14 $N=1$, $K=2$, $S(0)=5$, $S(1,\omega)=\begin{cases} 20/3 & \omega=\omega_1 \\ 40/9 & \omega=\omega_2 \end{cases}$

and $B(0)=1$, $r=1/9 \Rightarrow B(1)=\frac{10}{9}$

Solve
$$\max_H E[\mathcal{U}(V(1))]$$
subj. to $V(0)=v>0$

and $P(\omega_1)=p\in(0,1)$, $P(\omega_2)=1-p$

Note that
$$S^*(1,\omega)=\frac{S(1,\omega)}{B(1)}=\begin{cases} 6 & \omega=\omega_1 \\ 4 & \omega=\omega_2 \end{cases} \qquad \Delta S^*(1,\omega)=\begin{cases} 1 & \omega=\omega_1 \\ -1 & \omega=\omega_2 \end{cases}$$

$$V(1,\omega)=B(1)V^*(1,\omega)=B(1)\{V^*(0)+G^*\}=B(1)\{v+H_1\Delta S^*(1,\omega)\}$$
$$=\begin{cases} \frac{10}{9}(v+H_1) & \omega=\omega_1 \\ \frac{10}{9}(v-H_1) & \omega=\omega_2 \end{cases}$$

a) $u(x) = \log(x)$. Then

$$E[u(V(1))] = p \log\left(\frac{10}{9}(v + H_1)\right) + (1-p)\log\left(\frac{10}{9}(v - H_1)\right)$$
$$=: F(H_1)$$

The first order optimality condition

$$0 = \frac{\partial F}{\partial H_1}(H_1) = \frac{10}{9} p \frac{1}{\frac{10}{9}(v+H_1)} - \frac{10}{9}(1-p)\frac{1}{\frac{10}{9}(v-H_1)}$$

$$\Leftrightarrow \quad p(v - H_1) = (1-p)(v + H_1) \Leftrightarrow \boxed{H_1 = (2p-1)v}$$

Using $v = V(0) = H_0 + S(0)H_1 = H_0 + 5H_1$

$$\Leftrightarrow \quad H_0 = v - 5H_1 = v - 5(2p-1)v = (6-10p)v$$

$$E[u(V(1))] = p\log\left(\frac{20}{9}pv\right) + (1-p)\log\left(\frac{20}{9}(1-p)v\right)$$

15 $\quad u(w) = \log(w)$

Then, $u'(w) = \frac{1}{w} = i \iff w = \frac{1}{i}$

Therefore, $I(i) = (u')^{-1}(i) = \frac{1}{i}$

The optimal attainable wealth $\hat{W}$ is, then, given by

$$\hat{W} = I\left(\hat{\lambda}\frac{L}{B(1)}\right) = \frac{B(1)}{\hat{\lambda} L} \quad \text{where } L(\omega) = \frac{Q(\omega)}{P(\omega)}$$

To find the optimal Lagrange multiplier $\hat{\lambda}$ we use

$$E_Q\left[\frac{\hat{W}}{B(1)}\right] = v \iff E_Q\left[\frac{\cancel{B(1)}}{\hat{\lambda} L \cancel{B(1)}}\right] = v$$

$$\iff \hat{\lambda} = \frac{E_Q[L^{-1}]}{v} \iff \hat{\lambda} = \frac{\sum_{\omega\in\Omega} \frac{P(\omega)}{Q(\omega)} Q(\omega)}{v} = \frac{1}{v}$$

$\hat{\lambda}$ is constant

or $E_Q[L^{-1}] = E[L L^{-1}] = E[1] = 1$

Now we can compute an explicit expression for $\hat{W}$

$$\hat{W} = I\left(\hat{\lambda}\frac{L}{B(1)}\right) = I\left(\frac{L}{v\,B(1)}\right) = \frac{v\,B(1)}{L}$$

the optimal expected utility is

$$E[u(\hat{W})] = E[\log(\hat{W})] = E\left[\log\left(\frac{v\,B(1)}{L}\right)\right]$$

$$= \log(v) + E\left[\log\left(\frac{B(1)}{L}\right)\right]$$

$$= \log(v) - E\left[\log\left(\frac{L}{B(1)}\right)\right]$$

$N=2$, $K=2$, $r=1/9$, $B(0)=1$, $B(1)=\frac{10}{9}$, $S(0)=5$, $S(1)=\begin{cases} 20/9 & \omega=\omega_1 \\ 40/9 & \omega=\omega_2 \end{cases}$

$P(\omega_1)=3/5$, $P(\omega_2)=\frac{2}{5}$. One can check that $Q=(1/2, 1/2)^T$

Therefore,

$$L = \frac{Q}{P} = (5/6,\ 5/4)^T$$

$$\hat{w} = v\, B(1)\, \frac{1}{L} = \begin{cases} \frac{10}{9} v\, \frac{6}{5} = \frac{4}{3} v & \omega = \omega_1 \\ \frac{10}{9} v\, \frac{4}{5} = \frac{8}{9} v & \omega = \omega_2 \end{cases}$$

Next, we determine $H = (H_0, H_1)^T$ such that $V(1) = \hat{w}$.

That is,

$$H_0 B(1) + H_1 S(1) = \hat{w} \iff \begin{matrix} \frac{10}{9} H_0 + \frac{20}{3} H_1 = \frac{4}{3} v \\ \frac{10}{9} H_0 + \frac{40}{9} H_1 = \frac{8}{9} v \end{matrix}$$

$$\Rightarrow \boxed{\begin{matrix} H_0 = 0 \\ H_1 = v/5 \end{matrix}}$$

Note that

$$L/B(1) = \begin{cases} 5/6 \ \frac{9}{10} = 3/4 \\ 5/4 \ \frac{9}{10} = 9/8 \end{cases}$$

and therefore

$$E\left[\log\left(\frac{L}{B(1)}\right)\right] = \frac{3}{5}\log(3/4) + \frac{2}{5}\log(9/8)$$

Finally

$$E[U(\hat{w})] = \log(v) - \frac{3}{5}\log(3/4) - \frac{2}{5}\log(9/8)$$

List 4

(2) $\mathbb{F}$ filtration on a finite prob. space $(\Omega, \mathcal{F}, P)$

(2) Show that a $\mathbb{F}$-adapted process $M = (M(t))_{t=0,\dots,T}$ is a martingale $\Longleftrightarrow$ $E[M(t+1) \mid \mathcal{F}_t] = M(t)$

$t = 0, \dots, T-1.$

Our definition of Martingales:

(1) M is a Martingale $\Leftrightarrow$ M is $\mathbb{F}$-adapted and $E[M(t+s) \mid \mathcal{F}_t] = M(t)$

$t, s \geq 0.$

a) That (1) implies (2) follows from $s = 1$.

b) Follows from applying the tower property of conditional expectation. That is, If $\mathcal{G}_1 \subseteq \mathcal{G}_2$

(3) $\mathbb{E}[X \mid \mathcal{G}_1] = E[E[X \mid \mathcal{G}_2] \mid \mathcal{G}_1] = E[E[X \mid \mathcal{G}_1] \mid \mathcal{G}_2]$

Since $\mathbb{F}$ is a filtration we have that $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$, $t = 0, \dots T-1$.

Fix $t \in \{0, \dots T\}$. and $s \in \{0, \dots, T-t\}$, then

$$
\begin{aligned}
E[M(t+s) \mid \mathcal{F}_t] &\overset{(3)}{=} E[E[M(t+s) \mid \mathcal{F}_{t+s-1}] \mid \mathcal{F}_t] \\
&\overset{(2)}{=} E[M(t+s-1) \mid \mathcal{F}_t] \\
&\vdots \\
&\overset{(3)}{=} E[E[M(t+2) \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t] \\
&\overset{(2)}{=} E[M(t+1) \mid \mathcal{F}_t] \\
&\overset{(2)}{=} M(t)
\end{aligned}
$$

c) Prove that a martingale has constant expectation equal to $E[M(0)]$.

$$E[M(0)] = E[E[M(t) | \mathcal{F}_0]] = E[M(t)] \quad t = 1, \dots T.$$

law of total expectation.

Moreover, note that if $\mathcal{F}_0 = \{\Omega, \emptyset\}$, then $M(0)$ is a constant and $E[M(0)] = M(0)$.

(2) $\mathbb{F}$ a filtration on a finite prob. space $(\Omega, \mathcal{F}, P)$.

Show that if $H = \{H(t)\}_{t=1,\dots,T}$ is $\mathbb{F}$-predictable and

$S = \{S(t)\}_{t=0,\dots,T}$ is an $\mathbb{F}$-adapted martingale. Then

$$M(t) = \sum_{n=1}^{t} H(n)\,(S(n) - S(n-1)) \qquad t = 0,\dots,T.$$

is an $\mathbb{F}$-adapted martingale.

First, we need to show that the process M is $\mathbb{F}$-adapted.

- H $\mathbb{F}$-predictable $\Longrightarrow$ H is $\mathbb{F}$-adapted.
- If X and Y are $\mathcal{G}$-measurable, then $X \cdot Y$ and $X + Y$ are $\mathcal{G}$-measurable.
- S is $\mathbb{F}$-adapted. $\Rightarrow$ $S(n) - S(n-1)$ is $\mathcal{F}_n$-measurable.
  $\mathbb{F}$ is a filtration
- $H(n)(S(n) - S(n-1))$ is $\mathcal{F}_n$-measurable $\Rightarrow$ $M(t)$ is adapted.
  $\mathbb{F}$ is a filtration

To prove the martingale property, first note

$$M(t+1) = M(t) + H(t+1)\,\big(S(t+1) - S(t)\big) \qquad (4)$$

Then,

$$E[M(t+1) \mid \mathcal{F}_t] \overset{(4)}{=} E[M(t) + H(t+1)\,(S(t+1) - S(t)) \mid \mathcal{F}_t]$$

linearity of cond. expect.

$$= E[M(t) \mid \mathcal{F}_t] + E[H(t+1)\,(S(t+1) - S(t)) \mid \mathcal{F}_t]$$

$$= M(t) + H(t+1)\,E[(S(t+1) - S(t)) \mid \mathcal{F}_t]$$

($M(t)$ is $\mathcal{F}_t$-measurable, $H(t+1)$ is $\mathcal{F}_t$-measurable + what is $\mathcal{F}_t$-measurable goes out of cond. expect.)

$$= M(t)$$

↑ S is a $\mathbb{F}$-martingale.

3 $\mathbb{F}$ - a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ a finite probability space.

X a random variable

Show that the process $Y = \{ Y(t) = E[X \mid \mathcal{F}_t] \}_{t=0,\dots,T}$

is a $\mathbb{F}$ - martingale.

- Y is trivially $\mathbb{F}$-adapted because for each t $Y(t)$ is a cond. expect. with respect to $\mathcal{F}_t$. $\Rightarrow$ $\mathcal{F}_t$ - measurable.
- To show the martingale property.

$$E[Y(t+1) \mid \mathcal{F}_t] = E[E[X \mid \mathcal{F}_{t+1}] \mid \mathcal{F}_t]$$

definition of Y

$$= E[X \mid \mathcal{F}_t] = Y(t)$$

$\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$ + the tower property of conditional expectations.