8. Cox-Ross-Rubinstein & Black-Scholes models

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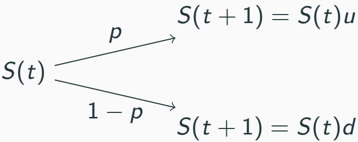
Outline

The Cox-Ross-Rubinstein Model

- The Cox-Ross-Rubinstein market model (CRR model), also known as the binomial model, is an example of a multi-period market model.
- At each point in time, the stock price is assumed to either go ‘up’ by a fixed factor u or go ‘down’ by a fixed factor d .



- Only four parameters are needed to specify the binomial asset pricing model: $u > 1 > d > 0$, $r > -1$ and $S(0) > 0$.
- The real-world probability of an ‘up’ movement is assumed to be the same $0 < p < 1$ for each period and is assumed to be independent of all previous stock price movements.

Definition 1

A stochastic process $X = \{X(t)\})_{t\in\{1,\ldots,T\}}$ defined on some probability space $(\Omega, \mathcal{F}, P)$ is said to be a (truncated) **Bernoulli process** with parameter $0 < p < 1$ (and time horizon T) if the random variables $X(1), X(2), \ldots, X(T)$ are independent and have the following common probability distribution

$$P(X(t) = 1) = 1 - P(X(t) = 0) = p, \qquad t \in \mathbb{N}.$$

- We can think of a Bernoulli process as the random experiment of flipping sequentially T coins.
- The sample space Ω is the set of vectors of zero's and one's of length T. Obviously, $\#\Omega = 2^T$.
- $X(t,\omega)$ takes the value 1 or 0 as ω_t, the t-th component of $\omega \in \Omega$, is 1 or 0, that is, $X(t,\omega) = \omega_t$.

The Bernoulli process

- $\mathcal{F}_t^X$ is the algebra corresponding to the observation of the first t coin flips.
- $\mathcal{F}_t^X = \mathfrak{a}(\pi_t)$ where π_t is a partition with 2^t elements, one for each possible sequence of t coin flips.
- The probability measure P is given by $P(\omega) = p^n (1-p)^{T-n}$, where ω is any elementary outcome corresponding to n "heads" and $T - n$ "tails".
- Setting this probability measure on Ω is equivalent to say that the random variables $X(1), \ldots, X(T)$ are independent and identically distributed.

Example

Consider $T = 3$. Let

$$\begin{aligned}
A_0 &= \{(0,0,0),(0,0,1),(0,1,0),(0,1,1)\}, \\
A_1 &= \{(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}, \\
A_{0,0} &= \{(0,0,0),(0,0,1)\}, \quad A_{0,1} = \{(0,1,0),(0,1,1)\}, \\
A_{1,0} &= \{(1,0,0),(1,0,1)\}, \quad A_{1,1} = \{(1,1,0),(1,1,1)\}.
\end{aligned}$$

We have that
$\pi_0 = \{\Omega\}, \pi_1 = \{A_0, A_1\}, \pi_2 = \{A_{0,0}, A_{0,1}, A_{1,0}, A_{1,1}\}, \pi_3 = \{\{\omega\}\}_{\omega \in \Omega}$ and $\mathcal{F}_t = \mathfrak{a}(\pi_t), t = 0, \ldots, 3$. In particular, $\mathcal{F}_3 = \mathcal{P}(\Omega)$.

The Bernoulli counting process

Definition 2

The **Bernoulli counting process** $N = \{N(t)\}_{t \in \{0,\dots,T\}}$ is defined in terms of the Bernoulli process X by setting $N(0) = 0$ and

$$N(t,\omega) = X(1,\omega) + \cdots + X(t,\omega), \qquad t \in \{1, \dots, T\}, \quad \omega \in \Omega.$$

- The Bernoulli counting process is an example of *additive random walk*.
- The random variable $N(t)$ should be thought as the number of heads in the first t coin flips.
- Since $\mathbb{E}[X(t)] = p$, $\mathrm{Var}[X(t)] = p(1-p)$ and the random variables $X(t)$ are independent, we have

$$\mathbb{E}[N(t)] = tp, \qquad \mathrm{Var}[N(t)] = tp(1-p).$$

- Moreover, for all $t \in \{1, \dots, T\}$ one has

$$P(N(t) = n) = \binom{t}{n} p^n (1-p)^{t-n}, \quad n = 0, \dots, t,$$

that is, $N(t) \sim \mathit{Binomial}(t,p)$.

The CRR market model

- The bank account process is given by $B = \left\{ B(t) = (1+r)^t \right\}_{t=0,\ldots,T}$.
- The binomial security price model features 4 parameters: p, d, u and $S(0)$, where $0 < p < 1, 0 < d < 1 < u$ and $S(0) > 0$.
- The time t price of the security is given by

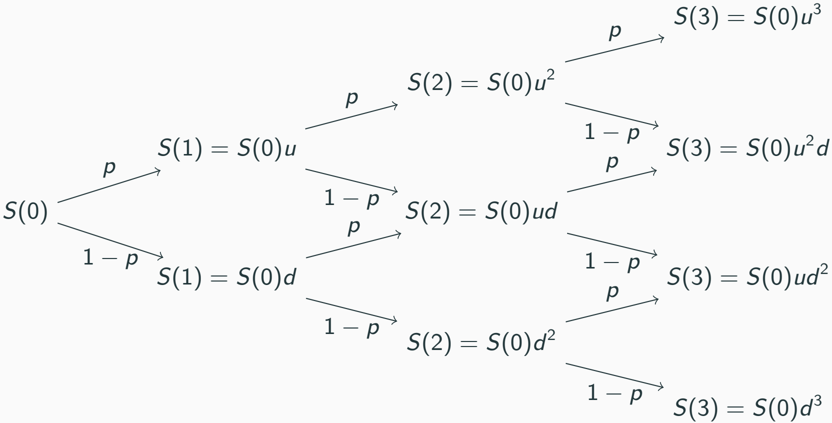
$$S(t) = S(0)\, u^{N(t)} d^{t-N(t)}, \quad t = 1, ..., T.$$

- The underlying Bernoulli process X governs the *up* and *down* movements of the stock. The stock price moves *up* at time t if $X(t,\omega) = 1$ and moves *down* if $X(t,\omega) = 0$.
- The Bernoulli counting process N counts the *up* movements. Before and including time t, the stock price moves up $N(t)$ times and down $t - N(t)$ times.
- The dynamics of the stock price can be seen as an example of a *multiplicative or geometric random walk*.

- The price process has the following probability distribution

$$P\left(S(t) = S(0)\,u^n d^{t-n}\right) = \binom{t}{n} p^n (1-p)^{t-n}, \quad n = 0, ..., t.$$

- Lattice representation



The CRR market model

- The event $\left\{S(t) = S(0)\, u^n d^{t-n}\right\}$ occurs if and only if exactly n out of the first t moves are *up*. The order of these t moves does not matter.
- At time t, there are 2^t possible sample paths of length t.
- At time t, the price process $S(t)$ can only take one of $t+1$ possible values.
- This reduction, from exponential to linear in time, in the number of relevant nodes in the lattice is crucial in numerical implementations.

Example

Consider $T = 2$. Let

$$\Omega = \{(d,d),(d,u),(u,d),(u,u)\}$$
$$A_d = \{(d,d),(d,u)\}, \quad A_u = \{(u,d),(u,u)\}.$$

We have that

$\pi_0 = \{\Omega\}, \pi_1 = \{A_d, A_u\}, \pi_2 = \{\{(d,d)\},\{(d,u)\},\{(u,d)\},\{(u,u)\}\}$, and $\mathcal{F}_t = \mathfrak{a}(\pi_t), t = 0, ..., 3$. Note that

$$\{S(2) = S(0)\, ud\} = \{(d,u),(u,d)\} \notin \pi_2.$$

Hence, the lattice representation is NOT the information tree of the model.

Arbitrage and completeness in the CRR model

Theorem 3

There exists a unique martingale measure in the CRR market model if and only if

$$d < 1 + r < u,$$

and is given by

$$Q(\omega) = q^n (1-q)^{T-n},$$

where ω is any elementary outcome corresponding to n up movements and $T-n$ down movement of the stock and

$$q = \frac{1+r-d}{u-d}.$$

Corollary 4

If $d < 1 + r < u$, then the CRR model is arbitrage free and complete.

Lemma 5

Let Z be a r.v. defined on some prob. space $(\Omega, \mathcal{F}, P)$, with $P(Z = a) + P(Z = b) = 1$ for $a, b \in \mathbb{R}$. Let $\mathcal{G} \subset \mathcal{F}$ be an algebra on Ω. If $\mathbb{E}[Z|\mathcal{G}]$ is constant then Z is independent of $\mathcal{G}$. (Note that the constant must be equal to $\mathbb{E}[Z]$).

Proof of Lemma 5.

Let $A = \{Z = a\}$ and $A^c = \{Z = b\}$. Then for any $B \in \mathcal{G}$

$$\mathbb{E}[Z\mathbf{1}_B] = \mathbb{E}[(a\mathbf{1}_A + b\mathbf{1}_{A^c})\mathbf{1}_B] = aP(A \cap B) + bP(A^c \cap B),$$

and

$$\mathbb{E}[\mathbb{E}[Z]\mathbf{1}_B] = \mathbb{E}[(aP(A) + bP(B))\mathbf{1}_B] = aP(A)P(B) + bP(A^c)P(B).$$

By the definition of cond. expect. we have that $\mathbb{E}[Z\mathbf{1}_B] = \mathbb{E}[\mathbb{E}[Z]\mathbf{1}_B]$. Using that $P(A^c) = 1 - P(A)$ and $P(A^c \cap B) = P(B) - P(A \cap B)$, we get that $P(A \cap B) = P(A)P(B)$ and $P(A^c \cap B) = P(A^c)P(B)$, which yields that $\mathfrak{a}(Z)$ is independent of $\mathcal{G}$. □

Proof of Theorem 3 .

Note that $S^*(t) = S(t)(1+r)^{-t}, t = 0, \ldots T$. Moreover

$$\frac{S(t+1)}{S(t)} = \frac{S(0)\,u^{N(t+1)}d^{t+1-N(t+1)}}{S(0)\,u^{N(t)}d^{t-N(t)}} = u^{N(t+1)-N(t)}d^{1-(N(t+1)-N(t))}$$
$$= u^{X(t+1)}d^{1-X(t+1)}, \qquad t = 0, \ldots, T-1.$$

Let Q be another probability measure on Ω.

We impose the martingale condition under Q

$$\mathbb{E}_Q\left[S^*(t+1)\middle|\mathcal{F}_t\right] = S^*(t) \Leftrightarrow \mathbb{E}_Q\left[u^{X(t+1)}d^{1-X(t+1)}\middle|\mathcal{F}_t\right] = 1+r.$$

This gives

$$(1+r) = \mathbb{E}_Q\left[u^{X(t+1)}d^{1-X(t+1)}\middle|\mathcal{F}_t\right]$$
$$= uQ\left(X(t+1)=1\middle|\mathcal{F}_t\right) + dQ\left(X(t+1)=0\middle|\mathcal{F}_t\right).$$

In addition,

$$1 = Q\left(X(t+1)=1\middle|\mathcal{F}_t\right) + Q\left(X(t+1)=0\middle|\mathcal{F}_t\right).$$

Proof of Theorem 3 .

Solving the previous equations we get the unique solution

$$Q\left(X\left(t+1\right)=1\middle|\,\mathcal{F}_t\right)=\frac{1+r-d}{u-d}=q,$$
$$Q\left(X\left(t+1\right)=0\middle|\,\mathcal{F}_t\right)=\frac{u-(1+r)}{u-d}=1-q.$$

Note that the r.v. $u^{X(t+1)}d^{1-X(t+1)}$ satisfies the hypothesis of Lemma 5 and, therefore, $u^{X(t+1)}d^{1-X(t+1)}$ is independent (under Q) of $\mathcal{F}_t$.

This means that

$$\begin{aligned}(1+r)&=\mathbb{E}_Q\left[u^{X(t+1)}d^{1-X(t+1)}\middle|\,\mathcal{F}_t\right]\\&=\mathbb{E}_Q\left[u^{X(t+1)}d^{1-X(t+1)}\right]\\&=uQ\left(X\left(t+1\right)=1\right)+dQ\left(X\left(t+1\right)=0\right),\end{aligned}$$

and we get that

$$Q\left(X\left(t+1\right)=1\right)=Q\left(X\left(t+1\right)=1\middle|\,\mathcal{F}_t\right),$$
$$Q\left(X\left(t+1\right)=0\right)=Q\left(X\left(t+1\right)=0\middle|\,\mathcal{F}_t\right).$$

Proof of Theorem 3.

As the previous unconditional probabilities does not depend on t we obtain that the random variables $X(1), \ldots X(T)$ are identically distributed under Q, i.e. $X(i) = \mathrm{Bernoulli}(q)$. Moreover, for $a \in \{0,1\}^T$ we have that

$$
\begin{aligned}
Q\left(\bigcap_{t=1}^{T}\{X(t)=a_t\}\right) &= \mathbb{E}_Q\left[\prod_{t=1}^{T}\mathbf{1}_{\{X(t)=a_t\}}\right] \\
&= \mathbb{E}_Q\left[\prod_{t=1}^{T-1}\mathbf{1}_{\{X(t)=a_t\}}\mathbb{E}_Q\left[\mathbf{1}_{\{X(T)=a_T\}}\middle|\mathcal{F}_{T-1}\right]\right] \\
&= \mathbb{E}_Q\left[\prod_{t=1}^{T-1}\mathbf{1}_{\{X(t)=a_t\}}Q\left(X(T)=a_T\middle|\mathcal{F}_{T-1}\right)\right] \\
&= \mathbb{E}_Q\left[\prod_{t=1}^{T-1}\mathbf{1}_{\{X(t)=a_t\}}\right]Q\left(X(T)=a_T\right) \\
&= Q\left(\bigcap_{t=1}^{T-1}\{X(t)=a_t\}\right)Q\left(X(T)=a_T\right).
\end{aligned}
$$

Proof of Theorem 3.

Iterating this procedure we get that

$$Q\left(\bigcap_{t=1}^{T}\{X(t)=a_t\}\right)=\prod_{t=1}^{T}Q\left(X(t)=a_t\right),$$

and we can conclude that $X(1),\ldots X(T)$ are also independent under Q.

Therefore, under Q, we obtain the same probabilistic model as under P but with $p=q$, that is,

$$Q(\omega)=q^n(1-q)^{T-n},\qquad n=\sum_{t=1}^{T}\omega_t.$$

The conditions for q are equivalent to $Q(\omega)>0$, which yields that Q is the unique martingale measure. □

- By the general theory developed for multiperiod markets we have the following result.

Proposition 6 (Risk Neutral Pricing Principle)

The arbitrage free price process of a European contingent claim X in the CRR model is given by

$$P_X(t) = B(t)\,\mathbb{E}_Q\left[\left.\frac{X}{B(T)}\right|\mathcal{F}_t\right] = (1+r)^{-(T-t)}\,\mathbb{E}_Q\left[X|\,\mathcal{F}_t\right], \qquad t = 0, \ldots, T,$$

where Q is the unique martingale measure characterized by $q = \frac{1+r-d}{u-d}$.

- If the contingent claim X is path-independent, $X = g(S(T))$, we have a more precise formula.
- Let $F_{p,g}(t,x)$ the function defined by

$$F_{p,g}(t,x) = \sum_{n=0}^{t} \binom{t}{n} p^n (1-p)^{t-n} g\left(xu^n d^{t-n}\right)$$

Proposition 7

Consider a European contingent claim X given by $X = g(S(T))$. Then, the arbitrage free price process $P_X(t)$ is given by

$$P_X(t) = (1+r)^{-(T-t)} F_{q,g}(T-t, S(t)), \qquad t = 0, ..., T,$$

where $q = \frac{1+r-d}{u-d}$.

Proof of Proposition 7.

Recall that

$$S(t) = S(0)\, u^{N(t)} d^{t-N(t)} = S(0) \prod_{j=1}^{t} u^{X_j} d^{1-X_j}, \quad t = 1, \ldots, T.$$

By Proposition 6 we have that

$$\begin{aligned}(1+r)^{(T-t)} P_X(t) &= \mathbb{E}_Q\left[g(S(T)) \middle| \mathcal{F}_t \right] = \mathbb{E}_Q\left[g\left(S(t) \prod_{j=t+1}^{T} u^{X_j} d^{1-X_j} \right) \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_Q\left[g\left(S(t) \prod_{j=t+1}^{T} u^{X_j} d^{1-X_j} \right) \right] = F_{q,g}(T-t, S(t)),\end{aligned}$$

where in the last equality we have used that $S(t)$ is $\mathcal{F}_t$-measurable and $X_{t+1}, \ldots, X_T$ are independent of $\mathcal{F}_t$.

Note that if X is $\mathcal{G}$-measurable and Y is independent of $\mathcal{G}$ then

$$\mathbb{E}\left[f(X,Y) \middle| \mathcal{G} \right] = \mathbb{E}\left[f(x,Y) \right]\big|_{x=X}.$$

□

Corollary 8

Consider a European call option with expiry time T and strike price K writen on the stock S. The arbitrage free price $P_C(t)$ of the call option is given by

$$P_C(t) = S(t) \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} \hat{q}^n (1-\hat{q})^{T-t-n} - \frac{K}{(1+r)^{T-t}} \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n},$$

where

$$\hat{n} = \inf\left\{ n \in \mathbb{N} : n > \log\left(K/(S(t)\, d^{T-t})\right) / \log(u/d) \right\}$$

and

$$\hat{q} = \frac{qu}{1+r} \in (0,1).$$

- This formula only involves two sums of $T - t - \hat{n} + 1$ binomial probabilities.
- Using the put-call parity relationship one can get a similar formula for European puts.

Proof of Corollary 8.

First note that

$$S(t)\, u^n d^{T-t-n} - K > 0 \iff n > \log\left(K/(S(t)\, d^{T-t})\right) / \log(u/d).$$

Let $g(x) = (x-K)^+$. If $\hat{n} > T-t$ then $F_{q,g}(T-t, S(t)) = 0$. If $\hat{n} \leq T-t$, then the formula in Proposition 7 yields

$$\begin{aligned}
&(1+r)^{T-t} P_C(t) \\
&= F_{q,g}(T-t, S(t)) \\
&= \sum_{n=0}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n} \left(S(t)\, u^n d^{T-t-n} - K\right)^+ \\
&= \sum_{n=0}^{\hat{n}} \binom{T-t}{n} q^n (1-q)^{T-t-n}\, 0 \\
&+ \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n} \left(S(t)\, u^n d^{T-t-n} - K\right)
\end{aligned}$$

Proof of Corollary 8.

$$
\begin{aligned}
&= \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n} S(t) u^n d^{T-t-n} \\
&\quad - \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n} K \\
&= S(t) \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} (qu)^n ((1-q)d)^{T-t-n} \\
&\quad - K \sum_{n=\hat{n}}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n}.
\end{aligned}
$$

The result follows by defining $\hat{q} = \frac{qu}{1+r}$ and noting that

$$
1 - \hat{q} = \frac{1+r-qu}{1+r} = \frac{qu+(1-q)d-qu}{1+r} = \frac{(1-q)d}{1+r},
$$

where we have used $qu + (1-q)d = \mathbb{E}_Q\left[u^{X(t+1)} d^{1-X(t+1)}\right] = 1 + r$. □

Hedging European options in the CRR model

- Let X be a contingent claim and $P_X = \{P_X(t)\}_{t=0,\dots,T}$ be its price process (assumed to be computed/known).
- As the CRR model is complete we can find a self-financing trading strategy $H = \{H(t)\}_{t=1,\dots,T} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,\dots,T}$ such that

$$P_X(t) = V(t) = H_0(t)(1+r)^t + H_1(t)S(t), \quad t = 1, \dots, T, \qquad (1)$$
$$P_X(0) = V(0) = H_0(1) + H_1(1)S(0).$$

- Given $t = 1, \dots, T$ we can use the information up to (and including) $t-1$ to ensure that H is predictable.
- Hence, at time t, we know $S(t-1)$ but we only know that

$$S(t) = S(t-1)u^{X(t)}d^{1-X(t)}.$$

- Using that $u^{X(t)}d^{1-X(t)} \in \{u, d\}$ we can solve equation (1) uniquely for $H_0(t)$ and $H_1(t)$.
- Making the dependence of P_X explicit on S we have the equations

$$P_X(t, S(t-1)u) = H_0(t)(1+r)^t + H_1(t)S(t-1)u,$$
$$P_X(t, S(t-1)d) = H_0(t)(1+r)^t + H_1(t)S(t-1)d.$$

Hedging European options in the CRR model

- The solution for these equations is

$$H_0(t) = \frac{uP_X(t, S(t-1)d) - dP_X(t, S(t-1)u)}{(1+r)^t (u-d)},$$
$$H_1(t) = \frac{P_X(t, S(t-1)u) - P_X(t, S(t-1)d)}{S(t-1)(u-d)}.$$

- The previous formulas only make use of the lattice representation of the model and not the information tree.

Proposition 9

Consider a European contingent claim $X = g(S(T))$. Then, the replicating trading strategy $H = \{H(t)\}_{t=1,\ldots,T} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,\ldots,T}$ is given by

$$H_0(t) = \frac{uF_{q,g}(T-t, S(t-1)d) - dF_{q,g}(T-t, S(t-1)u)}{(1+r)^T (u-d)},$$
$$H_1(t) = \frac{(1+r)^{T-t}\{F_{q,g}(T-t, S(t-1)u) - F_{q,g}(T-t, S(t-1)d)\}}{S(t-1)(u-d)}.$$

Hedging European options in the CRR model

- Let
$$C(\tau, x) = \sum_{n=0}^{\tau} \binom{\tau}{n} q^n (1-q)^{\tau-n} \left(xu^n d^{\tau-n} - K\right)^+ .$$
Then, $P_C(t) = (1+r)^{-(T-t)} C(T-t, S(t))$.

Proposition 10

The replicating trading strategy
$H = \{H(t)\}_{t=1,\dots,T} = \left\{(H_0(t), H_1(t))^T\right\}_{t=1,\dots,T}$ *for a European call option with strike* K *and expiry time* T *is given by*

$$H_0(t) = \frac{uC(T-t, S(t-1)d) - dC(T-t, S(t-1)u)}{(1+r)^T (u-d)},.$$
$$H_1(t) = \frac{(1+r)^{T-t}\{C(T-t, S(t-1)u) - C(T-t, S(t-1)d)\}}{S(t-1)(u-d)}.$$

- As $C(\tau, x)$ is increasing in x we have that $H_1(t) \geq 0$, that is, the replicating strategy does not involve short-selling.
- This property extends to any European contingent claim with increasing payoff g.

Hedging European options in the CRR model

- We can also use the value of the contingent claim X and backward induction to find its price process P_X and its replicating strategy H simultaneously.
- We have to choose a replicating strategy $H(T)$ based on the information available at time $T-1$.
- This gives raise to two equations

$$P_X(T, S(T-1)u) = H_0(T)(1+r)^T + H_1(T)S(T-1)u, \qquad (2)$$

$$P_X(T, S(T-1)d) = H_0(T)(1+r)^T + H_1(T)S(T-1)d. \qquad (3)$$

- The solution is

$$H_0(T) = \frac{uP_X(T, S(T-1)d) - dP_X(T, S(T-1)u)}{(1+r)^T(u-d)},$$

$$H_1(T) = \frac{P_X(T, S(T-1)u) - P_X(T, S(T-1)d)}{S(T-1)(u-d)}.$$

- Next, using that H is self-financing, we can compute

$$P_X(T-1, S(T-1)) = H_0(T)(1+r)^{T-1} + H_1(T)S(T-1),$$

  and repeat the procedure (changing T to $T-1$ in equations (2) and (3)) to compute $H(T-1)$.

The Black-Scholes model

- The Black-Scholes model is an example of continuous time model for the risky asset prices.

Let us summarize the underlying hypothesis of the Black-Scholes model on the prices of assets.

- The assets are traded continuously and their prices have continuous paths.
- The risk-free interest rate $r \geq 0$ is constant.
- The logreturns of the risky asset S_t are normally distributed:

$$\log\left(\frac{S_t}{S_u}\right) \sim \mathcal{N}\left(\left(\mu - \frac{\sigma^2}{2}\right)(t-u), \sigma^2 (t-u)\right).$$

- Moreover, the logreturns are independent from the past and are stationary.
- The model needs three parameters $\mu \in \mathbb{R}$,$\sigma > 0$ and $S_0 > 0$.

- Let Ω be a set with possibly infinite cardinality.

Definition 11

A σ-algebra $\mathcal{F}$ on Ω is a familly of subsets of Ω satisfying

1. $\Omega \in \mathcal{F}$.
2. If $A \in \mathcal{F}$ then $A^c = \Omega \setminus A \in \mathcal{F}$.
3. If $\{A_n\}_{n\geq 1} \subseteq \mathcal{F}$ then $\bigcup_{n\geq 1} A_n \in \mathcal{F}$.

Definition 12

A pair $(\Omega, \mathcal{F})$, where Ω is a set and $\mathcal{F}$ is a σ-algebra on Ω, is called a measurable space.

Definition 13

Given $\mathcal{G}$ a class of subsets of Ω we define $\sigma(\mathcal{G})$ the σ-algebra generated by $\mathcal{G}$ as the smallest σ-algebra containing $\mathcal{G}$, which coincides with the intersection of all σ-algebras containing $\mathcal{G}$.

- In $\mathbb{R}$, we can consider the Borel σ-algebra $\mathcal{B}(\mathbb{R})$, the σ-algebra generated by the open sets.

Definition 14

A probability measure on a measurable space $(\Omega, \mathcal{F})$ is a set function $P : \mathcal{F} \to [0,1]$ satisfying $P(\Omega) = 1$ and, if $\{A_n\}_{n\geq 1} \subseteq \mathcal{F}$ are pairwise disjoint then

$$P\left(\bigcup_{n\geq 1} A_n\right) = \sum_{n\geq 1} P(A_n).$$

Definition 15

A triple $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a σ-algebra on Ω and P is a probability measure on $(\Omega, \mathcal{F})$ is called a probability space.

Definition 16

Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ two measurable spaces. A function $X : E_1 \to E_2$ is said to be $(\mathcal{E}_1, \mathcal{E}_2)$-measurable if $X^{-1}(A) \in \mathcal{E}_1$ for all $A \in \mathcal{E}_2$.

Definition 17

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is a random variable if it is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable (usually one only write $\mathcal{F}$-measurable).

Definition 18

The σ-algebra generated by a random variable X is the σ-algebra generated by the sets of the form $\left\{X^{-1}(A) : A \in \mathcal{B}(\mathbb{R})\right\}$.

Definition 19

The law of a random variable X, denoted by $\mathcal{L}(X)$, is the image measure P_X on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, that is,

$$P_X(B) = P(X^{-1}B), \quad B \in \mathcal{B}(\mathbb{R}).$$

Definition 20

Let $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. Then the expectation of $g(X)$ is defined to be

$$\mathbb{E}\left[g(X)\right] = \int_{\Omega} g \circ X dP = \int_{\mathbb{R}} g dP_X.$$

If $P_X \ll \lambda$, with $\frac{dP_X}{d\lambda} = f_X$ then

$$\mathbb{E}\left[g(X)\right] = \int_{\mathbb{R}} g f_X d\lambda = \int_{\mathbb{R}} g(x) f_X(x) dx.$$

Definition 21

Let X be a random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}[|X|] < \infty$ and $\mathcal{G} \subset \mathcal{F}$ be a σ-algebra. The conditional expectation of X given $\mathcal{G}$, denoted by $\mathbb{E}[X|\mathcal{G}]$ is the unique random variable Z satisfying:

1. Z is $\mathcal{G}$-measurable.
2. For all $B \in \mathcal{G}$, we have $\mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[Z\mathbf{1}_B]$.

- As Ω does not need to be finite, the structure of the σ-algebras on Ω is not as easy as in the finite case. In particular, they are not always generated by partitions.
- This makes computing $\mathbb{E}[X|\mathcal{G}]$ much more difficult in general.
- However, $\mathbb{E}[X|\mathcal{G}]$ satisfies the same properties as when Ω was finite: tower law, total expectation, role of the independence,etc...

Definition 22

A (real-valued) stochastic process X indexed by $[0, T]$ is a family of random variables $X = \{X_t\}_{t \in [0,T]}$ defined on the same probability space $(\Omega, \mathcal{F}, P)$.

- We can think of a stochastic process as a function

$$X: \begin{array}{ccc} [0,T] \times \Omega & \longrightarrow & \mathbb{R} \\ (t, \omega) & \mapsto & X_t(\omega) \end{array}.$$

- For every $\omega \in \Omega$ fixed, the process X defines a function

$$X_{\cdot}(\omega): \begin{array}{ccc} [0,T] & \longrightarrow & \mathbb{R} \\ t & \mapsto & X_t(\omega) \end{array},$$

  which is called a *trajectory* or a *sample path* of the process.

- Hence, we can look at X as a mapping

$$X: \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^{[0,T]} \\ \omega & \mapsto & X_{\cdot}(\omega) \end{array},$$

  where $\mathbb{R}^{[0,T]}$ is the cartesian product of $[0, T]$ copies of $\mathbb{R}$ which is the set of all functions from $[0, T]$ to $\mathbb{R}$. That is, we can see X as a mapping from Ω to a space of functions.

- The canonical construction of a random variable consists on taking $X = Id$ and $(\Omega, \mathcal{F}, P) = (\mathbb{R}, \mathcal{B}(\mathbb{R}), P_X)$.
- For stochastic processes $Y = \{Y_t\}_{t\in[0,T]}$ this procedure is far from trivial. One can consider the measurable space $\left(\mathbb{R}^{[0,T]}, \mathcal{B}(\mathbb{R})^{[0,T]}\right)$ but to find P_Y one needs to do it consistently with the family of finite dimensional laws. (*Kolmogorov Extension Theorem*)
- Moreover, the space $\mathbb{R}^{[0,T]}$ is too big. One often wants to find a realization of the process in a nicer subspace as $C_0([0,T])$. (*Kolmogorov Continuity Theorem*)

Definition 23

A filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ is a family of nested σ-algebras, that is, $\mathcal{F}_s \subseteq \mathcal{F}_t$ if $s < t$.

Definition 24

A stochastic process $X = \{X_t\}_{t\in[0,T]}$ is $\mathbb{F}$-adapted if X_t is $\mathcal{F}_t$-measurable.

Definition 25

A stochastic process $X = \{X_t\}_{t \in [0,T]}$ is a $\mathbb{F}$-martingale if it is $\mathbb{F}$-adapted, $\mathbb{E}[|X_t|] < \infty$, $t \in [0, T]$ and

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s, \quad 0 \leq s < t \leq T.$$

Definition 26

A stochastic process $X = \{X_t\}_{t \in [0,T]}$ has independent increments if $X_t - X_s$ is independent of $X_r - X_u$, for all $u \leq r \leq s \leq t$.

Definition 27

A stochastic process $X = \{X_t\}_{t \in [0,T]}$ has stationary increments if for all $s \leq t \in \mathbb{R}_+$ we have that

$$\mathcal{L}(X_t - X_s) = \mathcal{L}(X_{t-s}).$$

Brownian motion

Definition 28

A stochastic process $W = \{W_t\}_{t\in[0,T]}$ is a (standard) Brownian motion if it satisfies

1. W has continuous sample paths P-a.s.,
2. $W_0 = 0$, P-a.s.,
3. W has independent increments,
4. For all $0 \leq s < t \leq T$, the law of $W_t - W_s$ is a $\mathcal{N}(0, (t-s))$.

Definition 29

A stochastic process $W = \{W_t\}_{t\in[0,T]}$ is a $\mathbb{F}$-Brownian motion if it satisfies

1. W has continuous sample paths P-a.s.,
2. $W_0 = 0$, P-a.s.,
3. For all $0 \leq s < t \leq T$, the random variable $W_t - W_s$ is independent of $\mathcal{F}_s$.
4. For all $0 \leq s < t \leq T$, the law of $W_t - W_s$ is a $\mathcal{N}(0, (t-s))$.

Definition 30

A stochastic process $L = \{L_t\}_{t\in[0,T]}$ is a Lévy process if it satisfies:

1. $L_0 = 0, P$-a.s.,
2. L has independent increments,
3. L has stationary increments, i.e., for all $0 \leq s < t$, the law of $L_t - L_s$ coincides with the law of L_{t-s}.
4. X is stochastically continuous, i.e., $\lim_{s\to t} P(|L_t - L_s| > \varepsilon) = 0, \forall\varepsilon > 0, t \in [0, T]$.

- That L is stochastically continuous does not imply that L has continuous sample paths.
- A Brownian motion is a particular case of Lévy process.
- The class of Lévy processes, in particular exponential Lévy processes, is a natural class of processes to consider for modeling stock prices.

Brownian motion with drift and geometric Brownian motion

Definition 31

A stochastic process $Y = \{Y_t\}_{t \in [0,T]}$ is a Brownian motion with drift μ and volatility σ if it can be written as

$$Y_t = \mu t + \sigma W_t, \quad t \in [0, T],$$

where W is a standard Brownian motion.

Definition 32

A stochastic process $S = \{S_t\}_{t \in [0,T]}$ is a geometric Brownian motion (or exponential Brownian motion) with drift μ and volatility σ if it can be written as

$$S_t = \exp\left(\mu t + \sigma W_t\right), \quad t \in [0, T],$$

where W is a standard Brownian motion.

- Note that the paths S are continuous and strictly positive by construction.

Increments of a geometric Brownian motion

- The increments of S are not independent.
- Its relative increments

$$\frac{S_{t_n} - S_{t_{n-1}}}{S_{t_{n-1}}}, \frac{S_{t_{n-1}} - S_{t_{n-2}}}{S_{t_{n-2}}}, ..., \frac{S_{t_1} - S_{t_0}}{S_{t_0}}, \quad 0 \leq t_0 < t_1 < \cdots < t_n \leq T,$$

  are independent and stationary.
- Equivalently,

$$\frac{S_{t_n}}{S_{t_{n-1}}}, \frac{S_{t_{n-1}}}{S_{t_{n-2}}}, ..., \frac{S_{t_1}}{S_{t_0}}, \qquad 0 \leq t_0 < t_1 < \cdots < t_n \leq T,$$

  and

$$\log\left(\frac{S_{t_n}}{S_{t_{n-1}}}\right), \log\left(\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right), ..., \log\left(\frac{S_{t_1}}{S_{t_0}}\right), \qquad 0 \leq t_0 < t_1 < \cdots < t_n \leq T,$$

  are also independent and stationary.
- Moreover, the law of $S_t/S_s, 0 \leq s < t \leq T$ is lognormal with parameters $\mu(t-s)$ and $\sigma^2(t-s)$, that is, the law of $\log\left(S_t/S_s\right), 0 \leq s < t \leq T$ is $\mathcal{N}\left(\mu(t-s), \sigma^2(t-s)\right)$.

The Black-Scholes model

- The time horizon will be the interval $[0, T]$.
- The price of the riskless asset, denoted by $B = \{B_t\}_{t\in[0,T]}$, is given by $B_t = e^{rt}, 0 \leq t \leq T$.
- The price of the risky asset, denoted by $S = \{S_t\}_{t\in 0,T]}$, is modeled by a continuous time stochastic process satisfying the stochastic differential equation (SDE)

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t, \qquad t \in [0, T],\\
S_0 &= S_0 > 0.
\end{aligned}
$$

- One can check that the process

$$
S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right), \qquad t \in [0, T],
$$

satisfies the previous SDE.
- Therefore, S_t is a geometric Brownian motion with drift $\mu - \frac{\sigma^2}{2}$ and volatility σ.

The Black-Scholes model

- Consider the discounted price process $S^* = \left\{S_t^* = e^{-rt} S_t\right\}_{t \in [0,T]}$.
- Note that S^* satisfies

$$\begin{aligned}
\mathbb{E}\left[\frac{S_t^*}{S_s^*}\middle| \mathcal{F}_s\right] &= \mathbb{E}\left[\exp\left(\left(\mu - \frac{\sigma^2}{2} - r\right)(t-s) + \sigma\left(W_t - W_s\right)\right)\middle| \mathcal{F}_s\right] \\
&= \mathbb{E}\left[\exp\left(\left(\mu - \frac{\sigma^2}{2} - r\right)(t-s) + \sigma\left(W_t - W_s\right)\right)\right] \\
&= \exp\left(\left(\mu - \frac{\sigma^2}{2} - r\right)(t-s)\right)\mathbb{E}\left[\exp\left(\sigma W_{t-s}\right)\right] \\
&= \exp\left(\left(\mu - \frac{\sigma^2}{2} - r\right)(t-s) + \frac{\sigma^2}{2}(t-s)\right) = e^{(\mu - r)(t-s)},
\end{aligned}$$

  where we have used that $\mathbb{E}\left[e^{\theta Z}\right] = e^{\theta\mu + \frac{\theta^2\sigma^2}{2}}$ if $Z \sim N\left(\mu, \sigma^2\right)$.
- Hence, S^* is a martingale under P iff $\mu = r$.
- Does there exist a probability measure Q such that S^* is a martingale under Q?

- The answer is given by Girsanov's theorem. Let Q be given by

$$\frac{dQ}{dP} = \exp\left(-\frac{\mu - r}{\sigma} W_T - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 T\right),$$

  then the process

$$\widetilde{W}_t = \frac{\mu - r}{\sigma} t + W_t,$$

  is a Brownian motion under Q.

- Moreover, S^* is a martingale under Q.

Theorem 33 (Risk-neutral pricing principle)

Let X be a contingent claim such that $\mathbb{E}_Q[|X|] < \infty$. Then its arbitrage free price at time t is given by

$$P_X(t) = e^{-r(T-t)} \mathbb{E}_Q[X | \mathcal{F}_t], \qquad 0 \leq t \leq T.$$

Theorem 34

The prices of a call and a put options are given by

$$C(t, S_t) = S_t \Phi\left(d_1\left(S_t, T - t\right)\right) - K e^{-r(T-t)} \Phi\left(d_2\left(S_t, T - t\right)\right),$$
$$P(t, S_t) = K e^{-r(T-t)} \Phi\left(-d_2\left(S_t, T - t\right)\right) - S_t \Phi\left(-d_1\left(S_t, T - t\right)\right),$$

where

$$d_1(x, \tau) = \frac{\log(x/K) + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$
$$d_2(x, \tau) = \frac{\log(x/K) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$

and

$$\Phi(x) = \int_{-\infty}^{x} \phi(z) dz = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) dz.$$

Note also that $d_1(t, \tau) = d_2(t, \tau) + \sigma\sqrt{\tau}$.

Proof of Theorem 34.

We will prove the formula for the call option, $X = (S(T) - K)^+$. By the risk-neutral valuation principle we know that

$$\begin{aligned} P_X(t) &= e^{-r(T-t)} \mathbb{E}_Q\left[(S(T) - K)^+ \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_Q\left[\left(\frac{S^*(T)}{S^*(t)} S^*(t) - e^{-r(T-t)} K \right)^+ \middle| \mathcal{F}_t \right] \\ &= \mathbb{E}_Q\left[\left(\frac{S^*(T)}{S^*(t)} x - e^{-r(T-t)} K \right)^+ \right] \Bigg|_{x=S^*(t)} \triangleq \Gamma(x)|_{x=S^*(t)} . \end{aligned}$$

As

$$\frac{S^*(T)}{S^*(t)} = \exp\left(-\frac{\sigma^2}{2}(T-t) + \sigma\left(\widetilde{W}_T - \widetilde{W}_t \right) \right),$$

and $\widetilde{W}_T - \widetilde{W}_t \sim \mathcal{N}(0, (T-t))$ under Q, we have that

$$\Gamma(x) = \int_{-\infty}^{+\infty} \phi(z) \left(x e^{-\frac{\sigma^2(T-t)}{2} + \sigma\sqrt{T-t} z} - K e^{-r(T-t)} \right)^+ dz.$$

Proof of Theorem 34.

Note that

$$xe^{-\frac{\sigma^2(T-t)}{2}+\sigma\sqrt{T-t}z} - Ke^{-r(T-t)} \geq 0 \Longleftrightarrow z \geq -d_2\left(x, T-t\right).$$

Therefore,

$$\begin{aligned}
\Gamma\left(x\right) &= \int_{-d_2(x,T-t)}^{+\infty} \phi\left(z\right)\left(xe^{-\frac{\sigma^2(T-t)}{2}+\sigma\sqrt{T-t}z} - Ke^{-r(T-t)}\right) dz \\
&= x\int_{-d_2(x,T-t)}^{+\infty} \phi\left(z\right) e^{-\frac{\sigma^2(T-t)}{2}+\sigma\sqrt{T-t}z} dz \\
&\quad - Ke^{-r(T-t)}\int_{-d_2(x,T-t)}^{+\infty} \phi\left(z\right) dz \\
&= I_1 - I_2.
\end{aligned}$$

Using that

$$\phi\left(z\right) e^{-\frac{\sigma^2(T-t)}{2}+\sigma\sqrt{T-t}z} = \phi\left(z-\sigma\sqrt{T-t}\right),$$

and

$$d_1\left(x, T-t\right) = \sigma\sqrt{T-t} + d_2\left(x, T-t\right),$$

Proof of Theorem 34.

we get

$$\begin{aligned} I_1 &= x \int_{-d_2(x,T-t)}^{+\infty} \phi\left(z - \sigma\sqrt{T-t}\right) dz \\ &= x \int_{-\left(\sigma\sqrt{T-t}+d_2(x,T-t)\right)}^{+\infty} \phi\left(z\right) dz \\ &= x\left(1 - \Phi\left(-d_1\left(x, T-t\right)\right)\right). \end{aligned}$$

On the other hand,

$$I_2 = Ke^{-r(T-t)}\left(1 - \Phi\left(-d_2\left(x, T-t\right)\right)\right).$$

The result follows from the following well known property of Φ

$$\Phi\left(z\right) = 1 - \Phi\left(-z\right), \qquad z \in \mathbb{R}.$$

□

The Greeks or sensitivity parameters

- Note that the price of a call option $C(t, S_t)$ actually depends on other variables

$$C(t, S_t) = C(t, S_t; r, \sigma, K).$$

- The derivatives with respect to these variables/parameters are known as the Greeks and are relevant for risk-management purposes.
- Here, there is a list of the most important:
  - Delta:

$$\Delta = \frac{\partial C}{\partial S}(t, S_t) = \Phi\left(d_1\left(S_t, T - t\right)\right).$$

  - Gamma:

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\Phi'\left(d_1\left(S_t, T-t\right)\right)}{\sigma S_t \sqrt{T-t}} = \frac{\phi\left(d_1\left(S_t, T-t\right)\right)}{\sigma S_t \sqrt{T-t}}$$

  - Theta:

$$\begin{aligned} \Theta = \frac{\partial C}{\partial t} &= -\frac{\sigma S_t \Phi'\left(d_1\left(S_t, T-t\right)\right)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi\left(d_2\left(S_t, T-t\right)\right) \\ &= -\frac{\sigma S_t \phi\left(d_1\left(S_t, T-t\right)\right)}{2\sqrt{T-t}} - rKe^{-r(T-t)}\Phi\left(d_2\left(S_t, T-t\right)\right). \end{aligned}$$

  - Rho:

$$\rho = \frac{\partial C}{\partial r} = K(T-t)e^{-r(T-t)}\Phi\left(d_2\left(S_t, T-t\right)\right).$$

  - Vega:

$$\frac{\partial C}{\partial \sigma} = S_t\sqrt{T-t}\Phi'\left(d_1\left(S_t, T-t\right)\right) = S_t\sqrt{T-t}\phi\left(d_1\left(S_t, T-t\right)\right).$$

Convergence of the CRR pricing formula to the Black-Scholes pricing formula

Convergence of the CRR pricing formula to the Black-Scholes pricing formula

- We will consider a family of CRR market models indexed by $n \in \mathbb{N}$.
- Partition the interval $[0, T)$ into $[(j-1)\frac{T}{n}, j\frac{T}{n})$, $j = 1...., n$.
- $S_n(j)$ will denote the stock price at time $j\frac{T}{n}$ in the nth binomial model.
- Similarly $B_n(j)$ represents the bank account at time $j\frac{T}{n}$, in the nth binomial model.
- Let $r_n = r\frac{T}{n}$ be the interest rate, where $r > 0$ is the interest rate with continuous compounding, i.e.,

$$\lim_{n\to\infty} (1 + r_n)^n = e^{rT}.$$

- Let $a_n = \sigma\sqrt{\frac{T}{n}}$, where σ is interpreted as the instantaneous volatility.
- Set up the *up* and *down* factors by

$$u_n = e^{a_n}(1 + r_n),$$
$$d_n = e^{-a_n}(1 + r_n).$$

- For n sufficiently large $d_n < 1$. Moreover, note that $u_n > 1 + r_n$ and that $d_n < 1 + r_n$ for all n and, by Theorem 3, there exists a unique martingale measure in th nth binomial model for all n.

Convergence of the CRR pricing formula to the Black-Scholes pricing formula

- The martingale probability measure parameter in the nth model is

$$q_n = \frac{1+r_n-d_n}{u_n-d_n} = \frac{1-e^{-a_n}}{e^{a_n}-e^{-a_n}} = \frac{a_n - \frac{1}{2}a_n^2 + o\left(a_n^2\right)}{2a_n + \frac{1}{3}a_n^3 + o\left(a_n^3\right)} = \frac{1}{2} - \frac{1}{4}a_n + o\left(a_n\right),$$

  where $o(\delta)$ with $\delta > 0$ means $\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$.
- Let $\{X_n(j)\}_{j=1,\dots,n}$ be the Bernoullli r.v. underlying the nth market model. Note that $Q_n(X_n(j)=1) = q_n$ and

$$S_n(j) = S(0)\, u_n^{X_n(1)+\cdots+X_n(j)} d_n^{j-(X_n(1)+\cdots+X_n(j))}, \quad j = 1, \dots, n.$$

- The value at time zero of a put option with strike K in the nth binomial market is given by

$$P_{\mathrm{Put}}^n(0) = (1+r_n)^{-n}\, \mathbb{E}_{Q_n}\left[(K - S(n))^+\right] = \mathbb{E}_{Q_n}\left[\left(\frac{K}{(1+r_n)^n} - S(0)\, e^{Y_n}\right)^+\right],$$

  where

$$Y_n = \sum_{j=1}^n Y_n(j) = \sum_{j=1}^n \log\left(\frac{u_n^{X_n(j)} d_n^{1-X_n(j)}}{(1+r_n)}\right).$$

- For n fixed the random variable $Y_n(1), \ldots, Y_n(n)$ are i.i.d. with

$$\begin{aligned}
\mathbb{E}_{Q_n}[Y_n(j)] &= q_n \log\left(\frac{u_n}{1+r_n}\right) + (1-q_n)\log\left(\frac{d_n}{1+r_n}\right) \\
&= \left(\frac{1}{2} - \frac{1}{4}a_n + o(a_n)\right) a_n + \left(\frac{1}{2} + \frac{1}{4}a_n + o(a_n)\right)(-a_n) \\
&= -\frac{1}{2}a_n^2 + o\left(a_n^2\right), \\
\mathbb{E}_{Q_n}\left[Y_n^2(j)\right] &= a_n^2 + o\left(a_n^2\right), \\
\mathbb{E}_{Q_n}[|Y_n(j)|^m] &= o\left(a_n^2\right) \qquad m \geq 3.
\end{aligned}$$

Theorem 35 (Lévy's continuity theorem)

A sequence $\{Y_n\}_{n\geq 1}$ of r.v, possibly defined on different probability spaces $(\Omega_n, \mathcal{F}_n, Q_n)$, converges in distribution to Y, defined on a probability space $(\Omega, \mathcal{F}, Q)$, if and only if the sequence of corresponding characteristic functions $\left\{\varphi_{Y_n} = \mathbb{E}_{Q_n}\left[e^{i\theta Y_n}\right]\right\}_{n\geq 1}$ converges pointwise to the characteristic function $\varphi_Y(\theta) = \mathbb{E}_Q\left[e^{i\theta Y}\right]$ of Y.

Convergence of the CRR pricing formula to the Black-Scholes pricing formula

- Let Y be a random variable defined on some probability space $(\Omega, \mathcal{F}, Q)$ with law $\mathcal{N}\left(-\frac{\sigma^2 T}{2}, \sigma^2 T\right)$. Its characteristic function is

$$\varphi_Y(\theta) = \exp\left(-i\theta\frac{\sigma^2 T}{2} - \theta^2\frac{\sigma^2 T}{2}\right).$$

- As $Y_n(j), ..., Y_n(n)$ are i.i.d. we have that

$$\begin{aligned}
\varphi_{Y_n}(\theta) = \mathbb{E}_{Q_n}\left[e^{i\theta Y_n}\right] = \prod_{j=1}^{n}\mathbb{E}_{Q_n}\left[e^{i\theta Y_n(j)}\right] = \mathbb{E}_{Q_n}\left[e^{i\theta Y_n(1)}\right]^n \\
= \left(1 + i\theta\mathbb{E}_{Q_n}[Y_n(j)] - \frac{\theta^2}{2}\mathbb{E}_{Q_n}\left[Y_n^2(j)\right] + o\left(a_n^2\right)\right)^n \\
= \left(1 - \left(\frac{i\theta + \theta^2}{2}\right)a_n^2 + o\left(a_n^2\right)\right)^n \\
= \left(1 - \left(\frac{i\theta + \theta^2}{2}\right)\sigma^2\frac{T}{n} + o(1/n)\right)^n,
\end{aligned}$$

which converges to $\varphi_Y(\theta)$ as n tends to infinity.

- We can conclude that Y_n converges in distribution to a $\mathcal{N}\left(-\frac{\sigma^2 T}{2}, \sigma^2 T\right)$.

- A sequence $\{Y_n\}_{n\geq 1}$ of random variables, defined on $(\Omega_n, \mathcal{F}_n, Q_n)$, converges in distribution to Y, defined on $(\Omega, \mathcal{F}, Q)$, if and only if

$$\mathbb{E}_{P_n}\left[g\left(Y_n\right)\right] \longrightarrow \mathbb{E}_{P}\left[g\left(Y\right)\right], \tag{4}$$

when $n \to +\infty$, for all $g \in C_b\left(\mathbb{R}\right)$.

- Therefore, since we know that $\{Y_n\}_{n\geq 1}$ converge in law to Y, by applying (4) with $g\left(x\right) = \left(Ke^{-rT} - S\left(0\right)e^{x}\right)^{+}$, we have

$$\begin{aligned} &\lim_{n\to+\infty} \mathbb{E}_{Q_n}\left[\left(Ke^{-rT} - S\left(0\right)e^{Y_n}\right)^{+}\right] \\ &= \int_{-\infty}^{+\infty} \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \left(Ke^{-rT} - S\left(0\right)\exp\left(-\frac{\sigma^2 T}{2} + \sigma\sqrt{T}z\right)\right)^{+} dz \\ &= P_P\left(0\right), \end{aligned}$$

where we have used that $Y \sim \mathcal{N}\left(-\frac{\sigma^2 T}{2}, \sigma^2 T\right)$ if and only if $Y = -\frac{\sigma^2 T}{2} + \sigma\sqrt{T}Z$ with $Z \sim \mathcal{N}\left(0,1\right)$.

Convergence of the CRR pricing formula to the Black-Scholes pricing formula

- Recall that

$$P^n_{\mathrm{Put}}(0) = \mathbb{E}_{Q_n}\left[\left(\frac{K}{(1+r_n)^n} - S(0)\,e^{Y_n}\right)^+\right].$$

- One can check that

$$\left|P^n_{\mathrm{Put}}(0) - \mathbb{E}_{Q_n}\left[\left(Ke^{-rT} - S(0)\,e^{Y_n}\right)^+\right]\right| \le K\left|(1+r_n)^{-n} - e^{-rT}\right|,$$

and, therefore, $P^n_{\mathrm{Put}}(0)$ and $\mathbb{E}_{Q_n}\left[\left(Ke^{-rT} - S(0)\,e^{Y_n}\right)^+\right]$ converge to the same limit as n tends to infinity.

- Then, we can conclude that

$$\begin{aligned}\lim_{n\to+\infty} P^n_{\mathrm{Put}}(0) &= \lim_{n\to+\infty}\mathbb{E}_{Q_n}\left[\left(Ke^{-rT} - S(0)\,e^{Y_n}\right)^+\right]\\ &= P_{\mathrm{Put}}(0).\end{aligned}$$

- It is easy to check that

$$P_{\mathrm{Put}}(0) = Ke^{-rT}\Phi\left(-d_2\left(S(0), T\right)\right) - S(0)\,\Phi\left(-d_1\left(S(0), T\right)\right),$$

where Φ is the cumulative normal distribution and d_1 and d_2 are the same functions defined in Theorem 34.

Convergence of the CRR pricing formula to the Black-Scholes pricing formula

- By using the put-call parity relationship (on the binomial market and on the Black-Scholes market) one gets that

$$\begin{aligned}\lim_{n\to+\infty} P^n_{\mathrm{Call}}(0) &= \lim_{n\to+\infty}\left(P^n_{\mathrm{Put}}(0) + S(0) - (1+r_n)^{-n} K\right)\\ &= P_{\mathrm{Put}}(0) + S(0) - e^{-rT} K\\ &= P_{\mathrm{Call}}(0),\end{aligned}$$

  where

$$\begin{aligned}P^n_{\mathrm{Call}}(0) &= (1+r_n)^{-n}\,\mathbb{E}_{Q_n}\left[(S(n)-K)^+\right]\\ &= \mathbb{E}_{Q_n}\left[\left(S(0)\,e^{Y_n} - \frac{K}{(1+r_n)^n}\right)^+\right],\end{aligned}$$

  and

$$P_{\mathrm{Call}}(0) = S(0)\,\Phi\left(d_1(S(0),T)\right) - Ke^{-rT}\Phi\left(d_2(S(0),T)\right)$$

- One can modify the previous arguments to provide the formulas for $P_{\mathrm{Call}}(t)$ and $P_{\mathrm{Put}}(t)$.

Convergence of the CRR pricing formula to the Black-Scholes pricing formula

Theorem 36

Let $g \in C_b(\mathbb{R})$ and let $X = g(S(T))$ be a contingent claim in the Black-Scholes model. Then the price process of X is given by

$$P_X(t) = \lim_{t \to +\infty} P_X^n(t), \qquad 0 \leq t \leq T,$$

where $P_X^n(t), n \geq 1$ are the price processes of X in the corresponding CRR models.

- There exist similar proofs of the previous results using the normal approximation to the binomial law, based on the central limit theorem.
- However, note that here we have a triangular array of random variables $\{Y_n(j)\}_{j=1,\ldots,n}, n \geq 1$. Hence, the result does not follow from the basic version of the central limit theorem.
- Moreover, the asymptotic distribution of Y_n need not be Gaussian if we choose suitably the parameters of the CRR model.
- For instance, if we set $u_n = u$ and $d_n = e^{ct/n}, c < r$ we have that Y_n converges in law to a Poisson random variable.
- This lead to consider the exponential of more general Lévy process as underlying price process for the stock.