X\right) &= \sup_{\lambda \in \left(0, \frac{3}{8}\right)}\left\{\frac{11}{35}\left\{2\lambda + 3\right\}\right\} = \frac{11}{35}\left\{2\frac{3}{8} + 3\right\} = \frac{33}{28}. \end{aligned}$$

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Problem 3

a (weight 10p)

We first compute the partitions associated to $S_1(0), S_1(1)$ and $S_1(2)$. We have

$$\pi_{S_1(0)} = \{S_1(0) = 3\} = \{\Omega\},$$
$$\pi_{S_1(1)} = \{\{S_1(1) = 2\}, \{S_1(1) = 4\}\} = \{\{\omega_3, \omega_4\}, \{\omega_1, \omega_2\}\} =: \{A_{1,1}, A_{1,2}\},$$
$$\pi_{S_1(2)} = \{\{S_1(2) = 1\}, \{S_1(2) = 4\}, \{S_1(2) = 6\}\} = \{\{\omega_2, \omega_4\}, \{\omega_3\}, \{\omega_1\}\} =: \{A_{2,1}, A_{2,2}, A_{2,3}\}.$$

The partitions associated to $(S_1(0), S_1(1))$ and to $(S_1(0), S_1(1), S_1(2))$ are given by

$$\begin{aligned}
\pi_{(S_1(0),S_1(1))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} = \{\Omega \cap A_{1,1}, \Omega \cap A_{1,1}\} = \{A_{1,1}, A_{1,2}\}, \\
\pi_{(S_1(0),S_1(1),S_1(2))} &= \pi_{S_1(0)} \cap \pi_{S_1(1)} \cap \pi_{S_1(2)} = \pi_{S_1(0),S_1(1)} \cap \pi_{S_1(2)} \\
&= \{A_{1,1} \cap A_{2,1}, A_{1,1} \cap A_{2,2}, A_{1,1} \cap A_{2,3}, A_{1,2} \cap A_{2,1}, A_{1,2} \cap A_{2,2}, A_{1,2} \cap A_{2,3}\} \\
&= \{\{\omega_4\}, \{\omega_3\}, \emptyset, \{\omega_2\}, \emptyset, \{\omega_1\}\} = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}.
\end{aligned}$$

The filtrations are given by

$$\mathcal{F}_0 = \mathfrak{a}(S_1(0)) = \mathfrak{a}(\{\Omega\}) = \{\emptyset, \Omega\},$$
$$\mathcal{F}_1 = \mathfrak{a}(S_1(0), S_1(1)) = \mathfrak{a}(\{A_{1,1}, A_{1,2}\}) = \{\emptyset, \Omega, A_{1,1}, A_{1,2}\} = \{\emptyset, \Omega, \{\omega_3, \omega_4\}, \{\omega_1, \omega_2\}\},$$
$$\mathcal{F}_2 = \mathfrak{a}(S_1(0), S_1(1), S_1(2)) = \mathfrak{a}(\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}) = \mathcal{P}(\Omega),$$

where $\mathcal{P}(\Omega)$ is the set of all subsets of Ω.

b (weight 20p)

Since $M = \{Q\}$ the market is arbitrage free and complete, due to the first and second fundamental theorem of asset pricing. Then, we can use the martingale method to solve the optimal portfolio problem. In this setup,$M = \{Q\}$, the martingale method consists in the following two steps:

1. We first solve the constrained optimization problem

$$\max_W \mathbb{E}[U(W)]$$
$$\text{subject to } \mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v,$$

   and obtain the optimal attainable wealth $\widehat{W}$.

2. Given $\widehat{W}$, we find the optimal trading strategy $\widehat{H}$ such that its associated value process $\widehat{V}$ replicates $\widehat{W}$, that is, $\widehat{V}(2) = \widehat{W}$.

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The previous constrained problem can be solved using the Lagrange multipliers method. The optimal attainable wealth $\widehat{W}$ is given by

$$\widehat{W} = I\left(\frac{\widehat{\lambda} L}{B(2)}\right),$$

where I is the inverse of $U'(u)$, L is the state-price density vector $L = \frac{Q}{P}$, $B(2)$ is the price of the risk-less asset at time 2 and $\widehat{\lambda}$ is the optimal Lagrange multiplier associated to the constraint $\mathbb{E}_Q\left[\frac{W}{B(2)}\right] = v$. Taking into account that $r = 0, U(u) = \log(u)$, $P = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)^T$ and $Q = \left(\frac{3}{10}, \frac{1}{5}, \frac{1}{6}, \frac{1}{3}\right)^T$,we have that

$$\begin{aligned}
i = U'(u) = u^{-1} &\iff I(i) = u = i^{-1}, \\
L = \left(\frac{\frac{3}{10}}{\frac{1}{4}}, \frac{\frac{1}{5}}{\frac{1}{4}}, \frac{\frac{1}{6}}{\frac{1}{4}}, \frac{\frac{1}{3}}{\frac{1}{4}}\right)^T &= \left(\frac{6}{5}, \frac{4}{5}, \frac{2}{3}, \frac{4}{3}\right)^T, \\
B(2) = 1,
\end{aligned}$$

which yield $\widehat{W} = \left(\widehat{\lambda} L\right)^{-1}$. The optimal Lagrange multiplier $\widehat{\lambda}$ satisfies the equation

$$v = \mathbb{E}_Q\left[\frac{\widehat{W}}{B(2)}\right] = \mathbb{E}_Q\left[\frac{I\left(\frac{\widehat{\lambda} L}{B(2)}\right)}{B(2)}\right] = \mathbb{E}_Q\left[\left(\widehat{\lambda} L\right)^{-1}\right] = \left(\widehat{\lambda}\right)^{-1} \mathbb{E}_Q\left[L^{-1}\right].$$

Therefore, we get

$$\widehat{\lambda} = \frac{\mathbb{E}_Q\left[L^{-1}\right]}{v}, \qquad \widehat{W} = v\frac{L^{-1}}{\mathbb{E}_Q\left[L^{-1}\right]},$$

and the optimal objective value is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \mathbb{E}\left[\log\left(v\frac{L^{-1}}{\mathbb{E}_Q\left[L^{-1}\right]}\right)\right] = \log\left(\frac{v}{\mathbb{E}_Q\left[L^{-1}\right]}\right) + \mathbb{E}\left[\log\left(L^{-1}\right)\right].$$

Computing

$$\begin{aligned}
L^{-1} &= \left(\frac{5}{6}, \frac{5}{4}, \frac{3}{2}, \frac{3}{4}\right)^T, \\
\mathbb{E}\left[\log\left(L^{-1}\right)\right] &= \frac{1}{4}\left\{\log\left(\frac{5}{6}\right) + \log\left(\frac{5}{4}\right) + \log\left(\frac{3}{2}\right) + \log\left(\frac{3}{4}\right)\right\} = \frac{1}{4}\log\left(\frac{75}{64}\right). \\
\mathbb{E}_Q\left[L^{-1}\right] &= \mathbb{E}\left[LL^{-1}\right] = 1,
\end{aligned}$$

we obtain

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \log(v) + \frac{1}{4}\log\left(\frac{75}{64}\right) = \log\left(v\left(\frac{75}{64}\right)^{1/4}\right).$$

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and

$$\widehat{W} = \frac{v}{1}\left(\frac{5}{6},\frac{5}{4},\frac{3}{2},\frac{3}{4}\right)^T = \left(v\frac{5}{6},v\frac{5}{4},v\frac{3}{2},v\frac{3}{4}\right)^T.$$

Finally, we have to compute the optimal trading strategy $\widehat{H} = \left\{\left(\widehat{H}_0(t), \widehat{H}_1(t)\right)^T\right\}_{t=1,2}$, that is, a self-financing and predictable process such that its asociated value process $\widehat{V}$ satisfies $\widehat{V}(2) = \widehat{W}$. We first compute, taking into account that $r = 0$, the discounted increments of the risky asset

$$\Delta S_1^*(2) = \Delta S_1(2) = (2,-3,2,-1)^T,$$
$$\Delta S_1^*(1) = \Delta S_1(1) = (1,1,-1,-1)^T.$$

- For $t = 2$, using that $\widehat{H}$ must be self-financing we have that $\widehat{W} = \frac{\widehat{W}}{B(2)} = \widehat{V}^*(1) + \widehat{H}_1(2)\Delta S_1^*(2)$.
  - Assuming that $\omega \in A_{1,1} = \{\omega_3, \omega_4\}$ and the predictability of $\widehat{H}$ we get the equations

    $$\begin{aligned}\frac{3}{2}v &= \widehat{W}_3 = \widehat{V}^*(1,\omega_3) + \widehat{H}_1(2,\omega_3)\times 2,\\ \frac{3}{4}v &= \widehat{W}_4 = \widehat{V}^*(1,\omega_4) + \widehat{H}_1(2,\omega_4)\times(-1),\\ \widehat{V}^*(1,\omega_3) &= \widehat{V}^*(1,\omega_4),\\ \widehat{H}_1(2,\omega_3) &= \widehat{H}_1(2,\omega_4),\end{aligned}$$

    which, using that $r = 0$, yield

    $$\begin{aligned}\widehat{V}^*(1,\omega_3) &= \widehat{V}^*(1,\omega_4) = \widehat{V}(1,\omega_3) = \widehat{V}(1,\omega_4) = v,\\ \widehat{H}_1(2,\omega_3) &= \widehat{H}_1(2,\omega_4) = \frac{1}{4}v.\end{aligned}$$

  - Assuming that $\omega \in A_{1,2} = \{\omega_1, \omega_2\}$ and the predictability of $\widehat{H}$ we get the equations

    $$\begin{aligned}\frac{5}{6}v &= \widehat{W}_1 = \widehat{V}^*(1,\omega_1) + \widehat{H}_1(2,\omega_1)\times 2,\\ \frac{5}{4}v &= \widehat{W}_2 = \widehat{V}^*(1,\omega_2) + \widehat{H}_1(2,\omega_2)\times(-3),\\ \widehat{V}^*(1,\omega_1) &= \widehat{V}^*(1,\omega_2),\\ \widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_2),\end{aligned}$$

    which, using that $r = 0$, yield

    $$\begin{aligned}\widehat{V}^*(1,\omega_1) &= \widehat{V}^*(1,\omega_2) = \widehat{V}(1,\omega_1) = \widehat{V}(1,\omega_2) = v,\\ \widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_2) = -\frac{1}{12}v.\end{aligned}$$

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- For $t = 1$, the predictability assumption yields that $\widehat{H}_1(1)$ is constant. Moreover, using that $\widehat{H}$ must be self-financing we have that $\widehat{V}^*(1) = \widehat{V}^*(0) + \widehat{H}_1(1)\Delta S_1^*(1)$ and we get the following two equations

$$v = \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (-1), \qquad (\text{for } \omega \in A_{1,1})$$
$$v = \widehat{V}^*(1,\omega) = \widehat{V}^*(0) + \widehat{H}_1(1) \times (1), \qquad (\text{for } \omega \in A_{1,2})$$

which, using that $r = 0$, yield

$$\widehat{V}^*(0) = \widehat{V}(0) = v, \qquad \widehat{H}_1(1) = 0.$$

- Finally we compute $H_0(1)$ and $H_0(2)$ from the definition of value process. We have

$$\widehat{H}_0(1) = \widehat{V}^*(0) - \widehat{H}_1(1)\, S_1^*(0) = v,$$

and

$$\begin{aligned}\widehat{H}_0(2,\omega) &= \widehat{V}^*(1,\omega) - \widehat{H}_1(2,\omega)\, S_1^*(1,\omega) \\ &= \begin{cases} v - \frac{1}{4}v \times 2 = \frac{1}{2}v & \text{if} \quad \omega \in A_{1,1} \\ v + \frac{1}{12}v \times 4 = \frac{4}{3}v & \text{if} \quad \omega \in A_{1,2} \end{cases}\end{aligned}$$

Problem 4

a (weight 10p)

The conditional expectation of X given $\mathcal{G}$ is the unique $\mathcal{G}$- measurable random variable $\mathbb{E}[X|\mathcal{G}]$ satisfying

$$\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[X|\mathcal{G}]\,\mathbf{1}_B], \qquad B \in \mathcal{G}.$$

A process M is an $\mathbb{F}$-martingale if M is $\mathbb{F}$-adapted and satisfies

$$E[M(t+1)|\mathcal{F}_t] = M(t), \qquad t = 0, ..., T-1.$$

b (weight 10p)

In order to prove that if $\mathcal{H} \subset \mathcal{G}$ is also an algebra on Ω, then

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]|\mathcal{H}] = \mathbb{E}[X|\mathcal{H}].$$

we have to prove first that $\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable and secondly that

$$\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\,\mathbf{1}_A] = \mathbb{E}[\mathbb{E}[X|\mathcal{H}]\,\mathbf{1}_A], \qquad A \in \mathcal{H}. \tag{13}$$

That $\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable is a direct consequence of the definition of conditional expectation. Moreover, we have that

$$\begin{aligned}\mathbb{E}[\mathbb{E}[X|\mathcal{G}]\,\mathbf{1}_A] &\overset{(1)}{=} \mathbb{E}[\mathbb{E}[X\mathbf{1}_A|\mathcal{G}]] \overset{(2)}{=} \mathbb{E}[X\mathbf{1}_A] \overset{(3)}{=} \mathbb{E}[\mathbb{E}[X\mathbf{1}_A|\mathcal{H}]] \\ &\overset{(4)}{=} \mathbb{E}[\mathbb{E}[X|\mathcal{H}]\,\mathbf{1}_A],\end{aligned}$$

(Continued on page 10.)

where we have used that: (1) $\mathbf{1}_A$ is $\mathcal{G}$-measurable and goes in $\mathbb{E}[\cdot|\,\mathcal{G}]$, (2) Law of total expectation, (3) Law of total expectation but with respect $\mathbb{E}[\cdot|\,\mathcal{H}]$, (4) $\mathbf{1}_A$ is $\mathcal{H}$-measurable and goes out $\mathbb{E}[\cdot|\,\mathcal{H}]$.

c (weight 10p)

$\Rightarrow$) We have that $Z = XL$ is $\mathbb{F}$-adapted because it is the product of two $\mathbb{F}$-adapted processes. Regarding the martingale condition, we have that

$$\begin{aligned}
\mathbb{E}[X(t+1)L(t+1)|\,\mathcal{F}_t] &\overset{(1)}{=} \mathbb{E}[X(t+1)\,\mathbb{E}[L(T)|\,\mathcal{F}_{t+1}]|\,\mathcal{F}_t] \\
&\overset{(2)}{=} \mathbb{E}[\mathbb{E}[X(t+1)L(T)|\,\mathcal{F}_{t+1}]|\,\mathcal{F}_t] \\
&\overset{(3)}{=} \mathbb{E}[X(t+1)L(T)|\,\mathcal{F}_t] \\
&\overset{(4)}{=} \frac{\mathbb{E}[X(t+1)L(T)|\,\mathcal{F}_t]}{L(t)}L(t) \\
&\overset{(5)}{=} \mathbb{E}_Q[X(t+1)|\,\mathcal{F}_t]\,L(t) \\
&\overset{(6)}{=} X(t)L(t),
\end{aligned}$$

where we have used: (1) Definition of the process L, (2) $X(t+1)$ is $\mathcal{F}_{t+1}$-measurable and goes in $\mathbb{E}[\cdot|\,\mathcal{F}_{t+1}]$, (3) Tower law, (4) Divide and multiply by $L(t)$, (5) Formula for the conditional expectation under Q, (6) X is a martingale under Q.

$\Leftarrow$) We have that $X = Z/L$ is $\mathbb{F}$-adapted because it is the quotient of two $\mathbb{F}$-adapted processes with strictly positive denominator. Regarding the martingale condition, we have that

$$\begin{aligned}
\mathbb{E}_Q[X(t+1)|\,\mathcal{F}_t] &\overset{(a)}{=} \frac{\mathbb{E}[X(t+1)L(T)|\,\mathcal{F}_t]}{L(t)} \\
&\overset{(b)}{=} \frac{\mathbb{E}[X(t+1)\,\mathbb{E}[L(T)|\,\mathcal{F}_{t+1}]|\,\mathcal{F}_t]}{L(t)} \\
&\overset{(c)}{=} \frac{\mathbb{E}[X(t+1)L(t+1)|\,\mathcal{F}_t]}{L(t)} \\
&\overset{(d)}{=} \frac{X(t)L(t)}{L(t)} = X(t),
\end{aligned}$$

where we have used: (a) Formula for the conditional expectation under Q, (b) Tower law and $X(t+1)$ is $\mathcal{F}_{t+1}$-measurable and goes out $\mathbb{E}[\cdot|\,\mathcal{F}_{t+1}]$, (c) Definition of the process L, $(d)XL$ is a martingale under P.