(5) $\mathcal{W} = \{ X \in \mathbb{R}^K : X = G^* = \sum_{n=1}^{N} H_n \Delta S_n^*$, for some trading strategy$\}$

$\mathcal{W}^\perp = \{ Y \in \mathbb{R}^K : Y^T X = 0 \, , \forall X \in \mathcal{W} \}$

Show that they are linear spaces

Let $\hat{X} = \sum_{n=1}^{N} \hat{H}_n \Delta S_n^*$ and $\tilde{X} = \sum_{n=1}^{N} \tilde{H}_n \Delta S_n^*$ belong to $\mathcal{W}$

We need to show that $a\hat{X} + b\tilde{X} \in \mathcal{W}$, $a, b \in \mathbb{R}$.

Set $H = a\hat{H} + b\tilde{H}$. Then,

$$\underbrace{\sum_{n=1}^{N} H_n \Delta S_n^*}_{\in \mathcal{W}} = \sum_{n=1}^{N} (a\hat{H}_n + b\tilde{H}_n) \Delta S_n^*$$

$$= a \sum_{n=1}^{N} \hat{H}_n \Delta S_n^* + b \sum_{n=1}^{N} \tilde{H}_n \Delta S_n^*$$

$$= a\hat{X} + b\tilde{X}$$

Hence, $H = a\hat{H} + b\tilde{H}$ is a trading strategy with discounted gains process equal to $a\hat{X} + b\tilde{X} \Rightarrow a\hat{X} + b\tilde{X} \in \mathcal{W}$

To show that $W^\perp$ is a linear subspace.

Consider $\hat{y}$ and $\tilde{y} \in W^\perp$ and $a, b \in \mathbb{R}$

Set $y = a\hat{y} + b\tilde{y}$

For any $x \in W$

$$x^T y = x^T(a\hat{y} + b\tilde{y}) = a\, x^T\hat{y} + b\, x^T\tilde{y}$$

$\hat{y}, \tilde{y} \in W^\perp \longrightarrow \; = a \cdot 0 + b \cdot 0 = 0 \Rightarrow y \in W^\perp$

$\Rightarrow W^\perp$ is a linear subspace.

6) Find $\mathcal{W}$ and $\mathcal{W}^\perp$ in some cases.

a) $K=2$, $N=1$, $r=1/9$; $S_1(0)=5$, $S_1(1)=\begin{pmatrix} 20/3 \\ 40/9 \end{pmatrix}$

$$S_1^*(1) = \frac{1}{B(1)} S_1(1) = \frac{1}{10/9} \begin{pmatrix} 20/3 \\ 40/9 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$\Delta S_1^* = \begin{pmatrix} 6 \\ 4 \end{pmatrix} - 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$G^* = H_1 \Delta S_1^* = H_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} H_1 \\ -H_1 \end{pmatrix}$$

Hence, $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathcal{W} \iff$ for some $H_1 \in \mathbb{R}$ $X_1 = H_1$, $X_2 = -H_1$

$$\iff X_1 + X_2 = 0$$

Hence,

$$\mathcal{W} = \{ X = (X_1, X_2)^T \in \mathbb{R}^2 : X_1 + X_2 = 0 \}$$

Let $X = (X_1, X_2)^T \in \mathcal{W} \setminus \{0\}$, $Y = (Y_1, Y_2) \in \mathbb{R}^2$

We will find conditions on y_1, y_2 such that $x^T y = 0$

$$0 = x^T y = x_1 y_1 + x_2 y_2 \underset{\substack{\uparrow \\ x \in W \\ x_2 = -x_1 \\ x_1 \in \mathbb{R}.}}{=} x_1 y_1 - x_1 y_2 = x_1 (y_1 - y_2)$$

$$\Leftrightarrow y_1 - y_2 = 0$$

Hence,

$$W^\perp = \{ x = (y_1, y_2)^T \in \mathbb{R}^2 : \; y_1 - y_2 = 0 \}$$

$$\begin{pmatrix} y_1 \\ y_1 \end{pmatrix} = y_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \dim(W^\perp) = 1.$$