Exercise 8

$K = 2$, $N = 1$, $r \geq 0$, $B(0) = 1$, $B(1) = 1+r$

$S_1(0) = 1$, $S_1(1) = \begin{cases} u & w = w_1 \\ d & w = w_2 \end{cases}$ $u > d > 0$

Conditions under which $\exists\, Q \in \mathbb{R}^2$ a RNPM.

$Q_1 > 0$, $Q_2 > 0$

$Q_1 + Q_2 = 1$ (1)

$S_1^*(0) = E_Q[S_1^*(1)] \iff 1+r = u Q_1 + d Q_2$ (2)

Set $Q_1 = \lambda$, then by (1), we get $Q_2 = 1-\lambda$

Combining with (2) we obtain

$\lambda u + (1-\lambda) d = 1+r \iff \lambda(u-d) = 1+r-d$

$\iff \lambda = \dfrac{1+r-d}{u-d}$

To ensure $0 < \lambda < 1$

$$\frac{1+r-d}{u-d} > 0 \iff 1+r > d$$

$$\frac{1+r-d}{u-d} < 1 \iff 1+r-d < u-d \iff 1+r < u$$

$$\boxed{d < 1+r < u} \Rightarrow \exists! \text{ RNPM } Q = \left(\frac{1+r-d}{u-d}, \frac{u-(1+r)}{u-d}\right)^T$$

the market is arbitrage free and complete by the first and second F.T.A.P.

For the complementary values of the parameters describe A.O.

To find A.O. recall that $H = (H_0, H_1)^T$ is an A.O. if $V^*(0) = 0$, $V^*(1,\omega) \geq 0$, $\omega \in \Omega$, and $\exists \tilde{\omega}$ s.t. $V^*(1,\tilde{\omega}) > 0$

$$0 = V^*(0) = H_0 \cdot 1 + H_1 \cdot 1 \iff H_0 = -H_1$$

$$V^*(1,\omega) = H_0 + H_1 \frac{S_1(1)}{B(1)} = \begin{cases} H_0 + H_1 \frac{u}{1+r} & \omega = \omega_1 \\ H_0 + H_1 \frac{d}{1+r} & \omega = \omega_2 \end{cases}$$

$$= \begin{cases} H_1\left(\frac{u}{1+r} - 1\right) = H_1 \frac{u-1-r}{1+r} & \omega = \omega_1 \\ H_1\left(\frac{d}{1+r} - 1\right) = H_1 \frac{d-1-r}{1+r} & \omega = \omega_2 \end{cases}$$

If $\boxed{u \leq 1+r}$ $V^*(1,\omega) \geq 0 \iff H_1 \leq 0$ and $V^*(1,\omega) > 0$

$\Updownarrow$

$H_1 < 0$, and then $H = (-H_1, H_1)$, $H_1 < 0$ is A.O.

If $\boxed{d \geq 1+r}$ $V^*(1,\omega) \geq 0 \iff H_1 \geq 0$ and $V^*(1,\omega) > 0$

$\Updownarrow$

$H_1 > 0$, and then $H = (-H_1, H_1)$ $H_1 > 0$ is A.O.

1) Exercise 6. a)

$K = 2$, $\Omega = \{\omega_1, \omega_2\}$, $r = 1/9$, $B(0) = 1$, $B(1) = \frac{10}{9}$

$N = 1$, $S_1(0) = 5$, $S_1(1,\omega) = \begin{cases} 20/3 & \omega = \omega_1 \\ 40/9 & \omega = \omega_2 \end{cases}$

a) Price of a put option with strike 5.

b) What trading strategy generates this contingent claim.

a) $X(\omega) = (K - S(1,\omega))^+ = \begin{cases} K - S_1(1,\omega), & S_1(1,\omega) < K \\ 0, & S_1(1,\omega) > K \end{cases}$

$= \begin{cases} 0 & \omega = \omega_1 \\ 5 - \frac{40}{9} = \frac{5}{9} & \omega = \omega_2 \end{cases}$

In this market we have a unique R.N.P.M. $Q = (1/2, 1/2)^T$

⇒ the market is arbitrage free and complete.

The price p of the contingent claim is given by

$$p = E_Q\left[X/_{B(1)} \right] = E_Q\left[\frac{9}{10} X \right]$$

$$= \frac{9}{10} \left(\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{5}{9} \right) = 1/4$$

by the R.N.P principle.

e) We need to find $H = (H_0, H_1)^T$ such that

$$X = V(1) = H_0 B(1) + H_1 S_1(1) \iff \begin{cases} \omega = \omega_1 \Rightarrow \frac{10}{9} H_0 + \frac{20}{3} H_1 = 0 \\ \omega = \omega_2 \Rightarrow \frac{10}{9} H_0 + \frac{40}{9} H_1 = \frac{5}{9} \end{cases}$$

$$\iff \quad H_0 = 3/2$$

$$H_1 = -1/4$$

10 Let $r \geq 0$ $\quad$ C price of a call, P price of a put $\Big\}$ same strike K

1) Both are attainable or neither are attainable.

2) When they are attainable

$$C - P = S(0) - K(1+r)^{-1}$$

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1) Call option is attainable

$\Updownarrow$

$\exists\, H = (H_0, H_1)^{+}$ s.t. $\quad (S(1) - K)^{+} = H_0(1+r) + H_1 S(1)$

Note that we can write $\quad (K - S(1))^{+} = (S(1) - K)^{+} + K - S(1).$

Therefore,

$$(S(1) - K)^{+} + K - S(1) = H_0(1+r) + H_1 S(1) + K - S(1)$$

$\Updownarrow$

$$\Updownarrow$$

$$(K - S(1))^+ = \left(H_0 + \frac{K}{1+r}\right)(1+r) + (H_1 - 1)\, S(1)$$

$$\Updownarrow$$

Put option is attainable with

$$\tilde{H} = \left(H_0 + \frac{K}{1+r},\ H_1 - 1\right)^+$$

2) The prices are

$$C = H_0 + H_1 S(0) \quad (V(0))$$

$$P = \hat{H}_0 + \hat{H}_1 S(0) = H_0 + \frac{K}{1+r} + (H_1 - 1)\, S(0)$$

$$\Rightarrow C - P = S(0) - \frac{K}{1+r}$$

(E2) Exercise 2 a)

$$K = 3, \quad B(0) = 1, \quad S(1,\lambda) = \begin{pmatrix} 10/9 & 20/9 \\ 10/9 & 40/9 \\ 10/9 & 30/9 \end{pmatrix}$$

We know from (2) a) that $\mathcal{M} \neq \emptyset$ and

$$\mathcal{M} = \{ Q \in \mathbb{R}^3 : Q = (\lambda, 2-3\lambda, 2\lambda-1)^T, \lambda \in (1/2, 2/3) \}$$

Since $\mathcal{M} \neq \emptyset$

X is attainable $\iff E_Q[X/B(1)]$ does not depend on $Q \in \mathcal{M}$.

We compute

$$E_Q\left[\frac{X}{B(1)}\right] = \frac{9}{10}\left(\lambda X_1 + (2-3\lambda) X_2 + (2\lambda-1) X_3\right)$$

$$= \frac{9}{10} \lambda (X_1 - 3X_2 + 2X_3) + \frac{9}{10}(2X_2 - X_3)$$

Therefore,

X is attainable $\iff$ $\boxed{X_1 - 3X_2 + 2X_3 = 0}$ $*$

X is the payoff of a call option

$$X = (S_2(1) - K)^+ = \begin{cases} (20/3 - K)^+ & \omega = \omega_1 \\ (40/3 - K)^+ & \omega = \omega_2 \\ (30/9 - K)^+ & \omega = \omega_3 \end{cases}$$

- $K \geq 20/3 \Rightarrow X = (0, 0, 0) \Rightarrow (*)$ ✓ $\Rightarrow X$ is attainable.
- $\frac{40}{9} \leq K < \frac{20}{3} \Rightarrow X = \left(\frac{20}{3} - K, 0, 0\right)^+ \Rightarrow (*)$ doesn't hold $\Rightarrow X$ is not attainable.
- $\frac{30}{9} \leq K < \frac{40}{9} \Rightarrow X = \left(\frac{20}{3} - K, \frac{40}{9} - K, 0\right)^+$

  $(*)$ ✓ $\iff \frac{20}{3} - K - 3\left(\frac{40}{9} - K\right) = 0 \iff -\frac{20}{3} + 2K = 0$

  $\iff K = \frac{10}{3} = \frac{30}{9} \Rightarrow$ if $K > 30/9$ X is not attainable

  if $K = 30/9$ X is attainable.

- $K < \frac{30}{9} \Rightarrow X = \left(20/3 - K,\ \frac{40}{9} - K,\ \frac{30}{9} - K\right)^T$

$\Rightarrow$ (*) ✓ $\Leftrightarrow \frac{20}{3} - K - 3\left(\frac{40}{9} - K\right) + 2\left(\frac{30}{9} - K\right) = 0$

$\Leftrightarrow -\frac{20}{3} + 2K + \frac{60}{9} - 2K = 0$

$\Rightarrow X$ is attainable.

According to exercise (10) the put option is attainable

$\Leftrightarrow$ the call option is attainable.