An Introduction to
Mathematical Finance
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Lecture 3

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Basic Financial Derivatives

The valuation of financial derivatives will be based on the principle of no arbitrage.

Definition 1. Arbitrage means making of a guaranteed risk free profit with a trade or a series of trades in the market.

Definition 2. An arbitrage free market is a market which has no opportunities for risk free profit.

Definition 3. An arbitrage free price for a security is a price that ensure that no arbitrage opportunity can be designed with that security.

The principle of no arbitrage states that the markets must be arbitrage free. Some financial jargon will be used in what follows. One says that has/takes a long position on an asset if one owns/is going to own a positive amount of that asset. One says that has/takes a short position on an asset if one has/is going to have a negative amount of that asset. Being short on money means borrowing. You can take a short position on many financial assets by short selling.

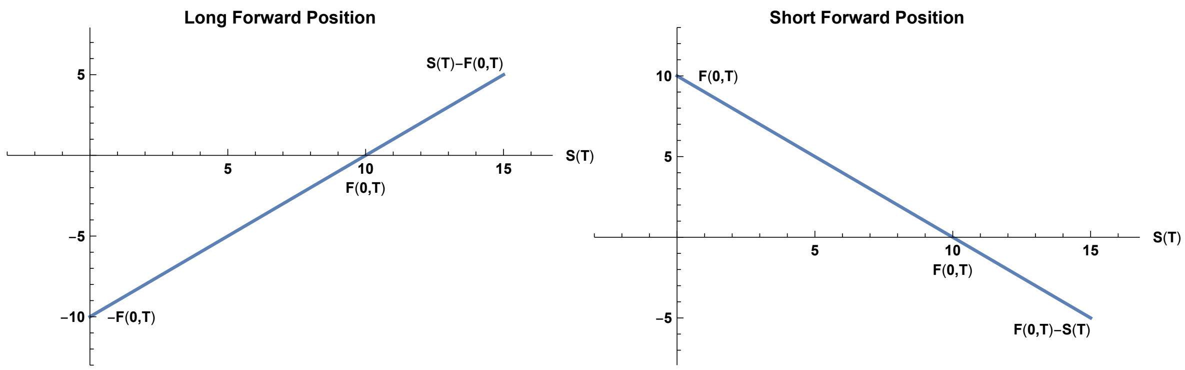
Example 4 (Short selling)**.** To implement some trading strategy you need to sell some amount of shares (to get money and invest in other assets). The problem is that you do not have any shares right now. Then, you can borrow the shares from another investor for a time period (paying interest) and sell the borrowed shares in the market to get the money you need for your strategy. To close this position, at the end of the borrowing period you must buy again the shares in the market and give them back to the lender.

1 Forward contracts

Definition 5. A forward contract is an agreement to buy or sell an asset on a fixed date in the future, called the delivery time, for a price specified in advance, called the forward price.

The party selling the asset is said to be taking a short forward position and the party buying the asset is said to be taking a long forward position. Both parties are obliged to fulfill the terms of the contract (no optionality). The main reason to enter into a forward contract agreement is to become independent of the unknown future price of a risky asset. Assume that the contract is entered at time $t = 0$, the delivery time is $t = T$ and denote by $F(0, T)$ the forward price. The time t price of the underlying asset is denoted $S(t)$. Due to the symmetry of the contract, no payment is made by either party at the beginning of the contract, $t = 0$.

At delivery, the party with the long position makes a profit if $F(0, T) < S(T)$,while the party with the short position will make a loss (exactly of the same magnitude as the profit of the other party). If $F(0, T) > S(T)$ the situation is reversed. The payoffs at delivery are:



If the contract is initiated at $t < T$, we will write $F(t, T)$ for the forward price and the payoffs will be $S(T) - F(t, T)$ (long position) and $F(t, T) - S(T)$(short position).

1.1 Forward price

As no payment is made at the beginning of the forward contract, the main problem is to find the forward price $F(0, T)$ such that both parties are willing to enter into such agreement. One possible approach would be to compute the present value, which we know that is zero, by discounting the expected payoff of the contract. That is,

$$0 = V(0) = e^{-rT}\mathbb{E}[S(T) - F(0, T)],$$

which yields $F(0, T) = \mathbb{E}[S(T)]$. Note that $F(0, T)$ would depend on the distribution of $S(T)$, hence, we would only have translated the problem to agree on which distribution use.

The solution comes from the fact that we can also invest in the money market and there exists only one value for $F(0, T)$ such avoid arbitrages. The price obtained does not depend on the distribution of $S(T)$.

Remark 6. An alternative to taking a long forward position is to borrow $S(0)$ NOK, to buy the asset at time zero and keep it until time T. At time T, the amount $S(0)e^{rT}$ to be paid to settle the loan is a natural candidate for $F(0, T)$.

Theorem 7. *The forward price $F(0, T)$ is given by*

$$F(0, T) = S(0)e^{rT}, \tag{1}$$

where r is the constant risk free interest rate under continuous compounding. If the contract is initiated at time $t \leq T$, then

$$F(t,T) = S(t)\, e^{r(T-t)}. \tag{2}$$

Remark 8. The formula in the previous theorem applies as long as the underlying asset does not generate an income (dividends) or a cost (storage and insurance costs for commodities). In this lecture we will, many times, be implicitly assuming that the underlying is a stock which does not pay dividends.

Proof. (Theorem 7, only formula (1).) Suppose that

$$F(0,T) > S(0)\, e^{rT}.$$

In this case, at time 0 we can:

- Borrow the amount $S(0)$ until T.
- Buy one share for $S(0)$.
- Take a short forward position with forward price $F(0,T)$ (at no cost).

Then, at time T :

- Sell the stock for $F(0,T)$, closing the forward position.
- Pay $S(0)\, e^{rT}$ to settle the loan.

This will bring a risk free profit of

$$F(0,T) - S(0)\, e^{rT} > 0,$$

which contradicts the principle of no arbitrage.

Next, suppose that

$$F(0,T) < S(0)\, e^{rT}.$$

In this case we can:

- Sell short one share for $S(0)$.
- Invest $S(0)$ at the risk free rate.
- Take a long forward position with forward price $F(0,T)$ (at no cost).

Then, at time T :

- Cash the risk free investment, collecting $S(0)\, e^{rT}$.
- Buy the stock for $F(0,T)$, closing the forward position
- Close the short selling position by returning the stock to the owner.

This will bring a risk free profit of

$$S(0)\, e^{rT} - F(0,T) > 0,$$

which contradicts the principle of no arbitrage. □

Remark 9. In the case considered here we always have

$$F(t,T) = S(t)\, e^{r(T-t)} > S(t).$$

Moreover, the difference $F(t,T) - S(t)$, called the basis, converges to 0 as t converges to T.

1.2 Value of a forward contract

Every forward contract has value zero when initiated. As time goes by, the price of the underlying asset may change and, along with it, the value of the forward contract.

Theorem 10. *The time t value of a long forward position with forward price $F(0,T)$ is given by*

$$V(t) = (F(t,T) - F(0,T))\, e^{-r(T-t)}, \qquad 0 \le t \le T,$$

where $F(t,T)$ is the forward price of a contract starting at t and with delivery date T.

Proof. Suppose that

$$V(t) < (F(t,T) - F(0,T))\, e^{-r(T-t)}. \tag{3}$$

Then, at time t:

- Borrow the amount $V(t)$.
- Pay $V(t)$ to enter a long forward position with forward price $F(0,T)$.
- Take a short forward position with forward price $F(t,T)$ (at no cost).

Next, at time T:

- Close the forward positions, getting:
  - $S(T) - F(0,T)$ for the long position,
  - $F(t,T) - S(T)$ for the short position.
- Pay $V(t)\, e^{r(T-t)}$ to settle the loan.

This will yield a risk free profit of

$$\begin{aligned} &S(T) - F(0,T) + F(t,T) - S(T) - V(t)\, e^{r(T-t)} \\ &= F(t,T) - F(0,T) - V(t)\, e^{r(T-t)} > 0. \end{aligned}$$

Consider the case $V(t) > (F(t,T) - F(0,T))\, e^{-r(T-t)}$ as an exercise. □

2 Futures

One of the two parties in a forward contract will be losing money. There is a risk of default (not being able to fulfill the contract) by the party losing money. Futures contracts are designed to eliminate such risk.

Definition 11. A futures contract is an exchange-traded standardized agreement between two parties to buy or sell an asset at an specified future time and at a price agreed today. The contract is marked to market daily and guaranteed by a clearing house.

Assume that time is discrete with steps of length τ, typically one day. The market dictates the so called *futures prices* $f(n\tau, T)$ for each time step $\{n\tau\}_{n\geq 0}$ such that $n\tau \leq T$. These prices are random (and not known at time 0) except for $f(0,T)$. As in the case of forward contracts, it cost nothing to take a futures position. However, a futures contract involves a random cash flow, known as *marking to market.* Namely, at each time step $n\tau \leq T, n \geq 1$, the holder of a long futures position will receive the amount $f(n\tau, T) - f((n-1)\tau, T)$ if positive, or he/she will have to pay it if negative. The opposite payments apply for a short futures position.

The following conditions are imposed:

1. The futures price at delivery is $f(T,T) = S(T)$.
2. At each $n\tau \leq T, n \geq 1$, the value of a futures position is reset to zero, after marking to market.

In particular 2. implies that it costs nothing to close, open or modify a futures position at any time step between 0 and T.

Remark 12. Each investor entering into a futures contract has to set up a deposit, called the *initial margin*, which is kept by the clearing house as collateral. In the case of a long futures position the amount is added/subtracted to the deposit, depending if it is positive/negative, at each time step. The opposite amount is added/subtracted for a short position. Any excess above the initial margin can be withdrawn. However, if the deposit drops below a certain level, called the *maintenance margin*, the clearing house will issue a *margin call*, requesting the investor to make a payment and restore the deposit to the level of the initial margin. If the investor fails to respond to a margin call the futures position is closed by the clearing house.

Remark 13. Futures markets are very liquid (high number of transactions) due to standardization and the presence of a clearing house. There is no risk of default, in contrast to forward contracts negotiated directly between two parties. A negative aspect of of the highly standardized contracts is that you may not find a contract that actually covers the risk you want to hedge. For example, you want protection against adverse movements of the share prices of a company, but in the market there are no futures on the share price of that company. In this case, you may use futures on a stock index containing the company, but the hedge is less perfect and more risky.

Theorem 14. *If the interest rate r is constant, then $f(0,T) = F(0,T)$.*

Proof. We will compare two strategies. Both having the same final payoff of $S(T)$ at time T. However, one starts with an amount of $e^{-rT}F(0,T)$ and uses forwards and the other starts with $e^{-rT}f(0,T)$ and uses futures. As both strategies have the same final payoff, the initial investments must coincide in order to avoid arbitrage. This yields that $F(0,T) = f(0,T)$.

Strategy I

- At time 0:
  - Take a long forward position with forward price $F(0,T)$ (at no cost).
  - Invest the amount $e^{-rT}F(0,T)$ risk free at rate r.

- At time T:
  - Close the risk free investment, collecting $F(0,T)$.
  - Use the forward contract to buy the stock for $F(0,T)$.
  - Sell the stock for the market price of $S(T)$.

Strategy II
Assume that $T = N\tau$.

- A time 0:
  - Open a fraction of $e^{-r(T-t_1)}$ of a long futures position (at no cost).
  - Invest the amount $e^{-rT} f(0,T)$ risk free, which will yield $v_0 = f(0,T)$ at time T.
- At time $n\tau$, $n = 1, ..., N-1$:
  - We receive (or pay) the amount $e^{-r(T-n\tau)} (f(n\tau, T) - f((n-1)\tau, T))$ as a result of marking to market.
  - We invest (or borrow) $e^{-r(T-n\tau)} (f(n\tau, T) - f((n-1)\tau, T))$ risk free, which will yield $v_n = f(n\tau, T) - f((n-1)\tau, T)$ at time T.
  - We increase our long futures position to $e^{-r(T-(n+1)\tau)}$ of a contract (at no cost).
- At time T:
  - We close the risk free investment, collecting the amount

$$\sum_{n=0}^{N-1} v_n = f(0,T) + \sum_{n=1}^{N-1} (f(n\tau, T) - f((n-1)\tau, T)) = f((N-1)\tau, T).$$

  - We close the futures position, receiving (or paying) the amount

$$S(T) - f((N-1)\tau, T).$$

  - The final balance will be

$$f((N-1)\tau, T) + S(T) - f((N-1)\tau, T) = S(T).$$

□

If we assume that the interest rates change unpredictably the strategy II cannot be implemented. Hence, in an economy with constant interest rate r we obtain a simple structure of futures prices

$$f(t,T) = S(t)\, e^{r(T-t)}. \tag{4}$$

Note that the futures are random but only due to S(t). Actually if the futures prices depart form the values given by (4), it is a reflection of the market's view of future interest rate changes. This is an example of price discovery using derivatives.

3 Options

Definition 15. A European call/put option is a contract giving the holder the right (but not the obligation) to buy/sell an asset for a price K fixed in advance, known as the exercise price or strike price, at a specified future date T, called the exercise time or expiry date.

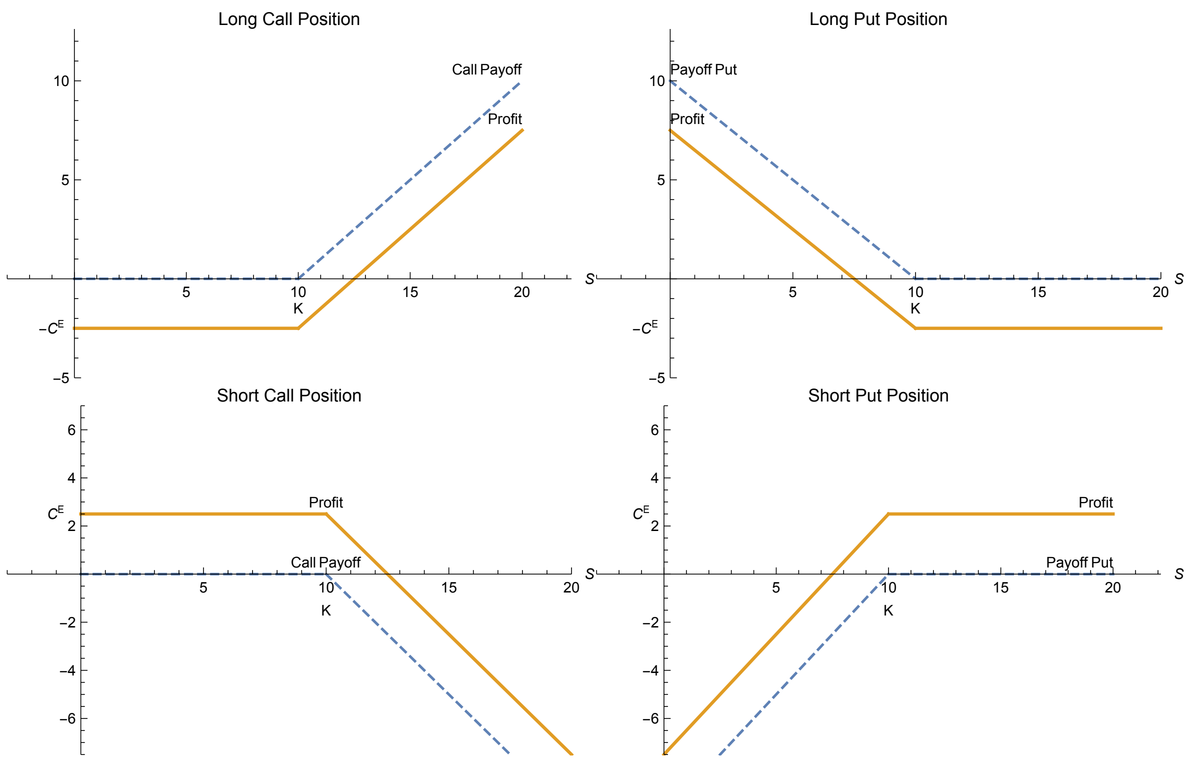
Definition 16. An American call/put option is a contract giving the holder the right (but not the obligation) to buy/sell an asset for a strike price K, also fixed in advance, at any time between now and the expiry date T.

Some underlying assets may be impossible to buy or sell (e.g., stock indices). The option then is cleared in cash. An option is determined by its payoff, which for a European call is

$$(S(T) - K)^{+} = \max(0, S(T) - K) = \begin{cases} S(T) - K & if \quad S(T) > K \\ 0 & if \quad S(T) \leq K \end{cases},$$

and for a put option is $(K - S(T))^{+}$. Since the payoffs are non negative, a premium must be paid to buy an option, otherwise there is an arbitrage opportunity. The *premium* is the market price of the option at time 0.

The prices of European calls and puts will be denoted by C^E and P^E. We will use the notation C^A and P^A for the American ones. The same constant interest rate r will apply for borrowing and lending money and we will use continuous compounding. The gain/profit of an option buyer (seller, also known as writer) is the payoff minus (plus) the premium C^E or P^E paid (received) for the option.



In what follows we will find bounds and some general properties for option prices. We will use the principle of no arbitrage alone, without any assumption on the evolution of the underlying asset prices.

3.1 Put call parity

If we take a long position with a European call and a short position with a European put, both with the same strike K and expiry time T, we get a portfolio having the same payoff as a long forward position with forward price K and delivery time T, that is,

$$(S(T) - K)^+ - (K - S(T))^+ = S(T) - K.$$

Remark 17. Consider a forward contract with forward price K instead of $F(0,T)$. The value of this contract at time t will be given by equation (3) in Theorem 10 with $F(0,T)$ replaced by K,

$$V_K(t) = (F(t,T) - K)\, e^{-r(T-t)}. \tag{5}$$

Such contract may have non-zero initial value

$$V_K(0) = (F(0,T) - K)\, e^{-rT} = S(0) - Ke^{-rT}. \tag{6}$$

As a result, the present value of such portfolio of options should be that of a forward contract with delivery price K, which is $S(0) - Ke^{-rT}$.

Theorem 18. *For a stock paying no dividends the following relation holds between the prices of European call and put options, both with the same strike price K and exercise time T,*

$$C^E - P^E = S(0) - Ke^{-rT}. \tag{7}$$

Proof. Suppose that

$$C^E - P^E > S(0) - Ke^{-rT}.$$

At time 0

- Buy one share for $S(0)$.
- Buy one put option for P^E.
- Write and sell one call option for C^E.
- Invest/borrow the amount $C^E - P^E - S(0)$, depending on the sign, risk free at rate r.

The value of this portfolio is zero.

At time T:

- Close the money market position, collecting (or paying) the amount

$$\left(C^E - P^E - S(0)\right) e^{rT}.$$

- Sell the share for K, either by:
  - exercising the put option if $S(T) \leq K$

- settling the short position in the call option if $S(T) > K$.

This will give a total profit of

$$\left(C^E - P^E - S(0)\right) e^{rT} + K > 0,$$

which is positive by assumption. Hence, we have an arbitrage.

Suppose that

$$C^E - P^E < S(0) - Ke^{-rT}.$$

At time 0

- Sell short one share for $S(0)$.
- Buy one call option for C^E.
- Write and sell one put option for P^E.
- Invest the amount $S(0) - C^E + P^E$, which is positive by assumption, risk free at rate r.

The value of this portfolio is zero.

At time T:

- Close the money market position, collecting the amount $\left(S(0) - C^E + P^E\right) e^{rT}$.
- Buy one share for K, either by:
  - exercising the call option if $S(T) > K$
  - settling the short position in the put option if $S(T) \leq K$.
  - Close the short selling position by returning the stock to the owner.

This will give a total profit of

$$\left(S(0) - C^E + P^E\right) e^{rT} - K > 0,$$

which is positive by assumption. Hence, we also have an arbitrage. □

Theorem 19. *(Put-call parity estimates) The price of American call and put options, with the same strike price K and expiry time T, satisfy*

$$C^A - P^A \leq S(0) - Ke^{-rT}, \tag{8}$$
$$C^A - P^A \geq S(0) - K. \tag{9}$$

Proof. ***Proof of*** (8). Suppose that (8) does not hold, that is

$$C^A - P^A - S(0) + Ke^{-rT} > 0.$$

Then,

- at time $t = 0$:
  - Sell a call, buy a put and buy a share, financing the transactions in the money market.

- If the American call is exercised at $0 < t \leq T$:
  - We get K for the share, closing the short call position.
  - We close the money market position.
  - We still have the put option which has a non negative value.

The final balance of this strategy is the value of the put at time t and the amount

$$
\begin{aligned}
K + \left(C^A - P^A - S(0)\right) e^{rt} &= \left(Ke^{-rt} + C^A - P^A - S(0)\right) e^{rt} \\
&\geq \left(Ke^{-rT} + C^A - P^A - S(0)\right) e^{rt} > 0.
\end{aligned}
$$

- If the American call is not exercised:
  - We sell the share for K, exercising the put option at time T.
  - We close the money market position.

The final balance of this strategy gives us

$$K + \left(C^A - P^A - S(0)\right) e^{rT} > 0.$$

Proof of (9). Suppose that (9) does not hold, that is

$$C^A - P^A - S(0) + K < 0.$$

Then,

- At time $t = 0$:
  - Sell a put, buy a call and sell short a share, financing the transactions in the money market.
- If the American put is exercised at time $0 < t \leq T$:
  - Borrow K from the money market and buy a share for K. This settles the short position on the American put.
  - Return the share to the owner, closing the short position on the share.
  - We still have the call option which has a non negative value.
  - We close the money market position.

The final balance of this strategy is the value of the call at time t and the amount

$$\left(P^A - C^A + S(0)\right) e^{rt} - K > Ke^{rt} - K > 0.$$

- If the American put is not exercised:
  - We buy the share for K, exercising the call option at time T.
  - We close the short sale of the stock.
  - We close the money market position.

The final balance of this strategy gives

$$\left(P^A - C^A + S(0)\right) e^{rT} - K > Ke^{rT} - K > 0.$$

□

3.2 Bounds on option prices

In this section we will assume that all the options have the same strike K and expiry time T. We start by noting the following obvious inequalities

$$\left.\begin{array}{l} 0 \leq C^E \leq C^A, \\ 0 \leq P^E \leq P^A. \end{array}\right\} \tag{10}$$

The option prices must be non-negative because they have non-negative payoff. Moreover, the American options should be more expensive because they give at least the same rights as the corresponding European options.

Proposition 20. *On a stock paying no dividends one has that*

$$\left(S(0) - Ke^{-rT}\right)^+ = \max\left(0, S(0) - Ke^{-rT}\right) \leq C^E < S(0), \tag{11}$$

$$\left(Ke^{-rT} - S(0)\right)^+ = \max\left(0, Ke^{-rT} - S(0)\right) \leq P^E < Ke^{-rT}. \tag{12}$$

Proof. ***Proof of*** (11). Suppose that $C^E \geq S(0)$. Then, we can sell the option and buy the stock, investing the balance on the money market. At time T, we sell the stock for $\min(S(T), K)$, settling the call option. Our risk-less profit will be

$$\left(C^E - S(0)\right) e^{rT} + \min(S(T), K) > 0.$$

Then, we have an arbitrage opportunity. On the other hand, from the put-call parity, see equation (7), and the bound $P^E \geq 0$, we obtain the lower bound

$$C^E = S\left(0 - Ke^{-rT} + P^E\right) \geq S(0) - Ke^{-rT}.$$

Proof of (12). From the put-call parity, and taking into account that $C^E < S(0)$, we get

$$P^E = C^E - S(0) + Ke^{-rT} < Ke^{-rT}.$$

Moreover, from the put-call parity again, and using that $C^E \geq 0$, we get

$$P^E = C^E - S(0) + Ke^{-rT} \geq Ke^{-rT} - S(0).$$

□

The following is an important (and counter-intuitive) result.

Theorem 21. *On a stock paying no dividends one has that*

$$C^E = C^A. \tag{13}$$

Proof. We already know that $C^A \geq C^E$. Suppose that $C^A > C^E$. Then, sell an American call and buy a European call, investing the balance in the money market.

- If the American call is exercised at $t \leq T$:
  - Borrow a share and sell it for K, closing the short position on the American call.
  - Invest K in the money market.

– At time T, use the European call to buy the share for K and close the short position on the stock.

The risk-less profit will be

$$\left(C^A - C^E\right) e^{rT} + Ke^{r(T-t)} - K > 0.$$

- If the American call is not exercised: You will end up with the European option (nonnegative value) and a risk-less profit of $\left(C^A - C^E\right) e^{rT}$.

□

Remark 22. As $C^A \geq C^E$ and $C^E \geq S(0) - Ke^{-rT}$, it follows that $C^A > S(0) - K$ if $r > 0$. Because the price of the American option is greater than its payoff, the option will sooner be sold than exercised at time 0. Similar inequalities hold for $t < T$ and one can repeat the arguments to conclude that the American option will never be exercised prior to the expiry time. This also shows that the American option is equivalent to the European option.

Proposition 23. *On a stock paying no dividends one has that*

$$\left(S(0) - Ke^{-rT}\right)^+ = \max\left(0, S(0) - Ke^{-rT}\right) \leq C^A < S(0), \tag{14}$$

$$(K - S(0))^+ = \max\left(0, Ke^{-rT} - S(0)\right) \leq P^A < K. \tag{15}$$

Proof. Exercise. □

3.3 Variables determining option prices

Here we will study how the option prices depend on variables such the strike K, the current price of the underlying $S(0)$, and the expiry time T. We shall analyse option prices as functions of one of the variables, keeping the remaining variables constant.

3.3.1 European options

Dependence on K

Proposition 24. *If $K_1 < K_2$, then*

1. *Monotonicity:*

$$C^E(K_1) > C^E(K_2),$$
$$P^E(K_1) < P^E(K_2).$$

2. *Lipschitz continuity:*

$$C^E(K_1) - C^E(K_2) < e^{-rT}(K_2 - K_1),$$
$$P^E(K_2) - P^E(K_1) < e^{-rT}(K_2 - K_1).$$

3. *Convexity: For $\alpha \in (0,1)$ we have*

$$C^E(\alpha K_1 + (1-\alpha) K_2) \leq \alpha C^E(K_1) + (1-\alpha) C^E(K_2),$$
$$P^E(\alpha K_1 + (1-\alpha) K_2) \leq \alpha P^E(K_1) + (1-\alpha) P^E(K_2).$$

Proof.

1. Obvious.

2. By the put-call parity we have

$$C^E(K_1) - P^E(K_1) = S(0) - K_1 e^{-rT},$$
$$C^E(K_2) - P^E(K_2) = S(0) - K_2 e^{-rT}.$$

   Subtracting the second equation from the first we get

$$C^E(K_1) - C^E(K_2) + P^E(K_2) - P^E(K_1) = (K_2 - K_1) e^{-rT}.$$

   As both terms on the left hand side are positive, each is strictly smaller than the right hand side.

3. Define $K = \alpha K_1 + (1-\alpha) K_2$. Suppose that

$$C^E(K) > \alpha C^E(K_1) + (1-\alpha) C^E(K_2).$$

   We sell an option with strike K, buy α options with strike K_1 and $1-\alpha$ options with strike K_2, investing the balance

$$C^E(K) - \alpha C^E(K_1) - (1-\alpha) C^E(K_2) > 0,$$

   risk free. If the option with strike K is exercised (at time T) we will have to pay $(S(T) - K)^+$. We can raise the amount

$$\alpha (S(T) - K_1)^+ + (1-\alpha)(S(T) - K_2)^+,$$

   by exercising the long position on the options. The risk-less profit follows from the inequality

$$(S(T) - K)^+ \leq \alpha (S(T) - K_1)^+ + (1-\alpha)(S(T) - K_2)^+,$$

   which is left as an exercise. Convexity for put options follow from that of call options, the put-call parity and the fact that the sum of convex functions is convex.

□

Dependence on the underlying asset price

The current price $S(0)$ of the underlying asset is given by the market and cannot be changed. But we can consider an option on a portfolio of x shares, worth $S = xS(0)$. The payoff of a European call with strike K on such portfolio, to be exercised at time T, will be $(xS(T) - K)^+$. We shall study the dependence of option prices on S. We will denote the call and put prices by $C^E(S)$ and $P^E(S)$.

Proposition 25. *If $S_1 < S_2$, then*

1. *Monotonicity:*

$$C^E(S_1) < C^E(S_2),$$
$$P^E(S_1) > P^E(S_2).$$

2. *Lipschitz continuity:*

$$C^E(S_2) - C^E(S_1) < S_2 - S_1,$$
$$P^E(S_1) - P^E(S_2) < S_2 - S_1.$$

3. *Convexity: For $\alpha \in (0,1)$ we have*

$$C^E(\alpha S_1 + (1-\alpha) S_2) \leq \alpha C^E(S_1) + (1-\alpha) C^E(S_2),$$
$$P^E(\alpha S_1 + (1-\alpha) S_2) \leq \alpha P^E(S_1) + (1-\alpha) P^E(S_2).$$

Proof.

1. Suppose that $C^E(S_1) \geq C^E(S_2)$ for some $x_1 S(0) = S_1 < S_2 = x_2 S(0)$. Sell a call on a portfolio with x_1 shares and buy a call on a portfolio with x_2 shares (same K and T). Invest the obtained amount $C^E(S_1) - C^E(S_2) > 0$ risk free. Since $x_1 < x_2$, the payoffs satisfy $(x_1 S(T) - K)^+ \leq (x_2 S(T) - K)^+$ with strict inequality if $K < x_2 S(T)$. If the shorted option is exercised at time T, then we use the other option to cover our liability and we are still left with a non-negative profit.

2. Follows by an analogous argument as the one used in Proposition 24, 2. based on the put-call parity.

3. The same argument as Proposition 24, 3., but selling a call on a portfolio of $x := \alpha x_1 + (1-\alpha) x_2$ and buying α calls on a portfolio with x_1 shares and buying $(1-\alpha)$ calls on a portfolio with x_2 shares.

□

3.3.2 American options

Dependence on K

Proposition 26. *If $K_1 < K_2$, then*

1. *Monotonicity:*

$$C^A(K_1) > C^A(K_2),$$
$$P^A(K_1) < P^A(K_2).$$

2. *Lipschitz continuity:*

$$C^A(K_1) - C^A(K_2) < (K_2 - K_1), \quad (16)$$
$$P^A(K_2) - P^A(K_1) < (K_2 - K_1).$$

3. *Convexity: For $\alpha \in (0,1)$ we have*

$$C^A(\alpha K_1 + (1-\alpha) K_2) \leq \alpha C^A(K_1) + (1-\alpha) C^A(K_2),$$
$$P^A(\alpha K_1 + (1-\alpha) K_2) \leq \alpha P^A(K_1) + (1-\alpha) P^A(K_2).$$

Proof.

1. Obvious.
2. We will only prove (16). Suppose that one has that

$$C^A(K_1) - C^A(K_2) \geq (K_2 - K_1), \qquad K_1 < K_2.$$

Then, we sell a call with strike K_1 and buy a call with strike K_2, investing $C^A(K_1) - C^A(K_2)$ risk free. If the sold call is exercised at time $t \leq T$, we exercise the other call and we get

$$(S(t) - K_2)^+ - (S(t) - K_1)^+ \geq -(K_2 - K_1),$$

with strict inequality if $S(t) < K_2$. If we combine this payoff with the risk free investment, which amounts to at least $(K_2 - K_1)e^{rt}$, we end up with a nonnegative amount of money.

3. Same strategy as European options, but now taking into account early exercise.

□

Dependence on the underlying asset price

As in the European case, we consider options on a portfolio of x shares worth $S = xS(0)$.

Proposition 27. *If $S_1 < S_2$ then*

1. *Monotonicity:*

$$\begin{aligned} C^A(S_1) &< C^A(S_2), \\ P^A(S_1) &> P^A(S_2). \end{aligned}$$

2. *Lipschitz continuity:*

$$\begin{aligned} C^A(S_2) - C^A(S_1) &< S_2 - S_1, \\ P^A(S_1) - P^A(S_2) &< S_2 - S_1. \end{aligned}$$

3. *Convexity: For $\alpha \in (0,1)$ we have*

$$\begin{aligned} C^E(\alpha S_1 + (1-\alpha)S_2) &\leq \alpha C^E(S_1) + (1-\alpha)C^E(S_2), \\ P^E(\alpha S_1 + (1-\alpha)S_2) &\leq \alpha P^E(S_1) + (1-\alpha)P^E(S_2). \end{aligned}$$

Proof. The proof of 1. is obvious. 2. and 3. are analogous to the respective sections in Proposition 26. □

Dependence on the expiry time

Proposition 28. *If* $T_1 < T_2$, *then*

$$C^A(T_1) \leq C^A(T_2),$$
$$P^A(T_1) \leq P^A(T_2).$$

Proof. We only do the argument for the calls (the one for the puts being analogous). Suppose that $C^A(T_1) > C^A(T_2)$. Then, sell the option with shorter time to expiry and buy the one with longer time to expiry, investing the balance risk free. If the sold option is exercised at time $t \leq T_1$, we can exercise the other option to cover our liability. We will have as risk less profit the amount $\left(C^A(T_1) - C^A(T_2)\right) e^{rt} > 0$. □

Remark 29. Note that the arguments in the previous proposition do not work for European options because early exercised is not allowed for European options.

3.4 Time value of options

Definition 30. We say that at time $0 \leq t \leq T$ a call option with strike K is

- (deep) in the money if $S(t) \overset{(\gg)}{>} K$,
- at the money if $S(t) = K$,
- (deep) out of the money if $S(t) \overset{(\ll)}{<} K$.

The same terminology applies to put options but with the inequalities reversed.

Definition 31. At time $0 \leq t \leq T$, the intrinsic value of a call (put) option with strike K is equal to $(S(t) - K)^+$ $((K - S(t))^+)$.

Remark 32.

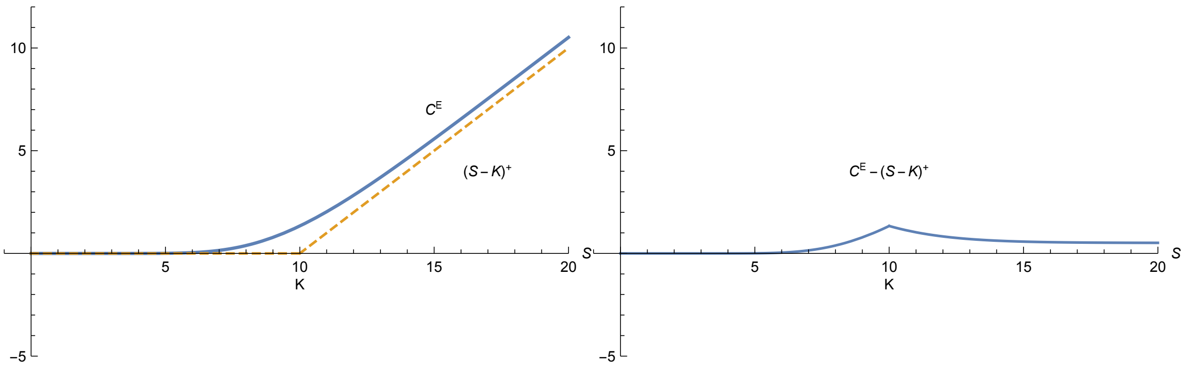
- The intrinsic value of out of the money or at the money options is zero.
- Options in the money have positive intrinsic value.
- The price of an American option prior to expiry must be greater than its intrinsic value.
- The price of a European option prior to expiry may be greater or smaller than its intrinsic value.

Definition 33. The time value of an option is the difference between the price of the option and its intrinsic value, that is,

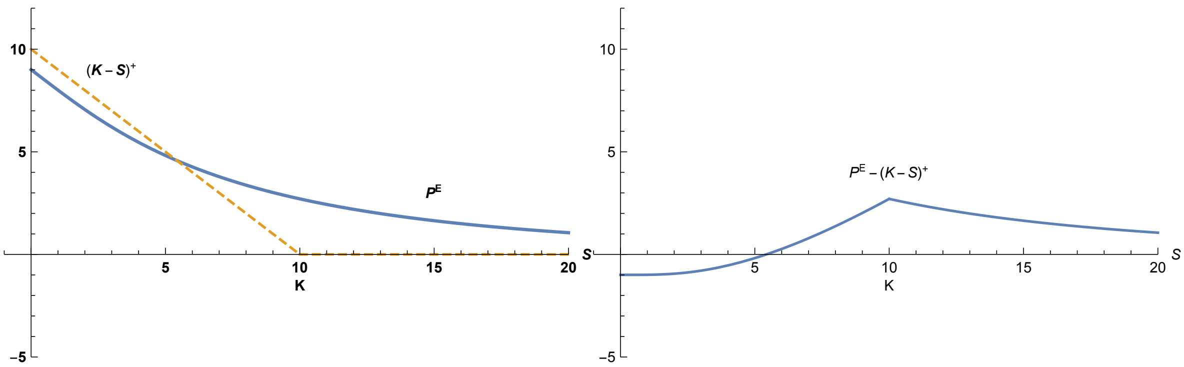
$$\begin{aligned} &C^E(t) - (S(t) - K)^+, && \text{European call} \\ &P^E(t) - (K - S(t))^+, && \text{European put} \\ &C^A(t) - (S(t) - K)^+, && \text{American call} \\ &P^A(t) - (K - S(t))^+. && \text{American put,} \end{aligned}$$

where, here, the argument t in the option prices denotes the current time, NOT the expiry time as in Proposition 28.

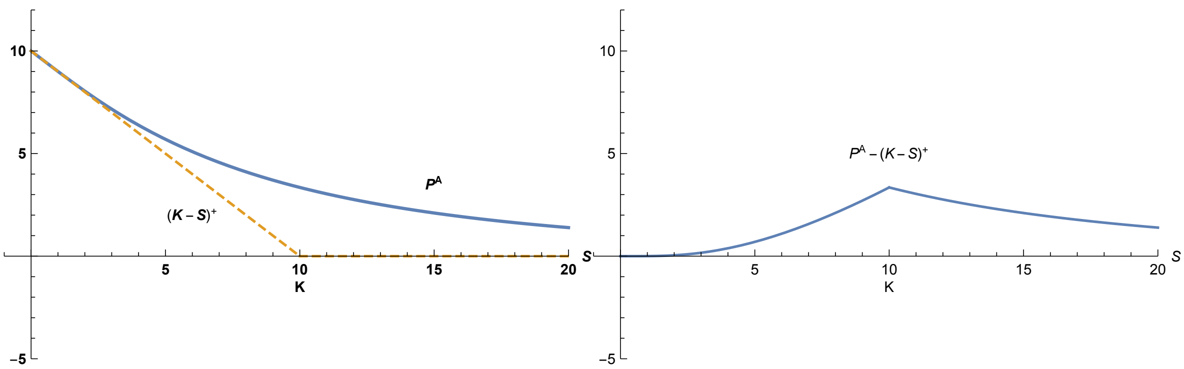
The time value of a European call as a function of $S(t)$ is always nonnegative and, for in the money calls, the time value is bigger than $K - Ke^{-r(T-t)}$, due to the inequality $C^E(t) \geq S(t) - Ke^{-r(T-t)}$ (this inequality is proved using analogous arguments as those in the proof of inequality (11) in Proposition 20). The same applies to an American call because their prices coincide.



The time value of a European put may be negative. This happens if the put option is deep in the money, because we can only exercise the option at time T and there is a considerable risk that in the meanwhile the stock price rises.



The time value of an American put is always nonnegative.



Proposition 34. *For any European or American call or put with strike price K, the time value attains its maximum at $S = K$.*

Proof. We do the proof only for European calls. If $S \leq K$ the intrinsic value is zero. Since $C^E(S)$ is an increasing function of S, this means that the time value is increasing for $S \leq K$. If $K \leq S_1 < S_2$, we have that

$$C^E(S_2) - C^E(S_1) \leq S_2 - S_1$$
$$\Updownarrow$$
$$C^E(S_2) - S_2 \leq C^E(S_1) - S_1$$
$$\Updownarrow$$
$$C^E(S_2) - (S_2 - K)^+ = C^E(S_2) - S_2 + K \leq C^E(S_1) - S_1 + K = C^E(S_1) - (S_1 - K)^+,$$

which yields that the time value of the call is a decreasing function of S if $S \geq K$. Hence, the maximum is at $S = K$. □

3.5 Hedging and speculating with options

Suppose that you are an investor with specific views on the future behavior of stock prices and you are willing to take/avoid risks. By building portfolios of calls, puts, underlying and bonds, you can replicate any piecewise linear terminal payoff function. Using European options, we will show some of the most popular strategies, used to hedge/speculate on stock prices.

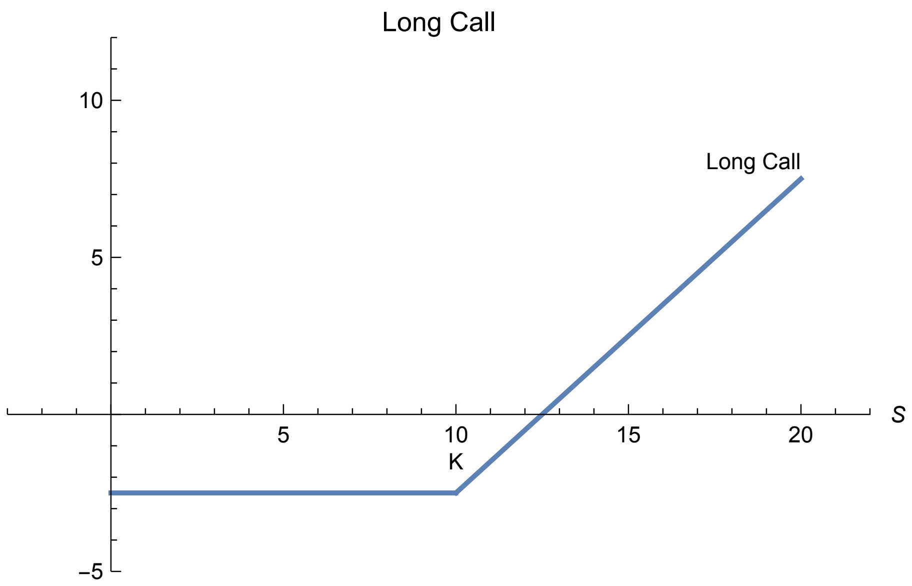
We will assume that all the options are on the same stock. They will also have the same strike K and expiry time T, unless stated otherwise. We will use a solid line to plot the total profit and dashed lines to plot individual option profits.

Thanks to the put-call parity different combinations of calls, puts, stocks and risk free investment may produce the same terminal profits. Hence, there is no unique way of implementing the following strategies.

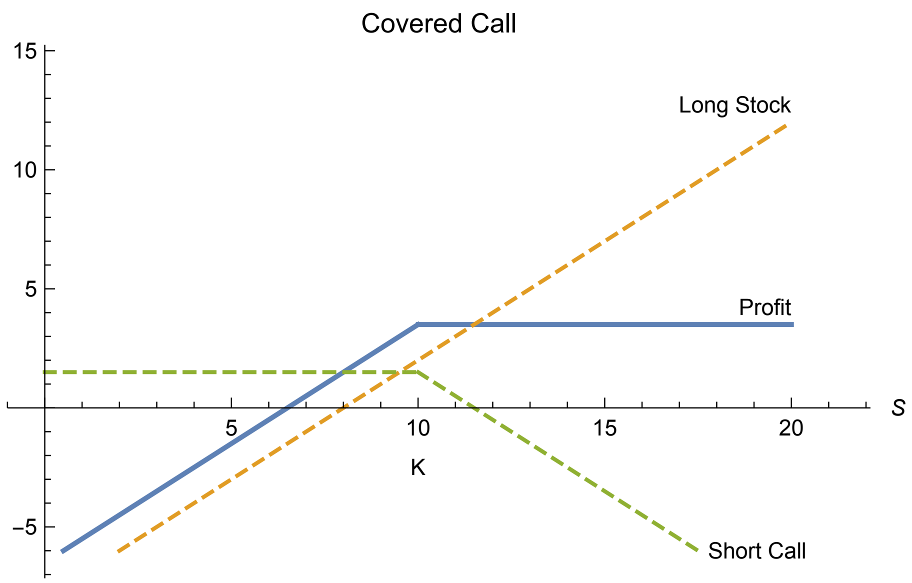
3.5.1 Bullish strategies

These strategies are used when a rise in the price of the stock is expected.

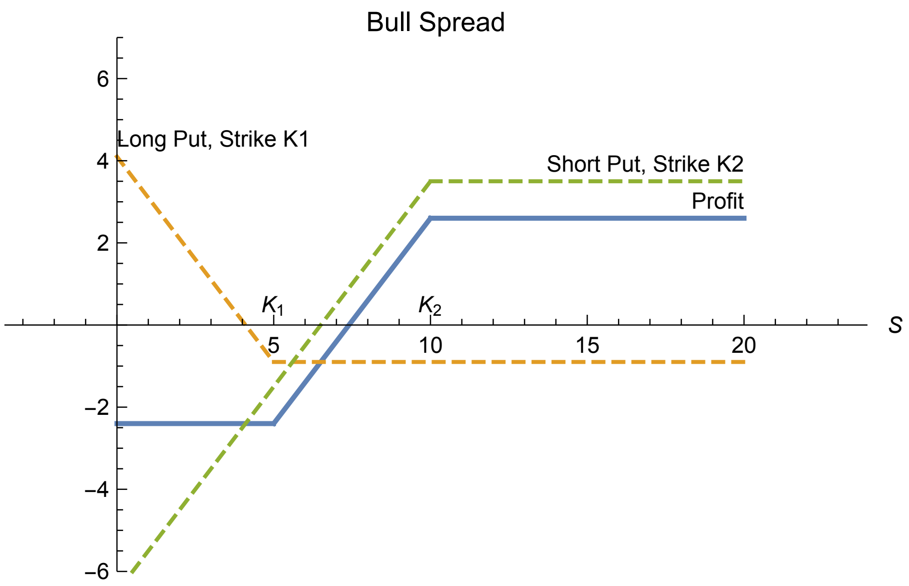
Long call In this strategy you buy a call option. You expect a high rise in the price of the stock. Loses are limited.



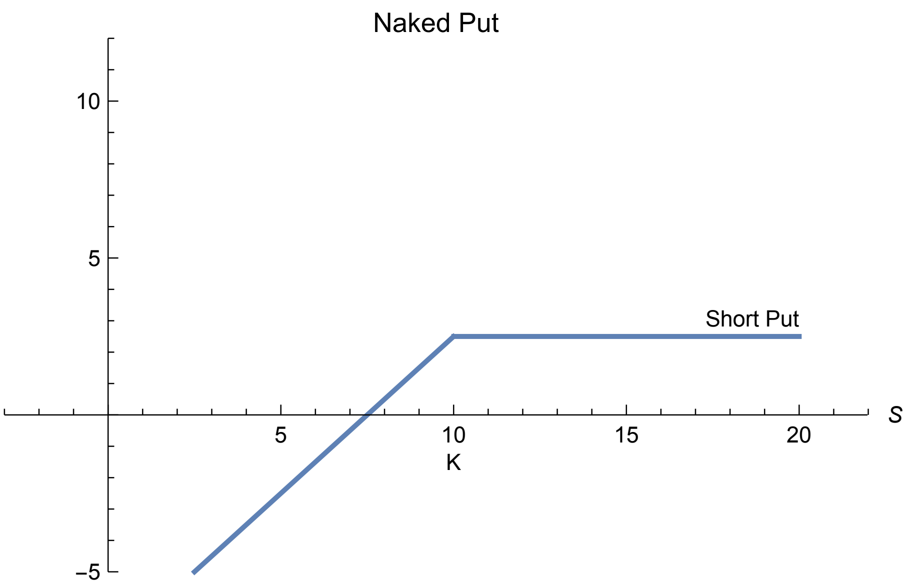
Covered call In this strategy you sell a call option and buy the stock. You expect a moderate rise in the price of the stock. Loses can be very high.



Bull spread Let $0 < K_1 < K_2$. In this strategy you buy a put option with strike K_1 and sell a put option with higher strike K_2. You expect a moderate rise in the price of the stock. Loses are limited. This strategy can also be implemented with calls.



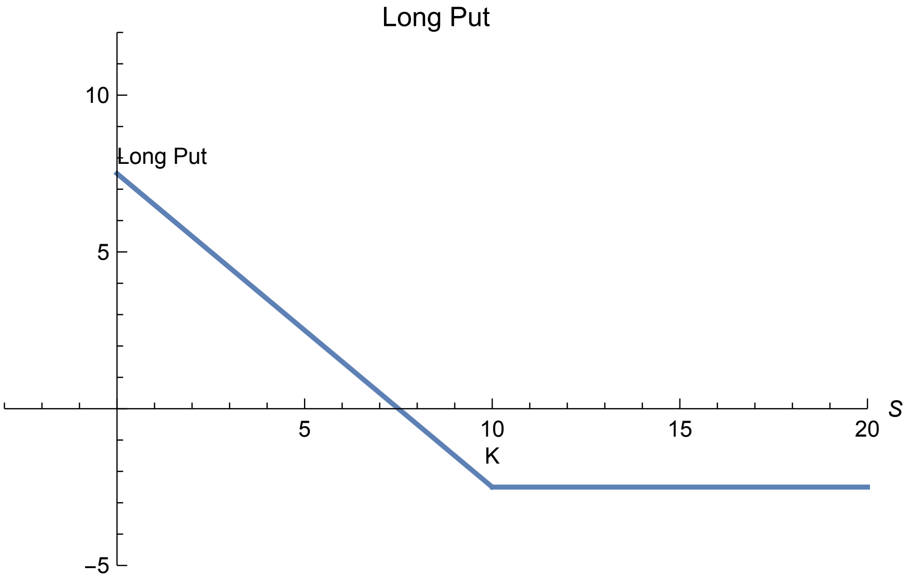
Naked put In this strategy you sell a put option. You expect a moderate rise in the price of the stock, in particular, you expect that at expiry time the stock price will be above the strike price of the put option . Loses can be very high.



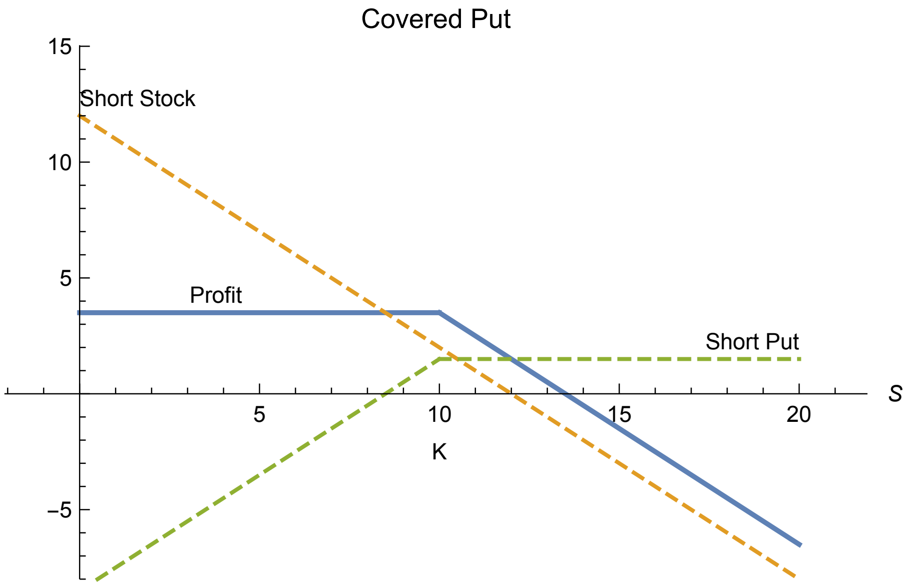
3.5.2 Bearish strategies

These strategies are used when a fall in the prices of the stock is expected.

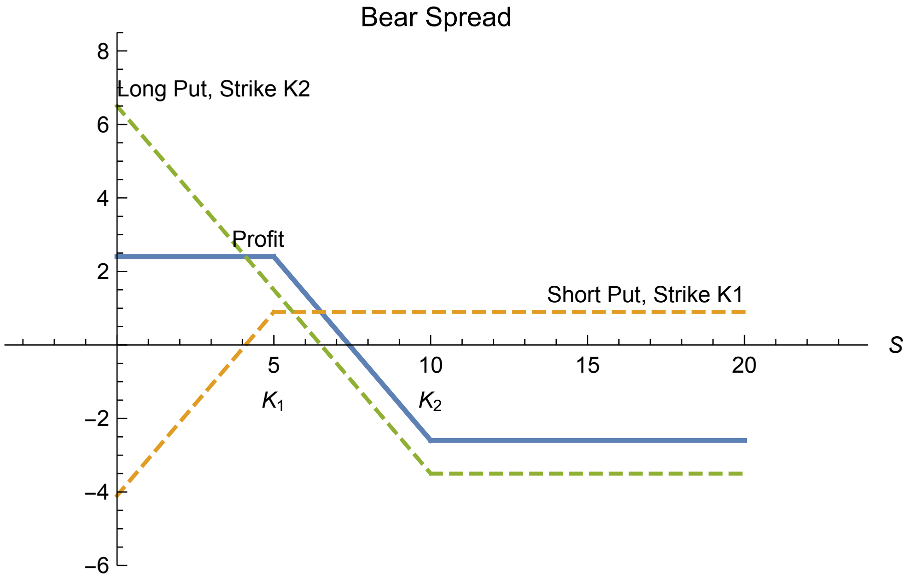
Long put In this strategy you buy a put option. You expect a big drop in the price of the stock. Loses are limited.



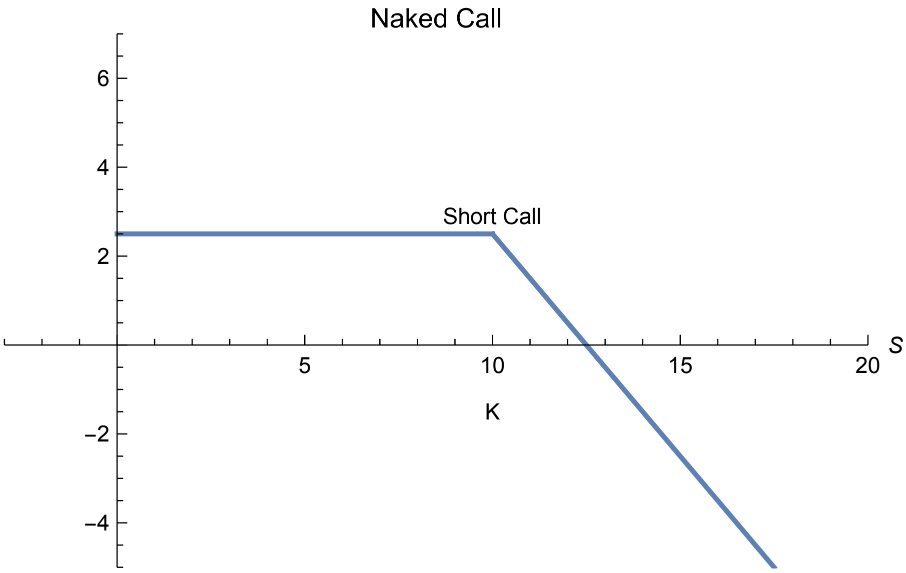
Covered put In this strategy you sell a put option and sell the stock. You expect a moderate drop in the price of the stock. Loses can be very high.



Bear spread Let $0 < K_1 < K_2$. In this strategy you buy a put option with strike K_2 and sell a put option with lower strike K_1. You expect a moderate drop in the price of the stock. Loses are limited. This strategy can also be implemented with calls.



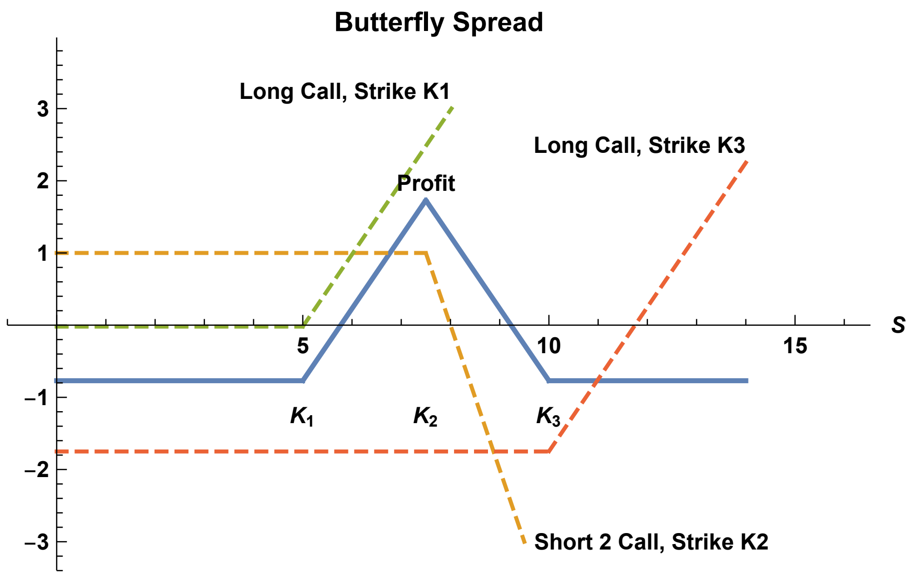
Naked call In this strategy you sell a call option. You expect a moderate drop in the price of the stock, in particular, you expect that at expiry time the stock price will be below the strike price of the call option . Loses can be very high.



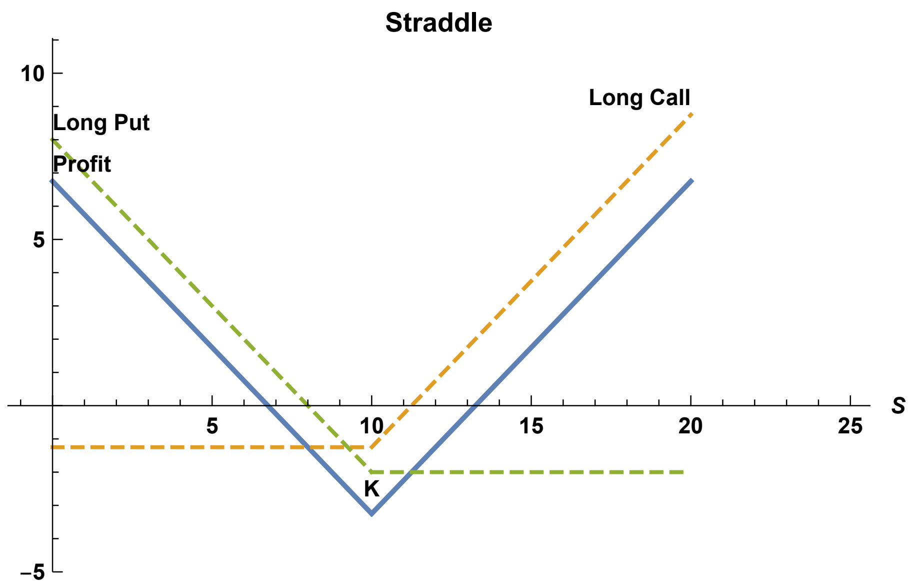
3.5.3 Neutral or non-directional strategies

These strategies are used when no clear direction in the price of the stock is expected. These strategies bet on the volatility (standard deviation) of the stock price.

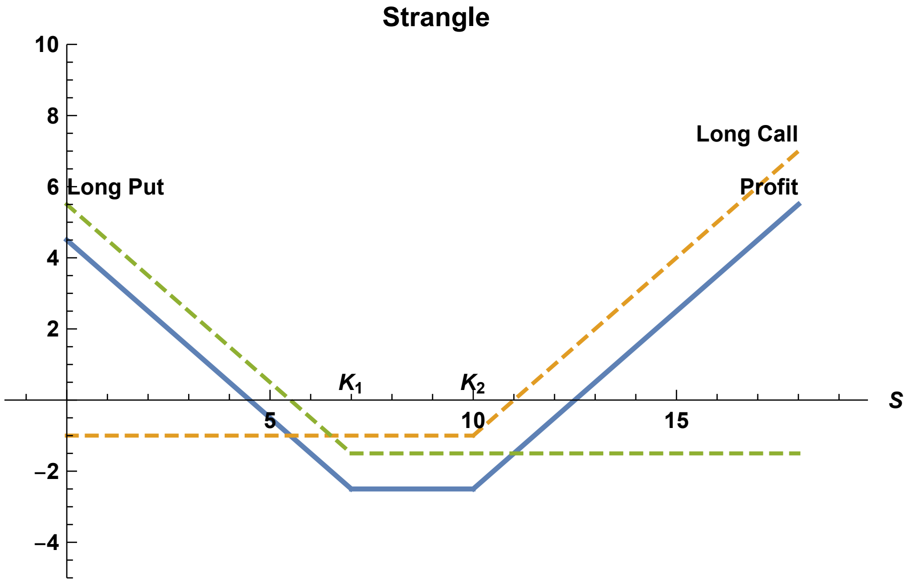
Butterfly spreads Let $0 < K_1 < K_2 < K_3$. In this strategy you buy a call option with strike K_1 and a call option with strike K_3 and sell two call options with strike K_2. Loses are limited. You make a profit if the stock price stays close to the strike K_2. Hence, you expect a low volatility for the stock price.



Straddles In this strategy you buy a call option and a put option with the same strike K. Loses are limited. You make a profit if the stock price ends far away from the strike price K. Hence, you expect a high volatility for the stock price.



Strangles Let $0 < K_1 < K_2$. In this strategy you buy a call option with strike K_2 and a put option with low strike K_1. Loses are limited. You make a profit if the stock price ends far out of the interval $[K_1, K_2]$. Hence, you expect a very high volatility for the stock price. Strangles are cheaper than straddles.



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