$\boxed{3}$ $K = 3$, $N = 2$, $B(0) = 1$, $S_1(0) = 4$, $S_2(0) = 7$

$$S(1,\Omega) = \begin{pmatrix} 1 & 8 & 10 \\ 1 & 6 & 8 \\ 1 & 3 & 4 \end{pmatrix}$$

1) $\exists$ dominant trading strategies ?

2) LOP holds ?

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1) Equivalent to $\nexists$ of linear pricing measures.

$\pi \in \mathbb{R}^K$, $\pi \geq 0$ is a linear pricing measure

$\Leftrightarrow$ $\pi \geq 0$, $\sum \pi_i = 1$ and $S_n^*(0) = E_\pi[S_n^*(1)]$

$(1, S_1^*(0), S_2^*(0)) = \pi^T S^*(1,\Omega)$, where $S^*(1,\Omega) = \frac{1}{B(1)} S(1,\Omega)$

In this exercise $B(0) = B(1) \Rightarrow r = 0 \Rightarrow S^*(1,\Omega) = S(1,\Omega)$

the previous equations read

$$\left.\begin{aligned} 1 &= \pi_1 + \pi_2 + \pi_3 \\ 4 &= 8\pi_1 + 6\pi_2 + 3\pi_3 \\ 7 &= 10\pi_1 + 8\pi_2 + 4\pi_3 \end{aligned}\right\} \Rightarrow \pi_3 = 1 - \pi_1 - \pi_2$$

$$\Downarrow$$

$$\left.\begin{aligned} 1 &= 5\pi_1 + 3\pi_2 \\ 3 &= 6\pi_1 + 4\pi_2 \end{aligned}\right\} \Rightarrow \pi_2 = \frac{1 - 5\pi_1}{3}$$

$$\Downarrow$$

$$9 = 18\pi_1 + 4(1 - 5\pi_1)$$

$\Rightarrow \pi_1 = -\frac{9}{2} < 0 \Rightarrow \nexists$ LPM

To actually find a D.T.S. is in general more difficult

We need $H = (H_0, H_1, H_2)^T$ such that

$$0 = V(0) = H_0 + H_1 S_1(0) + H_2 S_2(0) = H_0 + 4H_1 + 7H_2 \qquad (*)$$

and

$$0 < V(1) = S(1,\omega)\, H = \begin{pmatrix} H_0 + 8H_1 + 10H_2 \\ H_0 + 6H_1 + 8H_2 \\ H_0 + 3H_1 + 4H_2 \end{pmatrix} \qquad (**)$$

From $(*)$ we get that $H_0 = -4H_1 - 7H_2$ and combined with $(**)$ we obtain

$$\left.\begin{aligned} 4H_1 + 3H_2 &> 0 \\ 2H_1 + H_2 &> 0 \\ -H_1 - 3H_2 &> 0 \end{aligned}\right\} \begin{aligned} &(1) \\ &(2) \\ &(3) \end{aligned}$$

If $H_1 > 0$, $(3) \Rightarrow H_2 < -\frac{H_1}{3}$, (1) and (2) $H_2 > \max\left(-\frac{4H_1}{3}, -2H_1\right) = -\frac{4H_1}{3}$

i.e. $H_2 \in \left(-\frac{4H_1}{3}, -\frac{H_1}{3}\right)$

If $H_1 = 0 \Rightarrow (3) \Rightarrow H_2 < 0$, but then (1) and (2) cannot be satisfied. i.e. $\nexists$ DTS with $H_1 = 0$.

If $H_1 < 0$, $(3) \Rightarrow H_2 < -\frac{H_1}{3}$, (1) and (2) $\Rightarrow H_2 > \max(-\frac{H_1}{3}, -2H_1)$
$= -2H_1$

i.e., $\nexists$ D.T.S. with $H_1 \geq 0$.

2) LOP hold if $\nexists$ $\hat{H}$ and $\tilde{H}$ such that $\hat{V}(1,\omega) = \tilde{V}(1,\omega), \omega \in \Omega$ and $\hat{V}(0) > \tilde{V}(0)$.

Note that $\tilde{H} = \hat{H} \Rightarrow \hat{V}(0) = \tilde{V}(0)$, If we can show that for an arbitrary $V(1)$ there $\exists!$ H that generates $V(1)$ we can conclude that LOP holds

This condition boils down to show that $V(1) = S(1,\Omega)H$ has a unique solution for arbitrary $V(1)$.

Since in this exercise $S(1,\Omega)$ is a square matrix this is equivalent to $\det(S(1,\Omega)) \neq 0$ or $\operatorname{rank}(S(1,\Omega)) = 3$

$$\left(\begin{array}{ccc|c} 1 & 8 & 10 & V_1(t) \\ 1 & 6 & 8 & V_2(t) \\ 1 & 3 & 4 & V_3(t) \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 8 & 10 & V_1(t) \\ 0 & -2 & -2 & V_2(t)-V_1(t) \\ 0 & -5 & -6 & V_3(t)-V_1(t) \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 8 & 10 & V_1(t) \\ 0 & -2 & -2 & V_2(t)-V_1(t) \\ 0 & 0 & -1 & V_3(t)-V_1(t)-\frac{5}{2}\left(V_2(t)-V_1(t)\right) \end{array}\right)$$

$\Downarrow$

$\mathrm{rank}\,(J(4\,1)) = 3$

$\Rightarrow$ $H_2 = -V_3(t) + \frac{5}{2}V_2(t) - \frac{3}{2}V_1(t)$, $H_1 = 2V_1(t) - 3V_2(t) + V_3(t)$

$H_0 = -V_2(t) + 2V_3(t)$