List 3

$\boxed{4}$ $K = 3$, $N = 2$, $B(0) = 1$, $S_1(0) = 5$, $S_2(0) = 10$

$$S(1, \Omega) = \begin{pmatrix} 10/9 & 60/9 & 40/3 \\ 10/9 & 60/9 & 80/9 \\ 10/9 & 40/9 & 80/9 \end{pmatrix}$$

Show that $\nexists$ DTS but $\exists$ A.O.

Try to find a RNPM Q. If we cannot find it because some of the components are 0, then we will ~~have~~ found a LPM (they satisfy the same equation) and we will have proved that $\nexists$ DTS. but $\exists$ A.O.

Let $Q \in \mathbb{R}^3$. The equation for Q to be a RNPM is

$$Q^T S^*(1, \Omega) = (1, S_1^*(0), S_2^*(0))$$

$$\iff \quad E_Q[S^*_n(1)] = S^*_n(0) \quad n = 1,2$$

$$Q_1 + Q_2 + Q_3 = 1$$

Note that $S^*_1(0) = S_1(0) = 5$, $S^*_2(0) = S_2(0) = 10$

and
$$S^*(1,\Omega) = \frac{1}{\frac{10}{9}} S(1,\Omega) = \begin{pmatrix} 1 & 6 & 12 \\ 1 & 6 & 8 \\ 1 & 4 & 8 \end{pmatrix}$$

Hence,
$$(1, 5, 10) = (Q_1, Q_2, Q_3) \begin{pmatrix} 1 & 6 & 12 \\ 1 & 6 & 8 \\ 1 & 4 & 8 \end{pmatrix} \iff \left.\begin{array}{l} Q_1 + Q_2 + Q_3 = 1 \\ 6Q_1 + 6Q_2 + 4Q_3 = 5 \\ 12Q_1 + 8Q_2 + 8Q_3 = 10 \end{array}\right\}$$

$$\iff \left.\begin{array}{l} Q_3 = 1 - Q_1 - Q_2 \\ 2Q_1 + 2Q_2 = 1 \\ 4Q_1 = 2 \end{array}\right\} \implies \begin{array}{l} Q_1 = 1/2 \\ Q_2 = 0 \\ Q_3 = 1/2 \end{array} \implies \nexists \text{ RNPM } (Q_2 \not> 0)$$

but $\pi = (1/2, 0, 1/2)^T$ is a linear pricing measure.

$$\nexists \text{ RNPM} \iff \exists \text{ A.O.} \quad \Big| \quad \exists \text{ LPM} \iff \nexists \text{ DTS}$$

To find $H = (H_0, H_1, H_2)^T$ such that is an A.O., it must satisfy

- $0 = V^*(0) = V(0) = H_0 + 5H_1 + 10H_2 \Rightarrow H_0 = -5H_1 - 10H_2$

$$\begin{cases} \cdot\ V^*(1) = S^*(1,\omega)\, H = \begin{pmatrix} H_0 + 6H_1 + 12H_2 \\ H_0 + 6H_1 + 8H_2 \\ H_0 + 4H_1 + 8H_2 \end{pmatrix} \gneqq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \cdot\ E[V^*(1)] > 0 \end{cases}$$

$$H_0 + 6H_1 + 12H_2 \geq 0$$
$$H_0 + 6H_1 + 8H_2 \geq 0$$
$$H_0 + 4H_1 + 8H_2 \geq 2$$

+ At least one of the inequalities must be strict

(Notice that we do not know P)

$$\Updownarrow \quad H_0 = -5H_1 - 10H_2$$

(1) $H_1 + 2H_2 \geq 0$

(2) $H_1 - 2H_2 \geq 0$

(3) $-H_1 - 2H_2 \geq 0$

+ 〃

Take, for instance, $\boxed{H_1 = -2 H_2}$

Then, (1) and (3) are satisfied with equality

and (2) is satisfied only if $H_2 < 0$

Hence,

$$H = \left(-5(-2H_2) - 10H_2, -2H_2, H_2 \right)^T = (0, -2H_2, H_2)^T$$

with $H_2 < 0$

Actually, all A.O. in this market are of this form.

$$(1) + (3) \Rightarrow H_1 + 2H_2 = 0$$

11 Exercise 2. c)

$K = 4$, $B(0) = 1$, $S_1(0) = 5$, $S_2(0) = 10$

$$S(1,\Omega) = \begin{pmatrix} 10/9 & 60/9 & 40/3 \\ 10/9 & 60/9 & 80/9 \\ 10/9 & 40/9 & 80/9 \\ 10/9 & 20/9 & 120/9 \end{pmatrix} \qquad \text{Claim } X = \begin{pmatrix} 40 \\ 30 \\ 20 \\ 10 \end{pmatrix}$$

By exercise 7 we know that $M \neq \emptyset$

$$M = \{ Q \in \mathbb{R}^4 : Q = \left(\tfrac{1}{2}(1-\lambda), \lambda, 1/2 - \lambda, \lambda/2\right)^T, \ \lambda \in (0, 1/2) \}$$

Hence, by the second fundamental theorem of asset pricing the market is not complete.

Since $M \neq \emptyset$

X is attainable $\Leftrightarrow$ $E_{Q_\lambda}\left[\frac{X}{B(1)}\right] = \frac{9}{10} E_{Q_\lambda}[X]$ is constant for all λ

We have that

$$E_{Q_\lambda}[X] = \frac{1}{2}(1-\lambda)X_1 + \lambda X_2 + (1/2-\lambda)X_3 + \frac{\lambda}{2}X_4$$

$$= \lambda\left(-\frac{X_1}{2} + X_2 - X_3 + \frac{X_4}{2}\right) + \frac{1}{2}X_1 + \frac{1}{2}X_3$$

therefore,

$$E_{Q_\lambda}[X] \text{ does not depend on } \lambda \iff -\frac{X_1}{2} + X_2 - X_3 + \frac{X_4}{2} = 0$$

$$\iff \boxed{-X_1 + 2X_2 - 2X_3 + X_4 = 0} \quad (1)$$

and X is attainable $\iff$ (1) holds

For $X = (40, 30, 20, 10)^T$ we have that

$$-40 + 2\times 30 - 2\times 20 + 10 = -10 \neq 0 \text{ and, hence,}$$

X is not attainable $\Rightarrow$ the arbitrage free price of X belongs to the interval $[V_-(X), V_+(X)]$

$$V_-(X) = \inf_{Q \in M} E_Q\left[\frac{X}{D(4)} \right]$$

$$= \frac{9}{10} \inf_{0 < \lambda < 1/2} \left\{ \lambda \left(-\frac{X_2}{2} + X_2 - X_3 + \frac{X_1}{2} \right) + \frac{1}{2}(X_2 + X_3) \right\}$$

$$= \frac{9}{10} \inf_{0 < \lambda < 1/2} \left\{ -5\lambda + 30 \right\} = \frac{9}{10} \left\{ -\frac{5}{2} + 30 \right\} = \frac{9}{10} \cdot \frac{55}{2}$$

$$= 24'75$$

$$V_+(X) = \sup_{Q \in M} E_Q\left[\frac{X}{D(4)} \right] = \frac{9}{10} \sup_{0 < \lambda < 1/2} \left\{ -5\lambda + 30 \right\} = \frac{9}{10} \left\{ 0 + 30 \right\}$$

$$= 27$$