# 7. Multiperiod Securities Markets

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# Outline

# Model Specifications

# Model specifications

**Definition 1**

*A multiperiod model of financial markets is specified by the following ingredients:*

1. $T+1$ trading dates: $t = 0, \ldots, T$.
2. A finite probability space $(\Omega, \mathcal{P}(\Omega), P)$ with $\#\Omega = K$ and $P(\omega) > 0, \omega \in \Omega$.
3. A filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t=0,\ldots,T}$.
4. A bank account process $B = \{B(t)\}_{t=0,\ldots,T}$ with $B(0) = 1$ and $B(t, \omega) > 0$, $t \in \{0, \ldots, T\}$ and $\omega \in \Omega$. $B$ is assumed to be an $\mathbb{F}$-adapted process.
5. $N$ risky asset processes $S_n = \{S_n(t)\}_{t=0,\ldots,T}$, where $S_n$ is a nonnegative $\mathbb{F}$-adapted stochastic process for each $n = 1, \ldots, N$.

**Remark 2**

- *The filtration $\mathbb{F}$ represents the information available to the traders.*
- *In this course we will take $\mathbb{F}$ to be equal to $\mathbb{F}^{B,S}$, that is, the filtration generated by the bank account process and the $N$ risky asset processes:*

$$\mathcal{F}_t = \mathfrak{a}\left(\{B(u), S_1(u), \ldots, S_N(u)\}_{u \leq t}\right), \qquad t = 0, \ldots, T.$$

- *The bank account process $B$ is nondecreasing, which implies*

$$r(t) = (B(t) - B(t-1))/B(t-1) \geq 0, \qquad t = 1, \ldots, T$$

- *When $r(t) = r, t = 1, \ldots, T$, then $B(t) = (1+r)^t$, $t = 1, \ldots, T$ and*

$$\mathcal{F}_t = \mathfrak{a}\left(\{S_1(u), \ldots, S_N(u)\}_{u \leq t}\right), \qquad t = 0, \ldots, T$$

# Model specifications

**Definition 3**

A trading strategy $H = (H_0, H_1, ..., H_N)^T$ is a vector of stochastic processes $H_n = \{H_n(t)\}_{t=1,...,T}$, which are predictable with respect to $\mathbb{F}$. That is,

$$H_n(t) \text{ are } \mathcal{F}_{t-1}\text{-measurable}, \qquad n = 0, ..., N, \quad t = 1, ..., T.$$

**Remark 4**

- *Note that $H_n, n = 0, ..., N$, being $\mathbb{F}$-predictable processes, they are also $\mathbb{F}$-adapted processes.*
- *$H_n(0)$, $n = 0, ..., N$ is not specified because*
  - *$H_n(t), n \geq 1$ is the number of shares of the nth risky asset that the investor own from time $t-1$ to time $t$.*
  - *$H_0(t) B(t-1)$ is the amount of money that the trader invest/borrow in the money market (bank account) from time $t-1$ to time $t$.*
- *The trading position $H_n(t)$ is decided by the trader at time $t-1$ and then he/she only has the information associated to $\mathcal{F}_{t-1} \Rightarrow H_n(t)$ are $\mathbb{F}$-predictable.*

# Model specifications

**Definition 5**

The value process $V = \{V(t)\}_{t=0,\dots,T}$ is the stochastic process defined by

$$V(t) = \begin{cases} H_0(1)\,B(0) + \sum_{n=1}^{N} H_n(1)\,S_n(0) & \text{if} \quad t = 0, \\ H_0(t)\,B(t) + \sum_{n=1}^{N} H_n(t)\,S_n(t) & \text{if} \quad t \geq 1. \end{cases} \tag{1}$$

**Definition 6**

The gains process $G = \{G(t)\}_{t=1,\dots,T}$ is the stochastic process defined by

$$G(t) = \sum_{u=1}^{t} H_0(u)\,\Delta B(u) + \sum_{n=1}^{N}\sum_{u=1}^{t} H_n(u)\,\Delta S_n(u), \qquad t \geq 1, \tag{2}$$

where $\Delta B(u) = B(u) - B(u-1)$ and $\Delta S_n(u) = S_n(u) - S_n(u-1)$.

## Remark 7

- *Both $V$ and $G$ are $\mathbb{F}$-adapted processes.*
- $H_n(t)\,\Delta S_n(t)$ *represents the one-period gain or loss due to owning* $H_n(t)$ *shares of the security $n$ between times $t-1$ and $t$.*
- $G(t)$ *represents the cumulative gain or loss up to time $t$ of the portfolio.*
- $V(t)$ *represents the time-$t$ value of the portfolio* `before` *any transactions (changes in $H$) are made at time $t$.*
- *The time-$t$ value of the portfolio just* `after` *any time-$t$ transactions are made is*

$$H_0(t+1)\,B(t) + \sum_{n=1}^{N} H_n(t+1)\,S_n(t), \qquad t \geq 1. \tag{3}$$

- *In general these two portfolio values can be different, which means that we add or withdraw some money from the portfolio. If we do not allow this possibility we have a self-financing portfolio.*

## Model specifications

**Definition 8**

A trading strategy $H$ is self-financing if

$$V(t) = H_0(t+1)\,B(t) + \sum_{n=1}^{N} H_n(t+1)\,S_n(t), \qquad t = 1, \ldots, T-1. \tag{4}$$

**Remark 9**

- *It is easy to check that $H$ is self-financing if and only if*

$$V(t) = V(0) + G(t), \qquad t = 1, \ldots, T. \tag{5}$$

- *If no money is added or withdrawn from the portolio between time $t = 0$ and $t = T$, then any change in the portfolio's value is due to gain or loss in the investments*

# Model specifications

**Definition 10**

- The discounted price process $S_n^* = \{S_n^*(t)\}_{t=0,\dots,T}$ is defined by

$$S_n^*(t) = \frac{S_n(t)}{B(t)}, \qquad t = 0, \dots, T, \quad n = 1, \dots, N. \tag{6}$$

- The discounted value process $V^* = \{V^*(t)\}_{t=0,\dots,T}$ is defined by

$$V^*(t) = \begin{cases} H_0(1) + \sum_{n=1}^N H_n(1) S_n^*(0) & \text{if} \quad t = 0, \\ H_0(t) + \sum_{n=1}^N H_n(t) S_n^*(t) & \text{if} \quad t \geq 1. \end{cases} \tag{7}$$

- The discounted gains process $G^* = \{G^*(t)\}_{t=1,\dots,T}$ is defined by

$$G^*(t) = \sum_{n=1}^N \sum_{u=1}^t H_n(u) \Delta S_n^*(u), \quad t = 1, \dots, T, \tag{8}$$

where $\Delta S_n^*(u) = S_n^*(u) - S_n^*(u-1)$.

- It is easy to check that a trading strategy $H$ is self-financing if and only if

$$V^*(t) = V^*(0) + G^*(t), \qquad t = 0, \dots, T \tag{9}$$

**Example 11**

$N=1, K=4,\ B(t)=(1+r)^t,\ r\geq 0,\ S(0)=5,$

$$S(1,\omega)=\begin{cases} 8 & \text{if} \quad \omega=\omega_1,\omega_2 \\ 4 & \text{if} \quad \omega=\omega_3,\omega_4 \end{cases} = 8\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega)+4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega),$$

$$S(2,\omega)=\begin{cases} 9 & \text{if} \quad \omega=\omega_1 \\ 6 & \text{if} \quad \omega=\omega_2,\omega_3 \\ 3 & \text{if} \quad \omega=\omega_4 \end{cases} = 9\mathbf{1}_{\{\omega_1\}}(\omega)+6\mathbf{1}_{\{\omega_2,\omega_3\}}(\omega)$$
$$+3\mathbf{1}_{\{\omega_4\}}(\omega).$$

We have that $\mathcal{F}_0=\mathfrak{a}(S(0))=\mathfrak{a}\left(\pi_{S(0)}\right)=\{\emptyset,\Omega\},$

$$\begin{aligned}\mathcal{F}_1 &= \mathfrak{a}(S(0),S(1))=\mathfrak{a}\left(\pi_{S(0)}\cap\pi_{S(1)}\right)=\mathfrak{a}\left(\pi_{S(1)}\right)\\ &= \mathfrak{a}(\{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}\})=\{\emptyset,\Omega,\{\omega_1,\omega_2\},\{\omega_3,\omega_4\}\},\\ \mathcal{F}_2 &= \mathfrak{a}(S(0),S(1),S(2))=\mathfrak{a}\left(\pi_{S(0)}\cap\pi_{S(1)}\cap\pi_{S(2)}\right)\\ &= \mathfrak{a}\left(\pi_{S(1)}\cap\pi_{S(2)}\right)=\mathfrak{a}(\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}\})=\mathcal{P}(\Omega).\end{aligned}$$

**Example 11**

Let $H = \{H(t)\}_{t=1,2} = \left\{ (H_0(t), H_1(t))^T \right\}_{t=1,2}$ be a trading strategy. Since $H$ is predictable it has the form

$$\begin{aligned} H_0(1,\omega) &= H_0(1), \qquad H_1(1,\omega) = H_1(1), \\ H_0(2,\omega) &= H_0(2,\{\omega_1,\omega_2\})\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + H_0(2,\{\omega_3,\omega_4\})\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega), \\ H_1(2,\omega) &= H_1(2,\{\omega_1,\omega_2\})\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + H_1(2,\{\omega_3,\omega_4\})\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega). \end{aligned}$$

Then,

$$V(0) = H_0(1)B(0) + H_1(1)S(0) = H_0(1) + 5H_1(1),$$

$$\begin{aligned} V(1,\omega) &= H_0(1)B(1) + H_1(1)S(1) \\ &= (1+r)H_0 + H_1(1)\left(8\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + 4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega)\right) \\ &= \begin{cases} (1+r)H_0(1) + 8H_1(1) & \text{if} \quad \omega = \omega_1,\omega_2 \\ (1+r)H_0(1) + 4H_1(1) & \text{if} \quad \omega = \omega_3,\omega_4 \end{cases}, \end{aligned}$$

**Example 11**

$$
\begin{aligned}
&V(2,\omega)\\
&= H_0(2)B(2) + H_1(2)S(2)\\
&= \left(H_0(2,\{\omega_1,\omega_2\})\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + H_0(2,\{\omega_3,\omega_4\})\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega)\right)(1+r)^2\\
&+ \left(H_1(2,\{\omega_1,\omega_2\})\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + H_1(2,\{\omega_3,\omega_4\})\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega)\right)\\
&\quad\times\left(9\mathbf{1}_{\{\omega_1\}}(\omega) + 6\mathbf{1}_{\{\omega_2,\omega_3\}}(\omega) + 3\mathbf{1}_{\{\omega_4\}}(\omega)\right)\\
&= \begin{cases}
(1+r)^2 H_0(2,\{\omega_1,\omega_2\}) + 9H_1(2,\{\omega_1,\omega_2\}) & \text{if} \quad \omega=\omega_1\\
(1+r)^2 H_0(2,\{\omega_1,\omega_2\}) + 6H_1(2,\{\omega_1,\omega_2\}) & \text{if} \quad \omega=\omega_2\\
(1+r)^2 H_0(2,\{\omega_3,\omega_4\}) + 6H_1(2,\{\omega_3,\omega_4\}) & \text{if} \quad \omega=\omega_3\\
(1+r)^2 H_0(2,\{\omega_3,\omega_4\}) + 3H_1(2,\{\omega_3,\omega_4\}) & \text{if} \quad \omega=\omega_4
\end{cases}.
\end{aligned}
$$

We can also compute

$$
\begin{aligned}
\Delta B(1) &= 1+r-1 = r,\\
\Delta B(2) &= (1+r)^2 - (1+r) = r(r+1),
\end{aligned}
$$

# Model specifications

**Example 11**

$$\Delta S(1,\omega) = 8\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + 4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega) - 5 = \begin{cases} 3 & \text{if} \quad \omega = \omega_1, \omega_2 \\ -1 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases},$$

$$\begin{aligned} \Delta S(2,\omega) &= 9\mathbf{1}_{\{\omega_1\}}(\omega) + 6\mathbf{1}_{\{\omega_2,\omega_3\}}(\omega) + 3\mathbf{1}_{\{\omega_4\}}(\omega) \\ &\quad - \left(8\mathbf{1}_{\{\omega_1,\omega_2\}}(\omega) + 4\mathbf{1}_{\{\omega_3,\omega_4\}}(\omega)\right) \\ &= \begin{cases} 1 & \text{if} \quad \omega = \omega_1 \\ -2 & \text{if} \quad \omega = \omega_2 \\ 2 & \text{if} \quad \omega = \omega_3 \\ -1 & \text{if} \quad \omega = \omega_4 \end{cases}. \end{aligned}$$

Similarly we can compute

$$\begin{aligned} G(1,\omega) &= H_0(1)\,\Delta B(1) + H_1(1)\,\Delta S(1,\omega) \\ &= \begin{cases} rH_0(1) + 3H_1(1) & \text{if} \quad \omega = \omega_1, \omega_2 \\ rH_0(1) - H_1(1) & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}, \end{aligned}$$

# Model specifications

## Example 11

$$
\begin{aligned}
& G(2, \omega) \\
& = G(1, \omega) + H_0(2, \omega)\,\Delta B(2) + H_1(2, \omega)\,\Delta S(2, \omega) \\
& = \begin{cases}
rH_0(1) + 3H_1(1) + r(r+1)\,H_0(2, \{\omega_1, \omega_2\}) + H_1(2, \{\omega_1, \omega_2\}) & \text{if} \quad \omega = \omega_1 \\
rH_0(1) + 3H_1(1) + r(r+1)\,H_0(2, \{\omega_1, \omega_2\}) - 2H_1(2, \{\omega_1, \omega_2\}) & \text{if} \quad \omega = \omega_2 \\
rH_0(1) - H_1(1) + r(r+1)\,H_0(2, \{\omega_3, \omega_4\}) + 2H_1(2, \{\omega_3, \omega_4\}) & \text{if} \quad \omega = \omega_3 \\
rH_0(1) - H_1(1) + r(r+1)\,H_0(2, \{\omega_3, \omega_4\}) - 1H_1(2, \{\omega_3, \omega_4\}) & \text{if} \quad \omega = \omega_4
\end{cases}
\end{aligned}
$$

For $H$ to be self-financing we must have

$$
\begin{aligned}
V(1, \omega) &= \begin{cases}
(1+r)\,H_0(1) + 8H_1(1) & \text{if} \quad \omega = \omega_1, \omega_2 \\
(1+r)\,H_0(1) + 4H_1(1) & \text{if} \quad \omega = \omega_3, \omega_4
\end{cases} \\
&= \begin{cases}
(1+r)\,H_0(2, \{\omega_1, \omega_2\}) + 8H_1(2, \{\omega_1, \omega_2\}) & \text{if} \quad \omega = \omega_1, \omega_2 \\
(1+r)\,H_0(2, \{\omega_3, \omega_4\}) + 4H_1(2, \{\omega_3, \omega_4\}) & \text{if} \quad \omega = \omega_3, \omega_4
\end{cases}
\end{aligned}
$$

# Economic Considerations

**Definition 12**

An arbitrage opportunity is a trading strategy $H$ such that

1. $H$ is self-financing.
2. $V(0) = 0$.
3. $V(T) \geq 0$.
4. $\mathbb{E}[V(T)] > 0$.

Alternative equivalent formulations:

**Alternative 1**

$H$ is an arbitrage opportunity if

1. $H$ is self-financing.

b) $V^*(0) = 0$.

c) $V^*(T) \geq 0$.

d) $\mathbb{E}[V^*(T)] > 0$.

**Alternative 2**

$H$ is an arbitrage opportunity if

1. $H$ is self-financing.

b) $V^*(0) = 0$.

c') $G^*(T) \geq 0$.

d') $\mathbb{E}[G^*(T)] > 0$.

**Definition 13**

A risk neutral probability measure (martingale measure) is a probability measure $Q$ such that

1. $Q(\omega) > 0, \omega \in \Omega$.
2. $S_n^*$, $n = 1, ..., N$ are martingales under $Q$, that is,

$$\mathbb{E}_Q\left[S_n^*(t+s)\middle|\mathcal{F}_t\right] = S_n^*(t), \qquad t, s \geq 0, n = 1, ..., N. \tag{10}$$

**Remark 14**

- *It suffices to check* (10) *for* $s = 1$ *and* $t = 0, ..., T-1$, *that is,*

$$\mathbb{E}_Q\left[S_n^*(t+1)\middle|\mathcal{F}_t\right] = S_n^*(t).$$

- *If* $B(t) = (1+r)^t$, *then* (10) *is equivalent to*

$$\mathbb{E}_Q\left[S_n(t+1)\middle|\mathcal{F}_t\right] = (1+r)\, S_n(t). \tag{11}$$

**Example 15 (Continuation of Example 11)**

We will find $Q = (Q_1, Q_2, Q_3, Q_4)^T$ satisfying (11) for $t = 0, 1$.

- $t = 0$: We have $\mathcal{F}_0 = \{\emptyset, \Omega\}$ so the conditional expectation given $\mathcal{F}_0$ coincides with the ordinary expectation and the martingale measure condition is

$$S(0)(1+r) = \mathbb{E}_Q[S(1)|\mathcal{F}_0] = \mathbb{E}_Q[S(1)],$$

  that is

$$5(1+r) = 8(Q_1 + Q_2) + 4(Q_3 + Q_4).$$

- $t = 1$: We have $\mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$ so the conditional expectation given $\mathcal{F}_1$ is given by

$$\mathbb{E}_Q[S(2)|\mathcal{F}_1](\omega) = \mathbb{E}_Q[S(2)|\{\omega_1, \omega_2\}]\mathbf{1}_{\{\omega_1,\omega_2\}} + \mathbb{E}_Q[S(2)|\{\omega_3, \omega_4\}]\mathbf{1}_{\{\omega_3,\omega_4\}}.$$

**Example 15**

Using the rules for computing conditional expectation we get

$$\begin{aligned}\mathbb{E}_Q\left[S(2)\,|\,\{\omega_1,\omega_2\}\right] &= S(2,\omega_1)\frac{Q(\omega_1)}{Q(\{\omega_1,\omega_2\})} + S(2,\omega_2)\frac{Q(\omega_2)}{Q(\{\omega_1,\omega_2\})} \\ &= 9\frac{Q_1}{Q_1+Q_2} + 6\frac{Q_2}{Q_1+Q_2},\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}_Q\left[S(2)\,|\,\{\omega_3,\omega_4\}\right] &= S(2,\omega_3)\frac{Q(\omega_3)}{Q(\{\omega_3,\omega_4\})} + S(2,\omega_4)\frac{Q(\omega_4)}{Q(\{\omega_3,\omega_4\})} \\ &= 6\frac{Q_3}{Q_3+Q_4} + 3\frac{Q_4}{Q_3+Q_4}.\end{aligned}$$

The martingale measure condition is $(1+r)\,S(1) = \mathbb{E}_Q\left[S(2)\,|\,\mathcal{F}_1\right]$, and noting that $S(1,\omega) = 8\mathbf{1}_{\{\omega_1,\omega_2\}} + 4\mathbf{1}_{\{\omega_3,\omega_4\}}$ we get

$$\begin{aligned}9Q_1 + 6Q_2 &= 8(1+r)(Q_1+Q_2)\\ 6Q_3 + 3Q_4 &= 4(1+r)(Q_3+Q_4).\end{aligned}$$

**Example 15**

Combining the previous equations with the fact that $Q$ must be a probability we obtain the system

$$\begin{aligned} 8(Q_1 + Q_2) + 4(Q_3 + Q_4) &= 5(1 + r) \\ 9Q_1 + 6Q_2 &= 8(1 + r)(Q_1 + Q_2) \\ 6Q_3 + 3Q_4 &= 4(1 + r)(Q_3 + Q_4) \\ 1 &= Q_1 + Q_2 + Q_3 + Q_4, \end{aligned}$$

which has the solution

$$Q_1 = \frac{(1+5r)}{4}\frac{(2+8r)}{3}, \qquad Q_2 = \frac{(1+5r)}{4}\frac{(1-8r)}{3}$$
$$Q_3 = \frac{(3-5r)}{4}\frac{(1+4r)}{3}, \qquad Q_4 = \frac{(3-5r)}{4}\frac{(2-4r)}{3}.$$

Moreover,

$$Q > 0 \iff 0 \le r < 1/8.$$

**Remark 16**

*There is an alternative way for finding the martingale measure $Q$. This consists in decomposing the multiperiod market in a series of single period markets. One then find a risk neutral measure for each of these single period markets. The martingale measure for the multiple period market is contructed by "pasting together" these risk neutral measures. I showed this procedure on the blackboard.*

**Proposition 17**

*If $Q$ is a martingale measure and $H$ is a self-financing trading strategy, then $V^* = \{V^*(t)\}_{t=0,\ldots,T}$ is a martingale under $Q$.*

**Proof.**

Blackboard. □

**Theorem 18 (First Fundamental Theorem of Asset Pricing)**

*There do not exist arbitrage opportunities if and only if there exist a martingale measure.*

**Proof.**

Blackboard □

# Economic considerations

- All the concepts we saw for single period markets also extend to multiple period markets.

**Definition 19**

A linear pricing measure is a non-negative vector $\pi = (\pi_1, ..., \pi_K)^T$ such that for every self-financing trading strategy $H$ you have

$$V^*(0) = \sum_{k=1}^{K} \pi_k V_T^*(\omega_k).$$

- Clearly, if $Q$ is martingale measure then it is also a linear pricing measure.
- One can see that any strictly positive linear pricing measure $\pi$ must be a martingale measure.

**Theorem 20**

*A vector $\pi$ is a linear pricing measure if an only if $\pi$ is a probability measure on $\Omega$ under which all the discounted price processes are martingales.*

# Economic considerations

**Definition 21**

$H$ is a dominant self-financing trading strategy if there exists another self-financing trading strategy $\widehat{H}$ such that $V(0) = \widehat{V}(0)$ and $V(T,\omega) > \widehat{V}(T,\omega)$ for all $\omega \in \Omega$.

**Theorem 22**

*There exists a linear pricing measure if and only if there are no dominant trading strategies.*

**Definition 23**

We say the the law of one price holds for a multiperiod model if there do not exist two self-financing trading strategies, say $\widehat{H}$ and $\widetilde{H}$, such that $\widehat{V}(T,\omega) = \widetilde{V}(T,\omega)$ for all $\omega \in \Omega$ but $\widehat{V}(0) \neq \widetilde{V}(0)$.

- The existence of a linear pricing measure implies that the law of one price hold.

# Economic considerations

- Denote

$$
\begin{aligned}
W &= \left\{X \in \mathbb{R}^K : X = G^*, \text{ for some self-financing trading strategy } H\right\}, \\
W^\perp &= \left\{Y \in \mathbb{R}^K : X^T Y = 0, \text{ for all } X \in W\right\}, \\
A &= \left\{X \in \mathbb{R}^K : X \geq 0, X \neq 0\right\}, \\
P &= \left\{X \in \mathbb{R}^K : X_1 + \ldots + X_K = 1, X \geq 0\right\}, \\
P^+ &= \{X \in P : X_1 > 0, \ldots, X_K > 0\}.
\end{aligned}
$$

- As with single period markets:
  - We will denote by $M$ the set of all martingale measures.
  - The set of all linear pricing measures is $P \cap W^\perp$.
  - $M = P^+ \cap W^\perp$.
  - $W \cap A = \emptyset$ if and only if $M \neq \emptyset$.
  - $M$ is convex set whose closure is $P \cap W^\perp$, the set of all linear pricing measures.

# Risk Neutral Pricing

**Definition 24**

A contingent claim is a random variable $X$ representing the payoff at time $T$ of a financial contract which depends on the values of the risky assets in the market.

**Example 25**

Consider the market with $T = 2$, $K = 4$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}, \qquad S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases}.$$

- $X = (S(2) - 5)^+$. European call option with strike 5.

$$X = (\max(0, 9-5), \max(0, 6-5), \max(0, 6-5), \max(0, 3-5))^T$$
$$= (4, 1, 1, 0)^T.$$

**Example 25**

- $Y = \left(\frac{1}{3}\sum_{i=0}^{2} S(t) - 5\right)^{+}$. Asian call option with strike 5.

$$Y_1 = \left(\frac{1}{3}\sum_{i=0}^{2} S(t,\omega_1) - 5\right)^{+} = \max\left(0, \frac{1}{3}(5+8+9) - 5\right) = 7/3,$$

$$Y_2 = \left(\frac{1}{3}\sum_{i=0}^{2} S(t,\omega_2) - 5\right)^{+} = \max\left(0, \frac{1}{3}(5+8+6) - 5\right) = 4/3,$$

$$Y_3 = \left(\frac{1}{3}\sum_{i=0}^{2} S(t,\omega_3) - 5\right)^{+} = \max\left(0, \frac{1}{3}(5+4+6) - 5\right) = 0,$$

$$Y_4 = \left(\frac{1}{3}\sum_{i=0}^{2} S(t,\omega_3) - 5\right)^{+} = \max\left(0, \frac{1}{3}(5+4+3) - 5\right) = 0,$$

which yields $Y = (7/3, 4/3, 0, 0)^T$.

# Risk neutral pricing

**Assumption 26**

*The financial market model is arbitrage free, that is, there exist a martingale measure $Q$.*

**Definition 27**

A contingent claim $X$ is attainable (or marketable) if there exists $H$ a self-financing trading strategy sucht that $V(T,\omega) = X(\omega), \omega \in \Omega$. Such strategy is said to replicate or generate or hedge $X$.

**Theorem 28 (Risk Neutral Pricing)**

*The time $t$ value of an attainable contingent claim $X$, denoted by $P_X(t)$ ,is equal to $V(t)$, the time $t$ value of a portfolio generating $X$. Moreover,*

$$V(t) = \mathbb{E}_Q\left[\frac{B(t)}{B(T)}X \middle| \mathcal{F}_t\right], \qquad , t = 0, ..., T,$$

*for all martingale measures $Q$.*

**Proof.**

Blackboard. □

- In order to sell a contingent claim $X$ the seller must find the trading strategy that replicates/hedges $X$.
- We will see three methods for finding a hedging strategy.

**First method**

- We must know the value process $V = \{V(t)\}_{t=0,\ldots,T}$.
- We solve

$$V(t) = H_0(t) + \sum_{n=1}^{N} H_n(t) S_n(t), \qquad t = 1, \ldots, T,$$

taking into account that $H$ must be predictable.

## Second method

- All we know is $X$.
- In this method, we work backwards in time and find $V(t)$ and $H(t)$ simultaneously.
- Since $V(T) = X$, we first find $H(T)$ by taking into account that $H$ is predictable and solving

$$X = H_0(T)\,B(T) + \sum_{n=1}^{N} H_n(T)\,S_n(T).$$

- Using that $H$ is must be self-financing, we find $V(T-1)$ by computing

$$V(T-1) = H_0(T)\,B(T-1) + \sum_{n=1}^{N} H_n(T)\,S_n(T-1).$$

- Next, taking into account that $H$ is predictable, we find $H(T-1)$ by solving

$$V(T-1) = H_0(T-1)\,B(T-1) + \sum_{n=1}^{N} H_n(T-1)\,S_n(T-1).$$

- We repeat this procedure until computing $V(0)$.

# Risk neutral pricing

## Third method

- It relies on the fact that the self-financing condition

$$V^*(0) + G^*(t) = V^*(t),$$

  is equivalent to

$$V^*(t-1) + \sum_{n=1}^{N} H_n(t)\,\Delta S_n^*(t) = V^*(t).$$

- We can use this system of equations, together with the predictability condition on $H(t) = (H_1(t), ..., H_N(t))^T$, to find $V^*(t-1)$ and $H(t)$.
- Then, we can find

$$H_0(t) = V^*(t) - \sum_{n=1}^{N} H_n(t)\,S_n^*(t),$$

$$V(t-1) = B(t-1)\,V^*(t-1).$$

- We begin with $V^*(T) = X/B(T)$ and work backwards in time.

## Example 29 (Continuation Example 25)

Suppose $r = 0$. We know that $Q = (1/6, 1/12, 1/4, 1/2)^T$ is the unique martingale measure in this market.

- European call option $X = (4, 1, 1, 0)^T$. We have, by Theorem 28 and taking into account that $r = 0$, that

$$V(0) = \mathbb{E}_Q\left[\left.\frac{B(0)}{B(2)}X\right|\mathcal{F}_0\right] = \mathbb{E}_Q[X],$$

$$V(1) = \mathbb{E}_Q\left[\left.\frac{B(1)}{B(2)}X\right|\mathcal{F}_1\right] = \mathbb{E}_Q[X|\mathcal{F}_1],$$

$$V(2) = \mathbb{E}_Q\left[\left.\frac{B(2)}{B(2)}X\right|\mathcal{F}_2\right] = X.$$

Hence, computing

$$\mathbb{E}_Q[X] = 4\frac{1}{6} + 1\frac{1}{12} + 1\frac{1}{4} + 0\frac{1}{2} = 1,$$

**Example 29**

and

$$\mathbb{E}_Q\left[X|\left\{\omega_1,\omega_2\right\}\right]=\frac{\mathbb{E}_Q\left[X\mathbf{1}_{\{\omega_1,\omega_2\}}\right]}{Q\left(\left\{\omega_1,\omega_2\right\}\right)}=\frac{4\frac{1}{6}+1\frac{1}{12}+0\frac{1}{4}+0\frac{1}{2}}{\frac{1}{6}+\frac{1}{12}}=3,$$

$$\mathbb{E}_Q\left[X|\left\{\omega_3,\omega_4\right\}\right]=\frac{\mathbb{E}_Q\left[X\mathbf{1}_{\{\omega_3,\omega_4\}}\right]}{Q\left(\left\{\omega_3,\omega_4\right\}\right)}=\frac{0\frac{1}{6}+0\frac{1}{12}+1\frac{1}{4}+0\frac{1}{2}}{\frac{1}{4}+\frac{1}{2}}=\frac{1}{3},$$

$$\mathbb{E}_Q\left[X|\mathcal{F}_1\right]=3\mathbf{1}_{\{\omega_1,\omega_2\}}+\frac{1}{3}\mathbf{1}_{\{\omega_3,\omega_4\}},$$

note that $\mathcal{F}_1=\mathfrak{a}\left\{\left\{\omega_1,\omega_2\right\},\left\{\omega_3,\omega_4\right\}\right\}$, we obtain the values of the value process $V$.

We can compute $H$ using the first method.

For $t=2$ we have $V\left(2\right)=H_0\left(2\right)B\left(2\right)+H_1\left(2\right)S\left(2\right)$, which gives

$$V\left(2,\omega_1\right)=4=H_0\left(2,\omega_1\right)1+H_1\left(2,\omega_1\right)9,$$
$$V\left(2,\omega_2\right)=1=H_0\left(2,\omega_2\right)1+H_1\left(2,\omega_2\right)6,$$
$$V\left(2,\omega_3\right)=1=H_0\left(2,\omega_3\right)1+H_1\left(2,\omega_3\right)6,$$
$$V\left(2,\omega_4\right)=0=H_0\left(2,\omega_4\right)1+H_1\left(2,\omega_4\right)3,$$

**Example 29**

and the predictability constraint yields the following additional equations

$$H_0(2,\omega_1) = H_0(2,\omega_2), \qquad H_0(2,\omega_3) = H_0(2,\omega_4),$$
$$H_1(2,\omega_1) = H_1(2,\omega_2), \qquad H_1(2,\omega_3) = H_1(2,\omega_4).$$

Solving these equations we get

$$H_0(2,\omega) = \begin{cases} -5 & \text{if} \quad \omega = \omega_1,\omega_2 \\ -1 & \text{if} \quad \omega = \omega_3,\omega_4 \end{cases}, \qquad H_1(2,\omega) = \begin{cases} 1 & \text{if} \quad \omega = \omega_1,\omega_2 \\ 1/3 & \text{if} \quad \omega = \omega_3,\omega_4 \end{cases}.$$

For $t = 1$ we can write $V(1) = H_0(1)B(1) + H_1(1)S(1)$, which gives

$$\begin{array}{ll} V(1,\omega) = 3 = H_0(1,\omega)1 + H_1(1,\omega)8 & \text{if} \quad \omega = \omega_1,\omega_2 \\ V(1,\omega) = \frac{1}{3} = H_0(1,\omega)1 + H_1(1,\omega)4 & \text{if} \quad \omega = \omega_3,\omega_4 \end{array},$$

and the predicability constraint yields the following additional equations

$$H_0(1,\omega_1) = H_0(1,\omega_2) = H_0(1,\omega_3) = H_0(1,\omega_4),$$
$$H_1(1,\omega_1) = H_1(1,\omega_2) = H_1(1,\omega_3) = H_1(1,\omega_4).$$

Solving these equations we get $H_0(1,\omega) = -\frac{7}{3}$ and $H_1(1,\omega) = \frac{2}{3}$, $\omega \in \Omega$.

## Example 29

- Asian call option $Y = (7/3, 4/3, 0, 0)^T$. We will use the third method to simultaneously find $V$ and $H$. Recall that $\Delta S^*(2) = (1, -2, 2, -1)^T$ and $\Delta S^*(1) = (3, 3, -1, -1)^T$.
  For $t = 2$ we know that $\frac{Y}{B(2)} = V^*(2) = V^*(1) + H_1(2)\Delta S^*(2)$ wich gives

$$V^*(2, \omega_1) = \frac{7}{3} = V^*(1, \omega_1) + H_1(2, \omega_1)\,1,$$
$$V^*(2, \omega_2) = \frac{4}{3} = V^*(1, \omega_2) + H_1(2, \omega_2) \times (-2),$$
$$V^*(2, \omega_3) = 0 = V^*(1, \omega_3) + H_1(2, \omega_3)\,2,$$
$$V^*(2, \omega_4) = 0 = V^*(1, \omega_4) + H_1(2, \omega_4) \times (-1),$$

  and the predictability constraint for $H$ together with the adaptability of $V$ yield the additional equations

$$H_1(2, \omega_1) = H_1(2, \omega_2), \qquad H_1(2, \omega_3) = H_1(2, \omega_4),$$
$$V^*(1, \omega_1) = V^*(1, \omega_2), \qquad V^*(1, \omega_3) = V^*(1, \omega_4).$$

**Example 29**

Solving these equations we get

$$H_1(2,\omega) = \begin{cases} \frac{1}{3} & \text{if} \quad \omega = \omega_1, \omega_2 \\ 0 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}, \qquad V^*(1,\omega) = \begin{cases} 2 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 0 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}.$$

Note that

$$V(1,\omega) = V^*(1,\omega)\, B(1,\omega) = \begin{cases} 2 \times 1 = 2 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 0 \times 1 = 0 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}.$$

For $t = 1$ we know that $V^*(1) = V^*(0) + H_1(1)\,\Delta S^*(1)$ wich gives

$$\begin{aligned} V^*(1,\omega) &= 2 = V^*(0,\omega) + H_1(1,\omega)\, 3 && \text{if} \quad \omega = \omega_1, \omega_2 \\ V^*(1,\omega) &= 0 = V^*(0,\omega) + H_1(1,\omega) \times (-1) && \text{if} \quad \omega = \omega_3, \omega_4, \end{aligned}$$

and the predictability constraint for $H$ together with the adaptability of $V$ yield the additional equations

$$H_1(1,\omega_1) = H_1(1,\omega_2) = H_1(1,\omega_3) = H_1(1,\omega_4),$$
$$V^*(0,\omega_1) = V^*(0,\omega_2) = V^*(0,\omega_3) = V^*(0,\omega_4).$$

**Example 29**

Solving these equations we obtain

$$V^*(0,\omega) = \frac{1}{2}, \qquad H_1(1,\omega) = \frac{1}{2}, \qquad \omega \in \Omega.$$

Note that $V(0) = B(0)\,V^*(1) = \frac{1}{2}$.

Finally, to compute $H_0$, we use

$$H_0(1) = V^*(0) - H_1(1)\,S(0) = \frac{1}{2} - \frac{1}{2}5 = -2,$$

$$H_0(2) = V^*(1) - H_1(2)\,S(1) = \begin{cases} 2 - \frac{1}{3}\times 8 = -\frac{2}{3} & \text{if} \quad \omega = \omega_1, \omega_2 \\ 0 - 0\times 4 = 0 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}.$$

Note that $V(0) = \frac{1}{2}$ is the same value using the risk neutral approach

$$V(0) = \mathbb{E}_Q\left[\frac{B(0)}{B(2)}X \,\middle|\, \mathcal{F}_0\right] = \mathbb{E}_Q[X].$$

# Complete and Incomplete Markets

# Complete and incomplete markets

**Definition 30**

A market is complete if every contingent claim $X$ is attainable. Otherwise, it is called incomplete.

**Proposition 31**

*A multiperiod market is complete if and only if every underlying single period market is complete.*

**Proof.**

Blackboard. □

**Remark 32**

- *The backward procedures explained in the last section work if and only every underlying single period market is complete.*
- *The criterion given in Proposition 31, in general, is not a practical characterization of market completeness.*

**Theorem 33 (Second Fundamental Theorem of Asset Pricing)**

*Suppose that* $M \neq \emptyset$. *A multiperiod market is complete if and only if* $M = \{Q\}$.

**Proof.**

Blackboard. □

**Proposition 34**

*Suppose that* $M \neq \emptyset$. *A contingent claim* $X$ *is attainable if and only if* $\mathbb{E}_Q[X/B(T)]$ *takes the same value for every* $Q \in M$.

**Proof.**

Blackboard. □

**Example 35**

Consider the market with $K = 5$, $T = 2$, $r = 0$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2, \omega_3 \\ 4 & \text{if} \quad \omega = \omega_4, \omega_5 \end{cases}, \qquad S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 7 & \text{if} \quad \omega = \omega_2 \\ 6 & \text{if} \quad \omega = \omega_3, \omega_4 \\ 5 & \text{if} \quad \omega = \omega_5 \end{cases}.$$

One can check (exercise) that

$$M = \left\{ Q_\lambda = \left( \frac{\lambda}{4}, \frac{(2-3\lambda)}{4}, \frac{(2\lambda-1)}{4}, \frac{1}{4}, \frac{1}{2} \right)^T, \frac{1}{2} < \lambda < \frac{2}{3} \right\}.$$

A contingent claim $X = (X_1, X_2, X_3, X_4, X_5)^T$ is attainable if and only if

$$\mathbb{E}_Q\left[\frac{X}{B(2)}\right] = \mathbb{E}_Q[X] = X_1 \frac{\lambda}{4} + X_2 \frac{(2-3\lambda)}{4} + X_3 \frac{(2\lambda-1)}{4} + X_4 \frac{1}{4} + X_5 \frac{1}{2}$$
$$= \frac{\lambda}{4}(X_1 - 3X_2 + 2X_3) + \frac{1}{4}(2X_2 - X_3 + X_4 + 2X_5),$$

does not depend on $\lambda$, i.e., if and only if $X_1 - 3X_2 + 2X_3 = 0$.

# Optimal Portfolio Problem

- Let $U$ be an utility function as in section 5.1.
- We are interested in the following optimization problem:

$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(V\left(T\right)\right)\right] \\ \text{subject to} & V\left(0\right)=v, \\ & H\in\mathcal{H}, \end{array}\right\} \qquad (12)$$

  where $v\in\mathbb{R}$ and $\mathcal{H}:=\{\text{set of all self-financing trading strategies}\}$.
- Recall that $V\left(T\right)=V^{*}\left(T\right)B\left(T\right)$, $V^{*}\left(T\right)=V^{*}\left(0\right)+G^{*}\left(T\right)$. Therefore, (12) is equivalent to

$$\left.\begin{array}{cc} \max & \mathbb{E}\left[U\left(B\left(T\right)\left\{v+G^{*}\left(T\right)\right\}\right)\right] \\ \text{subject to} & H=\left(H_1,...,H_N\right)^{T}\in\mathcal{H}_P, \end{array}\right\} \qquad (13)$$

  where $v\in\mathbb{R}$ and
  $\mathcal{H}_P:=\left\{\text{set of all predictable processes taking values in } \mathbb{R}^N\right\}$.
- If $(\widehat{H}_1,...,\widehat{H}_N)^T$ is a solution of (13), then one can find $\widehat{H}_0$ such that $\widehat{H}=(\widehat{H}_0,\widehat{H}_1,...,\widehat{H}_N)^T$ is self-financing and $V\left(0\right)=v$, giving a solution to (12).

**Proposition 36**

*If $H$ is a solution of (12) and $V$ is its associated porfolio value process then*

$$Q(\omega) = \frac{B(T,\omega)\,U'(V(T,\omega),\omega)}{\mathbb{E}[B(T)\,U'(V(T))]} P(\omega), \qquad \omega \in \Omega,$$

*is a martingale measure.*

**Proof.**

Blackboard. □

# Optimal portfolio problem

- There are several methods to solve the optimal portfolio problem:
  - Direct approach (classical optimization problem taking into account predictability)
  - Dynamic programming.
  - Martingale method.
- We will only consider the martingale method in these lectures.
- This method is analogous to the risk neutral computational approach in single period financial markets.
- We will assume that:
  - The market is arbitrage free and complete: $M = \{Q\}$.
  - $U$ does not depend on $\omega$.
- The martingale method can be split in 3 steps.

**Step 1**

- Identify the set $W_v$ of attainable wealths:

$$W_v = \left\{ W \in \mathbb{R}^K : W = V(T) \text{ for some } H \in \mathcal{H} \text{ with } V(0) = v \right\}.$$

- If the model is complete

$$W_v = \left\{ W \in \mathbb{R}^K : \mathbb{E}_Q [W/B(T)] = v \right\}.$$

# Optimal portfolio problem

## Step 2

- We need to solve the problem

$$\left.\begin{array}{ll} \max & \mathbb{E}\left[U\left(W\right)\right] \\ \text{subject to} & W \in W_v, \end{array}\right\} \tag{14}$$

- To solve (14) we will use the method of Lagrange multipliers.
- Consider the Lagrange function

$$\begin{aligned} \mathcal{L}\left(W;\lambda\right) &= \mathbb{E}\left[U\left(W\right)\right] - \lambda\left(\mathbb{E}_Q\left[W/B\left(T\right)\right] - v\right) \\ &= \mathbb{E}\left[U\left(W\right)\right] - \lambda\left(\mathbb{E}\left[LW/B\left(T\right)\right] - v\right) \\ &= \mathbb{E}\left[U\left(W\right) - \lambda L\left(\frac{W}{B\left(T\right)} - v\right)\right]. \end{aligned}$$

- The first optimality condition gives

$$0 = \frac{\partial \mathcal{L}}{\partial \lambda}\left(W;\lambda\right) = \mathbb{E}_Q\left[W/B\left(T\right)\right] - v$$

$$0 = \frac{\partial \mathcal{L}}{\partial W_k}\left(W;\lambda\right) = P\left(\omega_k\right)\left\{U'\left(W\left(\omega_k\right)\right) - \lambda\frac{L\left(\omega_k\right)}{B\left(T,\omega_k\right)}\right\} \qquad k = 1, ..., K.$$

## Step 2

- Then the optimum $(\widehat{\lambda}, \widehat{W})$ satisfies

$$\mathbb{E}_Q\left[\widehat{W}/B(T)\right] = v, \qquad U'\left(\widehat{W}\right) = \widehat{\lambda}\frac{L}{B(T)}$$

- To solve these equations, we consider $I(y) := (U')^{-1}(y)$ and compute $\widehat{W} = I\left(\widehat{\lambda}\frac{L}{B(T)}\right)$, then $\widehat{\lambda}$ is chosen so that

$$\mathbb{E}_Q\left[I\left(\widehat{\lambda}LB^{-1}(T)\right)B^{-1}(T)\right] = v,$$

holds.

## Step 3

- Given the optimal wealth $\widehat{W}$, find a self-financing trading strategy $\widehat{H}$ that generates $\widehat{W}$.
- We use the second method for findind a replicating strategy.

**Example 37**

Consider the market with $T = 2$, $K = 4$, $S(0) = 5$,

$$S(1,\omega) = \begin{cases} 8 & \text{if} \quad \omega = \omega_1, \omega_2 \\ 4 & \text{if} \quad \omega = \omega_3, \omega_4 \end{cases}, \qquad S(2,\omega) = \begin{cases} 9 & \text{if} \quad \omega = \omega_1 \\ 6 & \text{if} \quad \omega = \omega_2, \omega_3 \\ 3 & \text{if} \quad \omega = \omega_4 \end{cases},$$

$0 \leq r < 1/8$ and $P = (1/4, 1/4, 1/4, 1/4)^T$.

We know that the unique martingale measure is

$$Q = \left( \frac{(1+5r)(2+8r)}{12}, \frac{(1+5r)(1-8r)}{12}, \right.$$
$$\left. \frac{(3-5r)(1+4r)}{12}, \frac{(3-5r)(2-4r)}{12} \right)^T$$

We want to solve the optimal portfolio problem with $U(u) = \log(u)$. Hence,

$$U'(u) = \frac{1}{u} \Longrightarrow I(y) = \left(U'\right)^{-1}(y) = \frac{1}{y}.$$

**Example 37**

We compute

$$L = \frac{Q}{P} = \left( \frac{(1+5r)(2+8r)}{3}, \frac{(1+5r)(1-8r)}{3}, \right.$$
$$\left. \frac{(3-5r)(1+4r)}{3}, \frac{(3-5r)(2-4r)}{3} \right)^T.$$

Next, we find the optimal wealth

$$\widehat{W} = I\left(\widehat{\lambda}\frac{L}{B(2)}\right) = \frac{B(2)}{\widehat{\lambda}L}$$

and the optimal multiplier $\hat{\lambda}$

$$\mathbb{E}_Q\left[\frac{\widehat{W}}{B(2)}\right] = v \iff \mathbb{E}_Q\left[\frac{B(2)}{\widehat{\lambda}LB(2)}\right] = v \iff \widehat{\lambda} = \frac{\mathbb{E}_Q\left[L^{-1}\right]}{v} = v^{-1},$$

where we have used that

$$\mathbb{E}_Q\left[L^{-1}\right] = \mathbb{E}_P\left[LL^{-1}\right] = 1.$$

**Example 37**

Hence,

$$\widehat{\lambda} = v^{-1}, \qquad \widehat{W} = vB(2)\,L^{-1},$$

and the optimal expected utility is given by

$$\mathbb{E}\left[U\left(\widehat{W}\right)\right] = \mathbb{E}\left[\log\left(\widehat{W}\right)\right] = \log(v) + \mathbb{E}\left[\log\left(B(2)\,L^{-1}\right)\right].$$

Since $B(2) = (1+r)^2$ is deterministic we have

$$\begin{aligned}
\mathbb{E}\left[U\left(\widehat{W}\right)\right] &= \log(v) + \log(B(2)) + \mathbb{E}\left[\log\left(L^{-1}\right)\right] \\
&= \log\left(v(1+r)^2\right) - \mathbb{E}[\log(L)] \\
&= \log\left(v(1+r)^2\right) - \frac{1}{4}\sum_{i=1}^{4}\log(L_i).
\end{aligned}$$

The last step is to compute the optimal strategy $\hat{H}$ that replicates the optimal wealth $\hat{W}$.

**Example 37**

- Recall that

$$\widehat{W} = vB(2)\,L^{-1} = \left( \frac{3v(1+r)^2}{(1+5r)(2+8r)}, \frac{3v(1+r)^2}{(1+5r)(1-8r)}, \frac{3v(1+r)^2}{(3-5r)(1+4r)}, \frac{3v(1+r)^2}{(3-5r)(2-4r)} \right)^T$$

- For $t = 2$, using that $\widehat{H}$ must be predictable, i.e., $\widehat{H}(2) \in \mathcal{F}_1$-measurable, we have that

$$\begin{aligned}
\frac{3v(1+r)^2}{(1+5r)(2+8r)} &= \widehat{W}_1 = \widehat{H}_0(2,\omega_1)(1+r)^2 + \widehat{H}_1(2,\omega_1)\,S(2,\omega_1) \\
&= (1+r)^2\,\widehat{H}_0(2,\omega_1) + 9\widehat{H}_1(2,\omega_1), \\
\frac{3v(1+r)^2}{(1+5r)(1-8r)} &= \widehat{W}_2 = \widehat{H}_0(2,\omega_2)(1+r)^2 + \widehat{H}_1(2,\omega_2)\,S(2,\omega_2) \\
&= (1+r)^2\,\widehat{H}_0(2,\omega_2)(1+r)^2 + 6\widehat{H}_1(2,\omega_2), \\
\widehat{H}_0(2,\omega_1) &= \widehat{H}_0(2,\omega_2), \\
\widehat{H}_1(2,\omega_1) &= \widehat{H}_1(2,\omega_2).
\end{aligned}$$

**Example 37**

Hence, for $\omega \in \{\omega_1, \omega_2\}$ we get

$$\widehat{H}_0(2,\omega) = \frac{12(1+10r)v}{(1+5r)(1-8r)(2+8r)},$$
$$\widehat{H}_1(2,\omega) = -\frac{(1+r)^2(1+16r)v}{(1+5r)(1-8r)(2+8r)}.$$

Moreover, since $\widehat{H}$ is self-financing, for $\omega \in \{\omega_1, \omega_2\}$

$$\begin{aligned}\widehat{V}(1,\omega) &= \widehat{H}_0(2,\omega)B(1) + \widehat{H}_1(2,\omega)S(1,\omega) \\ &= \frac{12(1+10r)v}{(1+5r)(1-8r)(2+8r)}(1+r) \\ &- \frac{(1+r)^2(1+16r)v}{(1+5r)(1-8r)(2+8r)}8 \\ &= \frac{2v(1+r)}{1+5r}.\end{aligned}$$

**Example 37**

We also have

$$\frac{3v\,(1+r)^2}{(3-5r)\,(1+4r)} = \widehat{W_3} = \widehat{H}_0\,(2,\omega_3)\,(1+r)^2 + \widehat{H}_1\,(2,\omega_3)\,S\,(2,\omega_3)$$
$$= (1+r)^2\,\widehat{H}_0\,(2,\omega_3) + 6\widehat{H}_1\,(2,\omega_3)\,,$$
$$\frac{3v\,(1+r)^2}{(3-5r)\,(2-4r)} = \widehat{W_4} = \widehat{H}_0\,(2,\omega_4)\,(1+r)^2 + \widehat{H}_1\,(2,\omega_4)\,S\,(2,\omega_4)$$
$$= (1+r)^2\,\widehat{H}_0\,(2,\omega_4) + 3\widehat{H}_1\,(2,\omega_4)\,,$$
$$\widehat{H}_0\,(2,\omega_3) = \widehat{H}_0\,(2,\omega_3)\,,$$
$$\widehat{H}_1\,(2,\omega_4) = \widehat{H}_1\,(2,\omega_4)\,.$$

Hence, for $\omega \in \{\omega_3, \omega_4\}$ we get

$$\widehat{H}_0\,(2,\omega) = \frac{36rv}{(3-5r)\,(2-4r)\,(1+4r)},$$
$$\widehat{H}_1\,(2,\omega) = \frac{(1+r)^2\,(1-8r)\,v}{2\,(3-5r)\,(2-4r)\,(1+4r)}.$$

**Example 37**

Moreover, since $\widehat{H}$ is self-financing, for $\omega \in \{\omega_3, \omega_4\}$

$$\begin{aligned}\widehat{V}(1,\omega) &= \widehat{H}_0(2,\omega)\,B(1) + \widehat{H}_1(2,\omega)\,S(1,\omega)\\ &= \frac{36rv}{(3-5r)(2-4r)(1+4r)}(1+r)\\ &+ \frac{(1+r)^2(1-8r)\,v}{2(3-5r)(2-4r)(1+4r)}8\\ &= \frac{2v(1+r)}{3-5r}.\end{aligned}$$

**Example 37**

- For $t = 1$, using that $\widehat{H}$ must be predictable, i.e., $\widehat{H}(1) \in \mathcal{F}_0$-measurable, we have that

$$\begin{aligned}
\frac{2v(1+r)}{(1+5r)} &= \widehat{V}(1,\omega_1) = \widehat{H}_0(1,\omega_1)(1+r) + \widehat{H}_1(1,\omega_1) S(1,\omega_3) \\
&= (1+r)\widehat{H}_0(2,\omega_1) + 8\widehat{H}_1(2,\omega_1), \\
\frac{2v(1+r)}{3-5r} &= \widehat{V}(1,\omega_3) = \widehat{H}_0(2,\omega_3)(1+r) + \widehat{H}_1(2,\omega_3) S(2,\omega_3) \\
&= (1+r)^2 \widehat{H}_0(2,\omega_3) + 4\widehat{H}_1(2,\omega_3), \\
\widehat{H}_0(1,\omega_1) &= \widehat{H}_0(1,\omega_2) = \widehat{H}_0(1,\omega_3) = \widehat{H}_0(1,\omega_4), \\
\widehat{H}_1(1,\omega_1) &= \widehat{H}_1(1,\omega_2) = \widehat{H}_1(1,\omega_3) = \widehat{H}_1(1,\omega_4).
\end{aligned}$$

Hence, for $\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\}$

$$\widehat{H}_0(1,\omega) = \frac{(30r-2)v}{(1+5r)(3-5r)}, \qquad \widehat{H}_1(1,\omega) = \frac{(1+r)(1-5r)v}{(1+5r)(3-5r)}.$$

**Example 37**

To double check

$$\begin{aligned}
\widehat{V}(0) &= \widehat{H}_0(1)\,B(0) + \widehat{H}_1(1)\,S(0)\\
&= \frac{(30r-2)\,v}{(1+5r)(3-5r)} + \frac{(1+r)(1-5r)\,v}{(1+5r)(3-5r)}5\\
&= v\frac{30r-2+(1+r)(1-5r)\,5}{(1+5r)(3-5r)}\\
&= v\frac{30r-2+5-25r+5r-25r^2}{3-5r+15r-25r^2}\\
&= v\frac{3+10r-25r^2}{3+10r-25r^2} = v.
\end{aligned}$$