$\boxed{7}$ Exercise 2. c) $K=3$, $N=2$

$S_1(0) = 5$, $S_2(0) = 10$

$$S(1,\Omega) = \begin{pmatrix} 10/9 & 60/9 & 40/3 \\ 10/9 & 60/9 & 80/9 \\ 10/9 & 40/9 & 80/9 \\ 10/9 & 20/9 & 120/9 \end{pmatrix}$$

$$S^*(1,\Omega) = \begin{pmatrix} 1 & 6 & 12 \\ 1 & 6 & 8 \\ 1 & 4 & 8 \\ 1 & 2 & 12 \end{pmatrix}$$

We search for $Q \in \mathbb{R}^4$, $Q > 0$ s.t. $E_Q[S_n^*(1)] = S_n^*(0)$ $n=1,2$.

$\Leftrightarrow$ $Q^T S^*(1,\Omega) = (1, S_1^*(0), S_2^*(0))$

$$\Leftrightarrow \left.\begin{array}{l} Q_1 + Q_2 + Q_3 + Q_4 = 1 \\ 6Q_1 + 6Q_2 + 4Q_3 + 2Q_4 = 5 \\ 12Q_1 + 8Q_2 + 8Q_3 + 12Q_4 = 10 \end{array}\right\} \quad Q_4 = 1 - Q_1 - Q_2 - Q_3$$

$$\Downarrow$$

$$\Rightarrow \left.\begin{array}{l} 4Q_1 + 4Q_2 + 2Q_3 = 3 \\ -4Q_2 - 4Q_3 = -2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{array}{l} Q_3 = 1/2 - Q_2 \\ 4Q_1 + 4Q_2 + 2Q_3 = 3 \end{array}\right\} \Rightarrow Q_1 = \frac{1}{4}\left(3 - 4Q_2 - 2\left(1/2 - Q_2\right)\right)$$

$$= \frac{1}{2}\left(1 - Q_2\right)$$

$Q_2 = \lambda$ and we get

$$Q = \left(\frac{1}{2}(1-\lambda),\ \lambda,\ 1/2 - \lambda,\ 1 - \frac{1}{2}(1-\lambda) - \lambda - \left(\frac{1}{2} - \lambda\right)\right)^T$$

$$= \left(\frac{1}{2}(1-\lambda),\ \lambda,\ \frac{1}{2} - \lambda,\ \lambda/2\right) \quad \text{with } \lambda \in (0, 1/2)$$

to ensure $Q > 0$

The market is arbitrage free because the set of RNPM is non empty (F.F.T.A.P)

$\boxed{1}$

- $S_n^*(t) = \dfrac{S_n(t)}{B(t)}$, $S(t) = (S_1(t), \dots, S_N(t))^T$

  $t = 0, 1.$

- $B^*(t) = \dfrac{B(t)}{B(t)} = 1$
- $V(t) = H_0 B(t) + \sum_{n=1}^{N} H_n S_n(t)$
- $V^*(t) = H_0 + \sum_{n=1}^{N} H_n S_n^*(t)$

a) Show that $V^*(t) = \dfrac{V(t)}{B(t)}$

$$V^*(t) = H_0 \frac{B(t)}{B(t)} + \sum_{n=1}^{N} H_n \frac{S_n(t)}{B(t)}$$

$$= \frac{1}{B(t)} \left(H_0 B(t) + \sum_{n=1}^{N} H_n S_n(t) \right) = \frac{V(t)}{B(t)} \checkmark$$

b) Show that $V^*(1) = V^*(0) + G^*$

Recall that $G^* = \sum_{n=1}^{N} H_n \Delta S_n^*$

Then.

$$V^*(1) = H_0 + \sum_{n=1}^{N} H_n S_n^*(1)$$

$$= H_0 + \sum_{n=1}^{N} H_n S_n^*(1) - \sum_{n=1}^{N} H_n S_n^*(0) + \sum_{n=1}^{N} H_n S_n^*(0)$$

$$= \underbrace{H_0 + \sum_{n=1}^{N} H_n S_n^*(0)}_{V^*(0)} + \underbrace{\sum_{n=1}^{N} H_n \Delta S_n^*}_{G^*}$$

2 a) $K = 3$, $N = 1$, $r = 1/9$, $S_1(0) = 5$

$$S_1(1) = \left(20/3,\ 40/9,\ 30/9 \right)^T$$

$$B(0) = 1, \quad B(1) = 10/9$$

$$H = (H_0, H_1)^T$$

$$S(1,\omega) = \begin{pmatrix} 10/9 & 20/3 \\ 10/9 & 40/9 \\ 10/9 & 30/9 \end{pmatrix}, \quad \Delta S(\omega) = \begin{pmatrix} 1/9 & 5/3 \\ 1/9 & -5/9 \\ 1/9 & -5/3 \end{pmatrix}$$

$$V(0) = V^*(0) = H_0 B(0) + H_1 S_1(0) = H_0 + 5 H_1$$

$$V(1) = S(1,\omega) H = \begin{pmatrix} 10/9\, H_0 + 20/3\, H_1 \\ 10/9\, H_0 + 40/9\, H_1 \\ 10/9\, H_0 + 30/9\, H_1 \end{pmatrix}$$

$$V^*(1) \underset{\uparrow}{=} \frac{V(1)}{B(1)} = \frac{1}{10/9} S(1,\omega) H = \begin{pmatrix} H_0 + 6 H_1 \\ H_0 + 4 H_1 \\ H_0 + 3 H_1 \end{pmatrix}$$

By Exercise 1

$$G = \Delta S(\Omega)\, H = \begin{pmatrix} 1/9\, H_0 + 5/3\, H_1 \\ 1/9\, H_0 - 5/9\, H_1 \\ 1/9\, H_0 - 5/3\, H_1 \end{pmatrix}$$

$$G^* \underset{\text{by Exercise}}{=} V^*(1) - V^*(0)\, \underset{\sim}{1}$$

$\uparrow$ vector of $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} H_0 + 6H_1 \\ H_0 + 4H_1 \\ H_0 + 3H_1 \end{pmatrix} - (H_0 + 5H_1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} H_1 \\ -H_1 \\ -2H_1 \end{pmatrix}$$

Alternatively, you can compute

$$S_1^*(1) = \frac{1}{B(1)} S_1(1) = \frac{9}{10} \begin{pmatrix} 20/3 \\ 40/9 \\ 30/9 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix}$$

and $\Delta S_1^* = S_1^*(1) - S_1^*(0) = S_1^*(1) - S_1(0) = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

and use the definition of $V^*(L)$ and G^*

$$V^*(L) = H_0 + H_L S_L^*(L) = H_0 + H_L \begin{pmatrix} 6 \\ 4 \\ 3 \end{pmatrix} = \begin{pmatrix} H_0 + 6 H_L \\ H_0 + 4 H_L \\ H_0 + 3 H_L \end{pmatrix}$$

$$G^*(L) = H_1 \Delta S_L^* = H_L \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} H_L \\ -H_L \\ -2 H_L \end{pmatrix}$$